Coinductive Predicates and Final Sequences in a Fibration

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Abstract

Coinductive predicates express persisting “safety” specifications of transition systems. Previous observations by Hermida and Jacobs identify coinductive predicates as suitable final coalgebras in a *fibration*—a categorical abstraction of predicate logic. In this paper we follow the spirit of a seminal work by Worrell and study final sequences in a fibration. Our main contribution is to identify some categorical “size restriction” axioms that guarantee stabilization of final sequences after ω steps. In its course we develop a relevant categorical infrastructure that relates fibrations and locally presentable categories, a combination that does not seem to be studied a lot. The genericity of our fibrational framework can be exploited for: binary relations (i.e. the logic of “binary predicates”) for which a coinductive predicate is bisimilarity; constructive logics (where interests are growing in coinductive predicates); and logics for name-passing processes.

Keywords: coalgebra; (co)recursive predicate; modal logic; fibration; locally presentable category

1 Introduction

Coinductive predicates postulate properties of state-based dynamic systems that persist after a succession of transitions. In computer science, *safety properties* of nonterminating, reactive systems are examples of paramount importance. This has led to an extensive study of specification languages in the form of fixed point logics and model-checking algorithms.

In this paper we follow [26,27] (further extended in [5,20]; see also [32, Chap. 6]) and take a categorical view on coinductive predicates. Here *coalgebras* represent transition systems; a *fibration* is a “predicate logic”; and a coinductive predicate is identified as a suitable coalgebra in a fibration. Our contribution is the study of *final sequences*—an iterative construction of final coalgebras that is studied notably in [2,44]—in such a fibrational setting.

Coalgebras have been successfully used as a categorical abstraction of transition systems (see e.g. [32,41]): by varying base categories and functors, coalgebras bring general results that work for a variety of systems at once. Fixed point logics (or modal logics in general), too, have been actively studied coalgebraically: coalgebraic modal logic is a prolific research field (see [12]); their base category is typically **Sets** but works like [34] go beyond and use presheaf categories for processes in name-passing calculi; and literature including [11, 13, 43] studies coalgebraic fixed point logics.

Unlike most of these works, we follow [26, 27] and parametrize the underlying "predicate logic" too with the categorical notion of *fibration*. The conventional setting of classical logic is represented by the fibration $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ (see Appendix A.3 for an introduction to fibrations).

However there are various other "logics" modeled as fibrations, and hence the fibrational language provides a uniform treatment of these different settings. An example is binary relations (instead of unary predicates) that form a fibration $\begin{smallmatrix}\mathbf{Rel}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ (see Appendix A.3). In this case coinductive predicates are bisimilarity (see the table, and Example 5.12 later).

fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$	$\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$	$\begin{smallmatrix}\mathbf{Rel}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$
coalgebra	invariant	bisimulation
final coalgebra	coindutive predicate	bisimilarity

Another example is predicates in constructive logics. They are modeled by the subobject fibration of a topos. In fact, coinductive predicates in constructive logics are an emerging research topic: coinduction is supported in the theorem prover Coq (based on the constructive *calculus of constructions*), see e.g. [6]; and, working in Coq, some interesting differences between classically equivalent (co)inductive predicates have been studied e.g. in [39].

Yet another example is modal logics for processes in various name-passing calculi. They are best modeled by the subobject fibration of a suitable (pre)sheaf category like $\mathbf{Sets}^{\mathbf{I}}$ and $\mathbf{Sets}^{\mathbf{F}}$.

1.1 Coinductive Predicates and Their Construction, Conventionally

In order to illustrate our technical contributions (§3) we first present a special case, with classical logic and Kripke models. We first introduce syntax.

Definition 1.1 (Rudimentary logic $\mathsf{R}\nu$) This fragment of the μ-calculus allows only one greatest fixed-point operator at the outermost position.

$$\mathsf{R}\nu_u \ni \alpha \ ::= \ a \mid \overline{a} \mid \Box u \mid \Diamond u \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \ ; \qquad \mathsf{R}\nu \ni \beta \ ::= \ \nu u.\, \alpha \ . \qquad (1)$$

Here a belongs to the set AP of *atomic propositions*; $\overline{a}$ stands for the negation of a; and u is the only fixed-point variable (with possibly multiple occurrences).

An $\mathsf{R}\nu$-formula can be thought of as a recursive definition of a coinductive predicate. Later we will model such a "definition" categorically as a predicate lifting. A specification expressible in $\mathsf{R}\nu$ is (may-) deadlock freedom ("there is an infinite path"). It is expressed by $\nu u.\, \Diamond u$ and is our recurring example.

An $\mathsf{R}\nu$-formula is interpreted in Kripke models. Let $c = (X, \to, V)$ be a Kripke model, where X is a state space, $\to \subseteq X \times X$ is a transition relation and $V : X \to \mathcal{P}(\mathsf{AP})$ is a valuation. The conventional interpretation $[\nu u.\alpha]_c$ of $\mathsf{R}\nu$-formulas in the

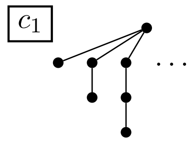
Kripke model c is given as follows (see e.g. [9]). Firstly, we interpret $\alpha \in \mathsf{R}\nu_u$ as a function $[\alpha]_c : \mathcal{P}X \to \mathcal{P}X$. Concretely:

$$\begin{array}{ll}
[a]_c(P) = \{x \mid a \in V(x)\} & [\overline{a}]_c(P) = \{x \mid a \notin V(x)\} \\
[\Box u]_c(P) = \{x \mid \forall y \in X.\, (x \to y \text{ implies } y \in P)\} & [\Diamond u]_c(P) = \{x \mid \exists y \in X.\, (x \to y \text{ and } y \in P)\} \\
[\alpha \wedge \alpha']_c(P) = [\alpha]_P \cap [\alpha']_P & [\alpha \vee \alpha']_c(P) = [\alpha]_P \cup [\alpha']_P
\end{array}$$

This function $[\alpha]_c$ is easily seen to be monotone, since u occurs only positively in α. Finally we define $[\nu u.\alpha]_c \subseteq X$ to be the greatest fixed point of the monotone function $[\alpha]_c : \mathcal{P}X \to \mathcal{P}X$.

The Knaster-Tarski theorem guarantees the existence of such a greatest fixed point $[\nu u.\alpha]_c$ in a complete lattice $\mathcal{P}X$. However its proof is highly nonconstructive. In contrast, a well-known construction [14] by Cousot and Cousot computes $[\nu u.\alpha]_c$ as the limit of the following descending chain (see also [9]). Here $\top$ denotes the subset $X \subseteq X$.

$$\top \;\geq\; [\alpha]_c \top \;\geq\; [\alpha]_c^2 \top \;\geq\; \cdots \tag{2}$$



An issue now is the length of the chain. If $[\alpha]_c$ preserves limits $\bigwedge$ (which is the case with $\alpha \equiv \Box u$), clearly ω steps are enough and yields $\bigwedge_{i\in\omega}([\alpha]_c^i \top)$ as the greatest fixed point. This is not the case with $\alpha \equiv \Diamond u$. Indeed, for the Kripke model c_1 on the right $[\nu u.\,\Diamond u]_{c_1} \neq \bigwedge_{i\in\omega}([\Diamond u]_{c_1}^i \top)$: there is no infinite path from the root; but it satisfies $[\Diamond u]_{c_1}^i \top$ ('there is a path of length $\geq i$') for each i.

Yet the chain (2) eventually stabilizes, bounded by the size of the poset $\mathcal{P}X$. Therefore the calculation of $[\nu u.\alpha]_c$ is, in general, via *transfinite* induction. This is what we call a *state space bound* for (2).

Besides a state space bound, another (possibly better and seemingly less known) bound can be obtained from a *behavioral view.* One realizes that not only the size of the state space X but also the *branching degree* can be used to bound the length of the chain (2). For example, a result similar to [24, Thm. 2.1] states that the chain stabilizes after ω steps if the Kripke model c is *finitely branching.* This holds however large the state space X is; and also for any $\mathsf{R}\nu$-formula $\nu u.\alpha$. Notice that the model c_1 (depicted above) is not finitely branching.

1.2 Final Sequences in a Fibration

This paper is about putting the observations in §1.1 in general categorical terms. Our starting observation is that the chain (2) resembles a *final sequence*, a classic construction of a final coalgebra.

In the theory of coalgebra a *final F-coalgebra* is of prominent importance since it is a fully abstract domain with respect to the *F-behavioral equivalence.* Therefore a natural question is if a final F-coalgebra exists; the well-known Lambek lemma prohibits e.g. a final $\mathcal{P}$-coalgebra. What matters is the *size* of F: when it is suitably bounded, it is known that a final coalgebra can be constructed via the following *final F-sequence.*

$$1 \xleftarrow{\;!\;} F1 \xleftarrow{F\,!} \cdots \xleftarrow{F^{i-1}\,!} F^i 1 \xleftarrow{F^i\,!} \cdots \tag{3}$$

Here 1 is a final object in $\mathbb{C}$, and $!$ is the unique arrow. In particular, if F is *finitary*, a final coalgebra arises as a suitable quotient of the ω-limit of the final sequence (3). This construction in **Sets** is worked out in [44]; it is further extended to locally presentable categories (those are categories suited for speaking of "size") with additional assumptions in [2].

Turning back to coinductive predicates, indeed, the fibrational view [26,27] identifies coinductive predicates as final coalgebras in a fibration. This leads us to scrutinize final sequences in a fibration. Our main result (Thm. 3.7) is a categorical generalization of the behavioral ω-bound (§1.1)—more precisely we axiomatize categorical "size restrictions" for that bound to hold.

The conditions are formulated in the language of locally presentable categories (see e.g. [4]; also Appendix A.2); and the combination of fibrations and locally presentable categories does not seem to have been studied a lot (an exception is [37, §5.3]). We therefore develop a relevant categorical infrastructure (§5.1). Our results there include a sufficient condition for the total category $\mathrm{Sub}(\mathbb{C})$ of a subobject fibration to be locally (finitely) presentable, and the same for a family fibration $\mathrm{Fam}(\Omega)$ too. Via these results, in §5.2 we list some concrete examples of fibrations to which our results in §3 on the behavioral bounds apply. They include: $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ (classical logic); $\begin{smallmatrix}\mathbf{Rel}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ (for bisimulation and bisimilarity); $\begin{smallmatrix}\mathrm{Sub}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{smallmatrix}$ for $\mathbb{C}$ that is locally finitely presentable and locally Cartesian closed (a topos is a special case); and $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ for a well-founded algebraic lattice Ω.

1.3 Summary and Future Work

To summarize, our contributions are: 1) combination of the mathematical observations in [26,27] and [32, Chap. 6] for a general formulation of coinductive predicates; 2) categorical behavioral bounds for final sequences that approximate coinductive predicates; and 3) a categorical infrastructure that relates fibrations and locally presentable categories.

While our focus is on coinductive predicates, inductive ones are just as important in system verification; so are their combinations. Such mixture of induction and coinduction is studied fibrationally in [25], but over mixed inductive and coinductive data types, and not over a coalgebra. We have obtained some preliminary fibrational observations in this direction.

Search for useful coinduction proof principles is an active research topic (see e.g. [8, 28]). We are interested in the questions of whether these principles are sound in a general fibrational setting, and what novel proof principles a fibrational view can lead to.

Coalgebraic modal logic is more and more often introduced based on a Stone-like duality (see e.g. [34]). Fibrational presentation of such dualities will combine the benefits of duality-based modal logics and the current results. We are also interested in the relationship to *coalgebraic infinite traces* [10, 30].

Kozen's *metric coinduction* [35] is a construction of coinductive predicates by the Banach fixed point theorem and is an alternative to the current paper's order-theoretic one. Its fibrational formulation is an interesting future topic.

Practical applications of our categorical behavioral bounds shall be pursued, too. Our results' precursor—the bounds for the final sequences in **Sets** [2,44]—have been used to bound execution of some algorithms e.g. for state minimization [3, 15, 16]. We aim at similar use. Finally, *games* are an extremely useful tool in fixed point logics (also in their coalgebraic generalization, see [11,13,43]; also [36]). We plan to investigate the use of games in the current (even more general) fibrational setting.

Organization of the Paper

In §2 we identify coinductive predicates as final coalgebras in a fibration, following the ideas of [26, 27, 32]. The main technical results are in §3, where we axiomatize size restrictions on fibrations and functors for a final sequence to stabilize after ω steps. These results are reorganized in §4 as a fibration of invariants. §5 is devoted to examples: first we develop a necessary categorical infrastructure then we discuss concrete examples.

In Appendix A we present minimal introductions to the theories of coalgebras, locally presentable categories and fibrations—the three topics that our technical developments rely on. Most proofs are deferred to Appendix B.

2 Coinductive Predicates as Final Coalgebras

In this section we follow the ideas in [26, 27, 32] and characterize coinductive predicates in various settings (for different behavior types, and in various underlying logics) in the language of fibration. An introduction to fibration is e.g. in [29]; see also Appendix A.3. In this paper for simplicity we focus on poset fibrations. It should however not be hard to move to general fibrations.

Definition 2.1 (Fibration) We refer to poset fibrations (where each fiber is a poset rather than a category) simply as *fibrations*.

Definition 2.2 (Predicate lifting) Let $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ be a fibration and F be an endofunctor on $\mathbb{C}$. A *predicate lifting* of F along p is a functor $\varphi : \mathbb{P} \to \mathbb{P}$ such that (φ, F) is an endomap of fibrations. This means: that the diagram on the right commutes; and that φ preserves Cartesian arrows, that is, $\varphi(f^*Q) = (Ff)^*(\varphi Q)$. See below.

$$\begin{tikzcd} \mathbb{P} \arrow[r, "\varphi"] \arrow[d, "p"'] & \mathbb{P} \arrow[d, "p"] \\ \mathbb{C} \arrow[r, "F"'] & \mathbb{C} \end{tikzcd} \tag{4}$$

$$\begin{tikzcd} \mathbb{P} \arrow[dd, "p"] & f^*Q \arrow[r, "\overline{f}Q"] & Q & \varphi(f^*Q) \arrow[r, "\varphi(\overline{f}Q)"] \arrow[d, phantom, "\shortparallel"] & \varphi Q \\ & & & (Ff)^*(\varphi Q) \arrow[ur, "\overline{Ff}(\varphi Q)"'] & \\ \mathbb{C} & X \arrow[r, "f"] & Y & FX \arrow[r, "Ff"] & FY \end{tikzcd} \tag{5}$$

In the prototype example $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$, the above definition coincides (see [32]) with the one used in coalgebraic modal logic (see e.g. [12])—presented as a (monotone) natural transformation $2^{(\_)} \overset{\varphi}{\Rightarrow} 2^{F(\_)} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$.

We think of predicate liftings as (co)recursive definitions of coinductive predicates (see Example 2.4). On top of it, we identify coinductive predicates (and invariants) as coalgebras in a fiber.

Definition 2.3 (Invariant, coinductive predicate) Let φ be a predicate lifting of F along $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$; and $X \xrightarrow{c} FX$ be a coalgebra in $\mathbb{C}$. They together induce an endofunctor (a monotone function) on the fiber $\mathbb{P}_X$, namely $\mathbb{P}_X \xrightarrow{\varphi} \mathbb{P}_{FX} \xrightarrow{c^*} \mathbb{P}_X$, where φ restricts to $\mathbb{P}_X \to \mathbb{P}_{FX}$ because of (4).

(i) A φ-*invariant* in c is a $(c^* \circ \varphi)$-coalgebra in $\mathbb{P}_X$, that is, an object $P \in \mathbb{P}_X$ such that $P \leq c^*(\varphi P)$ in $\mathbb{P}_X$.

(ii) The φ-*coinductive predicate* in c is the final $(c^* \circ \varphi)$-coalgebra (if it exists). Its carrier shall be denoted by $[\![\nu\varphi]\!]_c$. It is therefore the largest φ-invariant in c; Lambek's lemma yields that $[\![\nu\varphi]\!]_c = (c^* \circ \varphi)([\![\nu\varphi]\!]_c)$.

Example 2.4 ($\mathsf{R}\nu$) The conventional interpretation $[\nu u.\alpha]_c$ (described in §1.1) of $\mathsf{R}\nu$-formulas is a special case of Def. 2.3. Indeed, let us work in the fibration $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$, and with the endofunctor $F_{\mathrm{K}} = \mathcal{P}(\mathsf{AP}) \times \mathcal{P}(\_)$ on **Sets**. An F_{K}-coalgebra $X \xrightarrow{c} \mathcal{P}(\mathsf{AP}) \times \mathcal{P}X$ is precisely a Kripke model: c combines a valuation $X \to \mathcal{P}(\mathsf{AP})$ and the map $X \to \mathcal{P}X$ that carries a state to the set of its successors. To each formula $\alpha \in \mathsf{R}\nu_u$ we associate a predicate lifting φ_α of F_{K}. This is done inductively as follows.

$$\begin{array}{ll}
\varphi_a(U \subseteq X) = \big(\{V \in F_{\mathrm{K}}X \mid a \in \pi_1(V)\} \subseteq F_{\mathrm{K}}X\big) & \varphi_{\overline{a}}(U \subseteq X) = \big(\{V \mid a \notin \pi_1(V)\} \subseteq F_{\mathrm{K}}X\big)\\
\varphi_{\Box u}(U \subseteq X) = \big(\{V \mid \pi_2(V) \subseteq U\} \subseteq F_{\mathrm{K}}X\big) & \varphi_{\Diamond u}(U \subseteq X) = \big(\{V \mid \exists x \in U.\, x \in \pi_2(V)\} \subseteq F_{\mathrm{K}}X\big)\\
\varphi_{\alpha\wedge\alpha'}(U \subseteq X) = \big((\varphi_\alpha U \cap \varphi_{\alpha'} U) \subseteq F_{\mathrm{K}}X\big) & \varphi_{\alpha\vee\alpha'}(U \subseteq X) = \big((\varphi_\alpha U \cup \varphi_{\alpha'} U) \subseteq F_{\mathrm{K}}X\big)
\end{array} \tag{6}$$

In the above, π_1 and π_2 denote the projections from $F_{\mathrm{K}}X = \mathcal{P}(\mathsf{AP}) \times \mathcal{P}X$. Then it is easily seen by induction that $[\![\nu\varphi_\alpha]\!]_c$ in Def. 2.3 coincides with the conventional interpretation $[\nu u.\alpha]_c$ described in §1.1.

In fact, the predicate liftings φ_α in (6) are the ones commonly used in coalgebraic modal logic (where they are presented as natural transformations). We point out that the same definition of φ_α—they are written in the internal language of toposes—works for the subobject fibration $\begin{smallmatrix}\mathrm{Sub}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{smallmatrix}$ of any topos $\mathbb{C}$. Therefore the categorical definition of coinductive predicates (Def. 2.3) allows us to interpret the language $\mathsf{R}\nu$ in constructive underlying logics. Suitable completeness of $\mathbb{C}$ ensures that a final $(c^* \circ \varphi)$-coalgebra in Def. 2.3 exists.

Proposition 2.5 *Let φ be a predicate lifting of F along $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$; $X \xrightarrow{c} FX$ be a coalgebra in $\mathbb{C}$; and $P \in \mathbb{P}_X$. We have $P \leq [\![\nu\varphi]\!]_c$ if and only if there exists a φ-invariant Q such that $P \leq Q$.* □

The proposition is trivial but potentially useful. It says that an invariant can be used as a "witness" for a coinductive predicate. This is how bisimilarity is commonly established; and it can be used e.g. in [1, §6] as an alternative to the metric coinduction principle used there. [1]

[1] To be precise: only if we take PE in [1] as an atomic proposition (and that is essentially what is done in the proofs in [1, §6]). Our future work on nested μ's and ν's will more adequately address the situation.

Remark 2.6 The coalgebraic modal logic literature exploits the fact that there can be many predicate liftings (in the form of natural transformations) of the same functor F. Different predicate liftings correspond to different modalities (such as $\Box$ vs. $\Diamond$ for the same functor $\mathcal{P}$). This view of predicate liftings is also the current paper's (see Example 2.4).

In contrast, in fibrational studies like [5, 20, 26, 27], use of predicate liftings has focused on the validity of the *(co)induction proof principle*. For such purposes it is necessary to choose a predicate lifting φ that is "comprehensive enough," covering all the possible F-behaviors. In fact, it is common in these studies that "the" predicate lifting, denoted by $\mathrm{Pred}(F)$, is assigned to a functor F. An exception is [31].

3 Final Sequences in a Fibration

Here we present our main technical result (Thm. 3.7). It generalizes known behavioral ω-bounds (like [24, Thm. 2.1]; see §1.1); and claims that the chain (2) for a coinductive predicate stabilizes after ω steps, assuming that the behavior type functor F and the underlying logic $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ are "finitary" in a suitable sense (but no size restriction on φ).

3.1 Size Restrictions on a Fibration

We axiomatize finitariness conditions in the language of locally presentable categories (see Appendix A.2 for a minimal introduction). Singling out these conditions lies at the heart of our technical contribution.

Definition 3.1 (LFP category) A category $\mathbb{C}$ is *locally finitely presentable (LFP)* if it is cocomplete and it has a (small) set $\mathbb{F}$ of finitely presentable (FP) objects such that every object is a directed colimit of objects in $\mathbb{F}$.

Definition 3.2 (Finitely determined fibration) A (poset) fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ is *finitely determined* if it satisfies the following.

(i) $\mathbb{C}$ is LFP, with a set $\mathbb{F}$ of FP objects (as in Def. 3.1).

(ii) $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ has fiberwise limits and colimits.

(iii) For arbitrary $X \in \mathbb{C}$, let $(X_I)_{I\in\mathbb{I}}$ be the canonical diagram for X with respect to $\mathbb{F}$ (i.e. $\mathbb{I} = (\mathbb{F}\downarrow X)$), with a colimiting cocone $(X_I \overset{\kappa_I}{\to} X)_{I\in\mathbb{I}}$. Then for any $P, Q \in \mathbb{P}_X$,

$$P \le Q \quad\iff\quad \kappa_I^* P \le \kappa_I^* Q \text{ in } \mathbb{P}_{X_I} \text{ for each } I \in \mathbb{I}.$$

The intuition of Cond. iii) is that a predicate $P \in \mathbb{P}_X$ (over arbitrary $X \in \mathbb{C}$) is determined by its restrictions $(\kappa_I^* P)_{I\in\mathbb{I}}$ to FP objects X_I. One convenient sufficient condition for Cond. iii) is that the total category $\mathbb{P}$ is itself LFP, with its FP objects above the FP objects in $\mathbb{C}$ (Cor. 5.3). We note that Cond. i) guarantees, since LFP implies completeness, an (ω^{op}-)limit $F^\omega 1$ of the final F-sequence (3). However this does not mean (nor we need for later) that $F^\omega 1$ carries a final F-coalgebra (it fails for $F = \mathcal{P}_\omega$; see [44]).

Definition 3.3 (Well-founded fibration) A *well-founded fibration* is a finitely determined fibration that further satisfies:

(iv) If $X \in \mathbb{F}$ (hence FP), the fiber $\mathbb{P}_X$ is such that: the category $\mathbb{P}_X^{\mathrm{op}}$ consists solely of FP objects.

Since $\mathbb{P}_X$ is complete, this is equivalent to: there is no (ω^{op}-)chain $P_0 > P_1 > \cdots$ in $\mathbb{P}_X$ that is strictly descending.

We note that the following stronger variant of the condition

(iv') For *any* $X \in \mathbb{C}$, there is no strictly descending ω^{op}-chain in $\mathbb{P}_X$

rarely holds (it fails in $\begin{array}{c}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{array}$). The original Cond. iv) holds in many examples (as we will see later in §5) thanks to the restriction that X is FP.

The following trivial fact is written down for the record.

Lemma 3.4 *A finitely determined fibration* $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$ *is well-founded if $\mathbb{P}_X$ is a finite category for each $X \in \mathbb{F}$.* □

3.2 Final Sequences in a Fibration

The following result from [29, Prop. 9.2.1] is crucial in our development.

Lemma 3.5 *Let* $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$ *be a fibration, with $\mathbb{C}$ being complete. Then p has fiberwise limits if and only if $\mathbb{P}$ is complete and $p : \mathbb{P} \to \mathbb{C}$ preserves limits. If this is the case, a limit of a small diagram $(P_I)_{I\in\mathbb{I}}$ in $\mathbb{P}$ can be given by*

$$\textstyle\bigwedge_{I\in\mathbb{I}}(\pi_I^* P_I) \qquad \text{over } \mathrm{Lim}_{I\in\mathbb{I}} X_I .$$

Here $X_I := pP_I$; $(\mathrm{Lim}_{I\in\mathbb{I}} X_I \xrightarrow{\pi_I} X_I)_{I\in\mathbb{I}}$ is a limiting cone in $\mathbb{C}$; and $\bigwedge_{I\in\mathbb{I}}$ denotes the limit in the fiber $\mathbb{P}_{\mathrm{Lim}_I X_I}$. □

Fig. 1 presents two sequences. Here we assume that $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$ is finitely determined (Def. 3.2) and that φ is a predicate lifting of F. In the bottom diagram (in $\mathbb{C}$), the

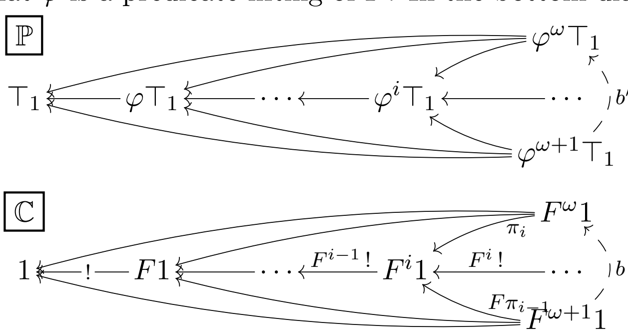


Figure 1. Final sequences in a fibration

object $1 \in \mathbb{C}$ is a final one (it exists since LFP implies completeness); $F1 \xrightarrow{!} 1$ is the unique map; $F^{\omega+1}1 := F(F^\omega 1)$; and b is a unique mediating arrow to the limit $F^\omega 1$. In the top diagram (in $\mathbb{P}$), the object $\top_1$ is the final object in the fiber $\mathbb{P}_1$; by Lem. 3.5 this is precisely a final object in the total category $\mathbb{P}$. Hence this diagram is nothing but a final sequence for the functor φ in $\mathbb{P}$. A limit $\varphi^\omega \top_1$ of this final sequence exists, again by Lem. 3.5, and moreover it can be chosen above $F^\omega 1$. We define $\varphi^{\omega+1}\top_1 := \varphi(\varphi^\omega \top_1)$.

Lemma 3.6 (Key lemma) *Let $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$ be a well-founded fibration; $F : \mathbb{C} \to \mathbb{C}$ be finitary; and φ be a predicate lifting of F. Then the final φ-sequence stabilizes after ω steps. More precisely: in Fig. 1, we have $\varphi^{\omega+1}\top_1 = b^*(\varphi^\omega \top_1)$.*

The object $\varphi^\omega \top_1$ is a "prototype" of φ-coinductive predicates in various coalgebras. This is one content of the following main theorem.

It is standard that a coalgebra $X \xrightarrow{c} FX$ in $\mathbb{C}$ induces a cone over the final F-sequence, and hence a mediating arrow $X \to F^\omega 1$ (see below). Concretely, $c_i : X \to F^i 1$ is defined inductively by: $X \xrightarrow{c_0} 1$ is $!$; and c_{i+1} is the composite $X \xrightarrow{c} FX \xrightarrow{Fc_i} F^{i+1}1$. The induced arrow to the limit $F^\omega 1$ is denoted by c_ω.

$$\begin{tikzcd}[row sep=small]
 & & & & F^\omega 1 \arrow[llll, bend right] \arrow[lll, bend right] \arrow[dl, "\pi_i"'] \\
1 & F1 \arrow[l, "!"'] & \cdots \arrow[l] & F^i 1 \arrow[l] & \cdots \arrow[l] \\
 & & & & X \arrow[llll, bend left] \arrow[lll, bend left] \arrow[ul, "c_i"] \arrow[uu, dashed, "c_\omega"']
\end{tikzcd} \tag{7}$$

Theorem 3.7 (Main result) *Let $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$ be a well-founded fibration; $F : \mathbb{C} \to \mathbb{C}$ be a finitary functor; φ be a predicate lifting of F along p; and $X \xrightarrow{c} FX$ be a coalgebra in $\mathbb{C}$.*

(i) *The φ-coinductive predicate $[\![\nu\varphi]\!]_c$ in c (Def. 2.3) exists. It is obtained by the following reindexing of $\varphi^\omega \top_1$, where c_ω is the mediating map in (7).*

$$[\![\nu\varphi]\!]_c \;=\; c_\omega^*(\varphi^\omega \top_1) \tag{8}$$

(ii) *Moreover, the predicate $[\![\nu\varphi]\!]_c$ is the limit of the following ω^{op}-chain in the fiber $\mathbb{P}_X$*

$$\top_X \;\geq\; (c^* \circ \varphi)(\top_X) \;\geq\; (c^* \circ \varphi)^2(\top_X) \;\geq\; \cdots \;, \tag{9}$$

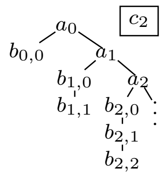
that stabilizes after ω steps. That is, $[\![\nu\varphi]\!]_c = \bigwedge_{i\in\omega}(c^* \circ \varphi)^i(\top_X)$. □

Example 3.8 ($\mathsf{R}\nu$) We continue Example 2.4 and derive from Thm. 3.7 the behavioral bound result described in §1.1: the chain (2) stabilizes after ω steps, for each $\alpha \in \mathsf{R}\nu_u$ and each *finitely branching* Kripke model c.

Indeed, the latter is the same thing as a coalgebra $X \xrightarrow{c} F_{\mathrm{fbK}}X$, where $F_{\mathrm{fbK}} = \mathcal{P}(\mathsf{AP}) \times \mathcal{P}_\omega(\_)$. Compared to F_{K} in Example 2.4 the powerset functor is restricted from $\mathcal{P}$ to $\mathcal{P}_\omega$; this makes F_{fbK} a finitary functor. Still the same definition of φ_α defines a predicate lifting of F_{fbK}. Thm. 3.7.ii can then be applied to the fibration $\begin{array}{c}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{array}$ (easily seen to be well-founded, Example 5.11), the finitary functor F_{fbK} and the predicate lifting φ_α for each α. It is not hard to see that the function $[\alpha]_c : \mathcal{P}X \to \mathcal{P}X$ in §1.1 coincides with $c^* \circ \varphi_\alpha : \mathbf{Pred}_X \to \mathbf{Pred}_X$ (note that

$\mathbf{Pred}_X \cong 2^X \cong \mathcal{P}X$); thus the chain (2) coincides with (9) that stabilizes after ω steps by Thm. 3.7.

Remark 3.9 The ω-bound of the length of the chain (9) is sharp. A (counter)example is given in the setting of Example 3.8, by the predicate lifting $\varphi_{\Diamond u}$ and the coalgebra (i.e. Kripke structure) c_2 on the right. There $b_{i,i}$ has no successors. Indeed, while $[\![\nu\varphi_{\Diamond u}]\!]_{c_2}$ is $\{a_i \mid i \in \omega\}$, its i-th approximant $((c_2)^*_i \circ \varphi^i_{\Diamond u})(\top_X)$ in (9) contains $b_{i,0}$ too.



Remark 3.10 It is notable that Thm. 3.7 imposes no size restrictions on $\varphi : \mathbb{P} \to \mathbb{P}$. Being a predicate lifting is enough.

Final F-sequences are commonly used for the construction of a final F-coalgebra. It is not always the case, however, that the limit $F^\omega 1$ is itself the carrier of a final coalgebra (even for finitary F; see [44, §5]). One obtains a final coalgebra either by: 1) quotienting $F^\omega 1$ by the behavioral equivalence (see e.g. [40]); or 2) continuing the final sequence till $\omega + \omega$ steps. The latter construction is worked out in [44] (in **Sets**) and in [2] (in LFP $\mathbb{C}$ with additional assumptions). Its relevance to the current work is yet to be investigated.

Coalgebra morphisms are compatible with coinductive predicates. This fact, like Prop. 2.5, is potentially useful in establishing coinductive predicates.

Proposition 3.11 *Let $f : X \to Y$ be a coalgebra morphism from $X \xrightarrow{c} FY$ to $Y \xrightarrow{d} FY$. In the setting of Lem. 3.6 and Thm. 3.7:*

(i) *If $Q \in \mathbb{P}_Y$ is a φ-invariant in d, so is $f^*Q \in \mathbb{P}_X$ in c.*

(ii) *We have $[\![\nu\varphi]\!]_c = f^*\big([\![\nu\varphi]\!]_d\big)$.* □

Remark 3.12 The current paper focuses on *finitely presentable* objects, *finitary* functors, etc.—i.e. the ω-presentable setting (see [4, §1.B]). This is for the simplicity of presentation: the results, as usual (as e.g. in [34]), can be easily generalized to the λ-presentable setting for an arbitrary regular cardinal λ. In such an extended setting we obtain a behavioral λ-bound.

4 A Fibration of Invariants

We organize the above observations in a more abstract fibered setting. The technical results are mostly standard; see e.g. [26, 27] and [32, Chap.6].

We write $\mathbf{Coalg}(F)$ for the category of F-coalgebras.

Proposition 4.1 *Let φ be a predicate lifting of F along $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$. Then the fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ is lifted to a fibration $\begin{smallmatrix}\mathbf{Coalg}(\varphi)\\ \downarrow \overline{p}\\ \mathbf{Coalg}(F)\end{smallmatrix}$, with two forgetful functors forming a map of fibrations from the latter to the former.* □

The next observation explains the current section's title.

Proposition 4.2 *Let* $\begin{smallmatrix}\mathbf{Coalg}(\varphi)\\ \downarrow\overline{p}\\ \mathbf{Coalg}(F)\end{smallmatrix}$ *be the lifted fibration in Prop. 4.1. For each coalgebra* $X \xrightarrow{c} FX$*, the fiber over* c *coincides with the poset of* φ*-invariants in* c*. That is:*

$$\begin{tikzcd}[column sep=small] \mathbf{Coalg}(\varphi)_{X \xrightarrow{c} FX} \arrow[rr, "\cong"] \arrow[dr] & & \mathbf{Coalg}(c^* \circ \varphi) \arrow[dl] \\ & \mathbb{P}_X & \end{tikzcd} .$$

□

Therefore Thm. 3.7.i) and Prop. 3.11.ii) state the fibration $\begin{smallmatrix}\mathbf{Coalg}(\varphi)\\ \downarrow\overline{p}\\ \mathbf{Coalg}(F)\end{smallmatrix}$ has fiberwise final objects. (At least part of) this statement itself is shown quite easily using the Knaster-Tarski theorem (each fiber is a complete lattice). Our contribution is its concrete construction as an ω^{op}-limit (Thm. 3.7.ii).

The following is an immediate consequence of Lem. 3.5.

Corollary 4.3 *Let* φ *be a predicate lifting of* F *along* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$*; and assume that a final* F*-coalgebra exists. The following are equivalent.*

(i) *The coinductive predicate* $[\![\nu\varphi]\!]_c$ *exists for each coalgebra* $c : X \to FX$*. Moreover they are preserved by reindexing (along coalgebra morphisms).*

(ii) *There exists a final* φ*-coalgebra that is above a final* F*-coalgebra.* □

5 Examples of Fibrations

5.1 Examples at Large

Here are several results that ensure a fibration to be finitely determined or well-founded, and hence enable us to apply Thm. 3.7. Some of them are well-known; others—especially those which relate fibrations and locally (finitely) presentable categories, including Lem. 5.4 and Cor. 5.7—seem to be new.

Lemma 5.1 [29, Prop. 5.4.7] *An (elementary) topos is a locally Cartesian closed category (LCCC).* □

The following results provide sufficient conditions for a fibration to be finitely determined (Def. 3.2). Recall that a full subcategory $\mathbb{F}$ of $\mathbb{P}$ is said to be *dense* if each object $P \in \mathbb{P}$ is a colimit of a diagram in $\mathbb{F}$.

Lemma 5.2 *Let* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ *be a fibration with fiberwise limits and colimits. Assume further that* $\mathbb{C}$ *is LFP with a set* $\mathbb{F}_\mathbb{C}$ *of FP objects (as in Def. 3.1). If the total category* $\mathbb{P}$ *has a dense subcategory* $\mathbb{F}_\mathbb{P}$ *such that every* $R \in \mathbb{F}_\mathbb{P}$ *is above* $\mathbb{F}_\mathbb{C}$ *(i.e.* $pR \in \mathbb{F}_\mathbb{C}$*), then* p *is finitely determined.* □

Corollary 5.3 *Let* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ *be a fibration with fiberwise limits and colimits, where* $\mathbb{C}$ *is LFP with a set* $\mathbb{F}_\mathbb{C}$ *of FP objects (in Def. 3.1). If the total category* $\mathbb{P}$ *is also LFP, with a set* $\mathbb{F}_\mathbb{P}$ *of FP objects (as in Def. 3.1) chosen so that every* $R \in \mathbb{F}_\mathbb{P}$ *is above* $\mathbb{F}_\mathbb{C}$*, then* p *is finitely determined.* □

The following is one of the results that are less trivial.

Lemma 5.4 *Let $\mathbb{C}$ be an LFP category with $\mathbb{F}$ being a set of FP objects (as in Def. 3.1). Assume that $\mathbb{C}$ is at the same time an LCCC. Then the total category* $\mathrm{Sub}(\mathbb{C})$ *of the subobject fibration is LFP: the set* $\mathbb{F}_{\mathrm{Sub}(\mathbb{C})} := \{ (P \rightarrowtail X) \mid P, X \in \mathbb{F}\}$ *consists of FP objects in* $\mathrm{Sub}(\mathbb{C})$*; and every object* $(Q \rightarrowtail Y) \in \mathrm{Sub}(\mathbb{C})$ *is a colimit of a directed diagram in* $\mathbb{F}_{\mathrm{Sub}(\mathbb{C})}$. □

It follows from Lem. 5.1, 5.4, and Cor. 5.3 that the internal logic of a topos that is LFP is finitely determined.

Corollary 5.5 *Let $\mathbb{C}$ be LFP and at the same time a topos (or more generally an LCCC). Then the subobject fibration* $\begin{smallmatrix}\mathrm{Sub}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{smallmatrix}$ *is finitely determined.* □

We turn to the family fibration $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ over a poset Ω (see Appendix A.3).

Lemma 5.6 *Let Ω be an* algebraic lattice*, i.e. a complete lattice in which each element is a join of compact elements. (Equivalently, Ω is LFP when considered as a category.) Then the set*

$$\mathbb{F}_{\mathrm{Fam}(\Omega)} := \{\, f : X \to \Omega \mid X \text{ is finite; for each } x \in X,\ f(x) \text{ is compact in } \Omega \,\} \tag{10}$$

consists of finitely generated objects and is dense in $\mathrm{Fam}(\Omega)$*. Therefore by Lem. 5.2,* $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ *is finitely determined.* □

It is known that the existence of a dense set of FG objects (like $\mathbb{F}_{\mathrm{Fam}(\Omega)}$ in Lem. 5.6) ensures the category to be locally λ-presentable. This is however for some regular cardinal λ that is possibly bigger than ω. See [4, Thm. 1.70].

Corollary 5.7 *Let Ω be an algebraic lattice. Then the total category* $\mathrm{Fam}(\Omega)$ *of* $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ *is locally presentable.* □

We turn to the notion of well-founded fibration (Def. 3.3; see also Lem. 3.4).

Example 5.8 (Presheaf categories) Let $\mathbb{A}$ be small. The presheaf category $\mathbf{Sets}^{\mathbb{A}}$ is LFP: the set $\mathbb{F}$ of finite colimits of representable presheaves $\mathbf{y}A$, where $\mathbf{y}A = \mathbb{A}(A, \_)$, satisfies the conditions of Def. 3.1.

The coming results are less trivial, too.

Lemma 5.9 *Let $\mathbb{A}$ be small. For any $X \in \mathbb{A}$,* $\mathrm{Sub}(\mathbf{y}X)$ *is finite if and only if the subset* $\{\mathrm{Im}(\mathbf{y}A \overset{\mathbf{y}f}{\to} \mathbf{y}X) \mid A \in \mathbb{A}, f\colon X \to A\} \subseteq \mathrm{Sub}(\mathbf{y}X)$ *is finite.*

As a special case, if every arrow f with domain $X \in \mathbb{A}$ factors $f = m \circ e$ as a split mono m followed by an epi e, then $\mathrm{Sub}(\mathbf{y}X)$ *is finite if and only if* $\mathrm{Quot}(X)$ *is finite. Here* $\mathrm{Quot}(X)$ *denotes the set of quotient objects of X.* □

Corollary 5.10 *If one of the conditions in Lem. 5.9 holds, the fibration* $\begin{smallmatrix}\mathrm{Sub}(\mathbf{Sets}^{\mathbb{A}})\\ \downarrow\\ \mathbf{Sets}^{\mathbb{A}}\end{smallmatrix}$ *is well-founded.* □

5.2 Concrete Examples

Example 5.11 (Pred) The fibration $\begin{smallmatrix}\mathbf{Pred}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ for the conventional setting of classical logic is easily seen to be well-founded. In particular, $\mathbf{Pred}_X \cong \mathcal{P}X$ is finite if X is FP (i.e. finite). Therefore to any finitary F and any predicate lifting φ, the results in §3 apply.

The (interpretations of the) formulas in $\mathsf{R}\nu$ (see Example 3.8) are examples of coinductive predicates in $\begin{smallmatrix}\mathbf{Pred}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$. Besides them, the study of coalgebraic modal logic has identified many predicate liftings for many functors F (probabilistic systems, neighborhood frames, strategy frames, weighted systems, etc.; see e.g. [12] and the references therein). These "modalities" all define coinductive predicates, to which the results in §3 may apply.

Example 5.12 (Rel) The fibration $\begin{smallmatrix}\mathbf{Rel}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ can be introduced from $\begin{smallmatrix}\mathbf{Pred}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ via change-of-base; concretely, an object of **Rel** is a pair (X, R) of a set X and a relation $R \subseteq X \times X$; an arrow $f : (X, R) \to (Y, S)$ is a function $f : X \to Y$ such that xRx' implies $f(x)Sf(x')$. See [29, p. 14].

This fibration is also easily seen to be well-founded; therefore to any finitary F the results in §3 apply. A predicate lifting φ along $\begin{smallmatrix}\mathbf{Rel}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ is more commonly called a *relation lifting* [27]; by choosing a suitable φ (a "sufficiently comprehensive" one) like in [27], a φ-invariant is precisely a bisimulation relation, and the φ-coinductive predicate is bisimilarity. We expect that the ω-behavioral bound in Thm. 3.7 can be used to bound execution of bisimilarity checking algorithms by partition refinement (for many different functors F).

In the following example, one can think of Ω as a Heyting algebra, and then the underlying logic becomes constructive.

Example 5.13 ($\mathrm{Fam}(\Omega)$) Let Ω be an algebraic lattice that has no strictly descending (ω^{op}-)chains. Then the family fibration $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ is well-founded (see Lem. 5.6). Therefore to any finitary F the results in §3 apply. It is not hard to interpret the language $\mathsf{R}\nu$ in this setting, by defining predicate liftings similar to (6). This gives examples of coinductive predicates in $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$.

Presheaf Examples

Let $\mathbf{F}$ be the category of natural numbers as finite sets (i.e. $n = \{0, 1, \dots, n-1\}$) and all functions between them; $\mathbf{F}_+$ be its full subcategory of nonzero natural numbers; and $\mathbf{I}$ be the category of natural numbers and injective functions. Coalgebras in the presheaf categories $\mathbf{Sets}^{\mathbf{F}}$, $\mathbf{Sets}^{\mathbf{F}_+}$ and $\mathbf{Sets}^{\mathbf{I}}$ are commonly used for modeling processes in various name-passing calculi. For the π-calculus $\mathbf{Sets}^{\mathbf{I}}$ has been found appropriate (see e.g. [17, 18]); while for the fusion calculus we do need non-injective functions in $\mathbf{F}$ or $\mathbf{F}_+$ (see [38, 42]).

Inspired by [34], we are interested in coinductive predicates for such processes. They are naturally modeled in the subobject fibration of a presheaf category. Here we find a distinction: the subobject fibrations of $\mathbf{Sets}^{\mathbf{F}}$ and $\mathbf{Sets}^{\mathbf{F}_+}$ are well-founded; but that of $\mathbf{Sets}^{\mathbf{I}}$ is not. In view of Cor. 5.5 and Example 5.8, the only condition to check is Cond. iv) in Def. 3.3.

Example 5.14 ($\mathrm{Sub}(\mathbf{Sets}^{\mathbf{F}})$, $\mathrm{Sub}(\mathbf{Sets}^{\mathbf{F}_+})$) The subobject fibration $\begin{array}{c}\mathrm{Sub}(\mathbf{Sets}^{\mathbf{F}_+})\\ \downarrow\\ \mathbf{Sets}^{\mathbf{F}_+}\end{array}$ is well-founded: this is shown by Cor. 5.10. An important fact here is that in $\mathbf{Sets}$ a mono with a nonempty domain splits.

The subobject fibration $\begin{array}{c}\mathrm{Sub}(\mathbf{Sets}^{\mathbf{F}})\\ \downarrow\\ \mathbf{Sets}^{\mathbf{F}}\end{array}$ is well-founded, too. To show that $\mathrm{Sub}(\mathbf{y}0)$ is finite, we appeal to the first half of Lem. 5.9: we observe that the set $\{\mathrm{Im}\,\mathbf{y}f \mid n \in \mathbf{F}, f\colon 0 \to n\}$ is equal to the two-element set $\big\{\mathrm{Im}(\mathbf{y}(0 \xrightarrow{\mathrm{id}_0} 0)), \mathrm{Im}(\mathbf{y}(0 \xrightarrow{!} 1))\big\}$ since $0 \xrightarrow{!} n$ and $0 \xrightarrow{!} m$ factor through each other, for each $n, m \geq 1$.

We turn to functors F and φ. In modeling processes of name-passing calculi as coalgebras in these categories, one typically uses endofunctors F that are constructed from the following building blocks. Let $\mathbf{N} \in \{\mathbf{F}, \mathbf{F}_+, \mathbf{I}\}$.

- Constant functors, binary sum $+$, binary product $\times$, and exponentials $(\_)^X$. These are much like for polynomial functors on $\mathbf{Sets}$. An important example of the first is the *name* presheaf $\mathcal{N} = \mathrm{Hom}(1, \_) \in \mathbf{Sets}^{\mathbf{N}}$.
- The *abstraction* functor $\delta : \mathbf{Sets}^{\mathbf{N}} \to \mathbf{Sets}^{\mathbf{N}}$ given by $\delta X = X(\_ + 1)$.
- The free semilattice functor $\mathcal{P}_{\mathrm{f}}$ for finite branching. This captures Kuratowski finiteness and suitable in $\mathbf{Sets}^{\mathbf{I}}$. See e.g. [17, 42].
- In $\mathbf{Sets}^{\mathbf{F}}$ and $\mathbf{Sets}^{\mathbf{F}_+}$, another choice of a "finite powerset functor" $\widetilde{K}$ is more appropriate. See [38]; also [42, p. 4].

All such functors are known to be finitary (see e.g. [38]).

Coinductive predicates in this setting can be introduced much like $\mathsf{R}\nu$ in Example 2.4 (note that $\mathbf{Sets}^{\mathbf{N}}$ is a topos), for properties like deadlock freedom. Such a language can be extended further through the modalities proposed in [34]: they correspond to constructions specific to presheaves and include the modality $\langle \overline{a}(b) \rangle$ for a binding 'input' operation. More examples will be worked out in our future paper.

Example 5.15 ($\mathrm{Sub}(\mathbf{Sets}^{\omega}), \mathrm{Sub}(\mathbf{Sets}^{\mathbf{I}})$) Consider the presheaf category $\mathbf{Sets}^{\omega}$ over the ordinal ω as a poset. The fibration $\begin{array}{c}\mathrm{Sub}(\mathbf{Sets}^{\omega})\\ \downarrow\\ \mathbf{Sets}^{\omega}\end{array}$ is finitely determined but not well-founded. It fails to satisfy Cond. iv) in Def. 3.3: let $P_n : \omega \to \mathbf{Sets}$ be the family of presheaves defined by

$$P_n(m) := \quad \big(0 \text{ if } m < n; \quad 1 \text{ if } n \leq m \big)$$

for each $n \in \omega$. Then $P_0 > P_1 > \cdots$ is a strictly descending chain in $\mathrm{Sub}(\mathbf{y}0)$. The same counterexample works for $\mathrm{Sub}(\mathbf{Sets}^{\mathbf{I}})$.

In contrast, the subobject fibration for $\mathbf{Sets}^{\omega^{\mathrm{op}}}$ is well-founded by Lem. 5.9.

Remark 5.16 Well-foundedness fails in $\mathrm{Sub}(\mathbf{Sets}^{\omega})$, $\mathrm{Sub}(\mathbf{Sets}^{\mathbf{I}})$, and in $\mathrm{Fam}(\Omega)$ for Ω that does have a strictly descending ω^{op}-chain. This means the logics modeled by the fibrations are inherently "big." Still, extensions of our results in §3 are possible from finitary (i.e. ω-presentable) to the λ-presentable setting for bigger λ, so that they apply to the (current) nonexamples.

Acknowledgments

Thanks are due to Kazuyuki Asada, Keisuke Nakano, Keiko Nakata, Ana Sokolova, and the participants of Dagstuhl Seminar 12411 "Coalgebraic Logics" (including Samson Abramsky, Vincenzo Ciancia, Corina Cîrstea, Ernst-Erich Doberkat, Clemens Kupke, Alexander Kurz, and Yde Venema) for useful discussions. We are grateful to the anonymous referees for their careful reading and useful suggestions, too. I.H., K.C. and T.K. are supported by Grants-in-Aid for Young Scientists (A) No. 24680001 and Grants-in-Aid for Challenging Exploratory Research No. 23654033, JSPS, and by Aihara Innovative Mathematical Modeling Project, FIRST Program, JSPS/CSTP.

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A Appendix: Preliminaries

A.1 Theory of Coalgebra

Given a category $\mathbb{C}$ and an endofunctor $F : \mathbb{C} \to \mathbb{C}$, an *$F$-coalgebra* is a pair of $X \in \mathbb{C}$ and an arrow $c : X \to FX$ (we shall denote a coalgebra simply by $X \xrightarrow{c} FX$). The notion has turned out to be a useful categorical abstraction of state-based dynamic systems. In an F-coalgebra $X \xrightarrow{c} FX$, the *carrier object* $X \in \mathbb{C}$ is understood as a state space; the functor F specifies the *behavior type*; and the arrow c represents actual dynamics. In the most common setting of $\mathbb{C} = \mathbf{Sets}$, examples of functors F (and the corresponding behavior types) are:

- $A \times (\_)$ for A-stream automata;
- $\mathcal{P}(\mathsf{AP}) \times \mathcal{P}(\_)$ for Kripke models;
- $\mathcal{P}(\mathsf{AP}) \times \mathcal{P}_\omega(\_)$ for finitely branching Kripke models, with where $\mathcal{P}_\omega$ is the finite powerset functor;
- $\mathcal{P}(A \times \_)$ for labeled transition systems;
- $\mathcal{D}(A \times \_)$ for generative probabilistic systems;

and so on. See [32, 41] for detailed introduction.

In the theory of coalgebra as a categorical theory of (state-based dynamical) systems, the notion of *final coalgebra* plays a prominent role. A final F-coalgebra $Z \xrightarrow{\zeta} FZ$ is one such that, for any F-coalgebra $X \xrightarrow{c} FX$, there is a unique morphism of coalgebras from c to ζ.

$$
\begin{tikzcd}
FX \arrow[r, dashed, "F\overline{c}"] & FZ \\
X \arrow[u, "c"] \arrow[r, dashed, "\overline{c}"] & Z \arrow[u, "\text{final}"', "\zeta"]
\end{tikzcd}
\tag{A.1}
$$

Its system-theoretic significance is that: 1 Z is often the collection of "all possible F-behaviors"; and 2 the induced arrow $\overline{c}$ assigns, to each state in X, its behavior. The "behaviors" here follow a black-box view on systems (it ignores internal states) and often captures the natural notion of "F-bisimilarity."

Therefore a question arises if a final F-coalgebra exists. The well-known Lambek lemma (that ζ is necessarily an iso) prohibits e.g. a final $\mathcal{P}$-coalgebra. What matters here is the *size* of F: when it is suitably bounded, a concrete construction of a final coalgebra is known. It obtains a final coalgebra via a *final F-sequence* (Here 1 is a final object in $\mathbb{C}$).

$$
1 \xleftarrow{\;!\;} F1 \longleftarrow \cdots \xleftarrow{F^{i-1}!} F^i 1 \xleftarrow{F^i !} \cdots
\tag{A.2}
$$

In particular, if F is *finitary* (a size restriction described later), a final coalgebra arises as a suitable quotient of the limit of the final sequence (3). This construction in **Sets** is worked out in [44]; it is further extended to locally presentable categories (those are categories suited for speaking of "size") with additional assumptions in [2]. The current paper's goal is to apply this construction also to coinductive predicates.

A.2 Locally Finitely Presentable Categories

The theory of coalgebra has been mainly developed in the base category $\mathbb{C} = \mathbf{Sets}$. Exceptions include the category of nominal sets or (pre)sheaf categories (e.g. [18,19]) for name-passing calculi, and Kleisli categories (e.g. [22,23]) for trace semantics and simulation. The current paper follows [2, 34] and finds locally finitely presentable categories a convenient abstract setting. Here we follow [4] and list a minimal set of definitions and results on locally finitely presentable categories.

The following is a categorical formalization of "finiteness" of objects. Examples are finite sets (in **Sets**), and algebras presented by finitely many generators and finitely many equations (in suitable categories of algebras).

Definition A.1 (Finitely presentable object) An object $X \in \mathbb{C}$ is *finitely presentable (FP)* if the functor $\mathbb{C}(X, \_) : \mathbb{C} \to \mathbf{Sets}$ preserves filtered colimits.

Definition A.2 (Locally finitely presentable category) A category $\mathbb{C}$ is *locally finitely presentable (LFP)* if it is cocomplete and it has a (small) set $\mathbb{F}$ of FP objects such that every object is a directed colimit of objects in $\mathbb{F}$.

Lemma A.3 *Let $\mathbb{C}$ be LFP, with a set $\mathbb{F}$ of FP objects as in Def. 3.1; and $X \in \mathbb{C}$. The* canonical diagram *for X with respect to $\mathbb{F}$*

$$(\mathbb{F}{\downarrow}X) \xrightarrow{\pi} \mathbb{F} \hookrightarrow \mathbb{C} \tag{A.3}$$

has X as its colimit. Here π is the projection.

Proof The proof of [4, Prop. 1.22] yields the claim. □

Lemma A.4 [4, Cor. 1.28 & Prop. 1.61] *Let $\mathbb{C}$ be LFP.*

(i) $\mathbb{C}$ *is complete.*

(ii) $\mathbb{C}$ *has* (StrongEpi, Mono)- *and* (Epi, StrongMono)*-factorization structures.* □

The following notion (which is already in Def. A.1) is about the "size" of functors. An intuition (when $\mathbb{C} = \mathbf{Sets}$) is: a functor F is finitary if F's action FX on an arbitrary set X is determined by its action FX' on all the finite subsets $X' \subseteq X$.

Definition A.5 (Finitary functor) An endofunctor $F : \mathbb{C} \to \mathbb{C}$ is *finitary* if it preserves filtered colimits.

This notion is commonly used to bound the "branching degree" of systems as F-coalgebras. For example, the finite powerset functor $\mathcal{P}_\omega$ is finitary; the (full) powerset functor $\mathcal{P}$ is not.

There are many LFP categories, among which are **Sets**, the category **Posets** of posets and monotone maps, and categories of algebras with finitary operations. See [4] for more examples.

Example A.6 (Presheaf categories) Let $\mathbb{A}$ be a small category. The presheaf category $\mathbf{Sets}^{\mathbb{A}}$ is LFP: the set

$$\mathbb{F} := \{\text{finite colimits of representable presheaves } \mathbf{y}A\} \ ,$$

where $\mathbf{y}A = \mathbb{A}(A, \_)$, satisfies the conditions of Def. A.1.

Definition A.7 (Finitely generated object) An object $X \in \mathbb{C}$ is *finitely generated (FG)* if the functor $\mathbb{C}(X, \_) : \mathbb{C} \to \mathbf{Sets}$ preserves directed colimits of monos—that is, directed colimits of diagrams in which every (connecting) arrow is a mono.

It is clear that FP implies FG. In algebraic terms, FP objects are algebras presented by finitely many generators and finitely many equations; while for FG objects only a set of generators is required to be finite. The two notions coincide in "non-algebraic" examples such as **Sets**. See [4, §1.E].

A.3 Fibrations

We follow [29], although we focus on the simpler notion of poset fibration.

Introduction (via Indexed Posets)

This paper's interest is in coinductive predicates, hence in predicate logic. The most straightforward formalization of predicate is as a subset $P \subseteq X$ of a set (a 'universe') X: an element $x \in X$ *satisfies* P if $x \in P$. Accompanying is the natural notion of entailment: P *entails* Q if $P \subseteq Q$. This way we obtain the poset $(2^X, \subseteq)$ of predicates over X.

However it is not on a single universe X that we consider predicates. For example, in a situation where there are two Kripke models $c = (X, \to, V_X)$, $d = (Y, \to, V_Y)$ and a "homomorphism" $f : X \to Y$, a natural question is if the interpretation of a formula $\nu u.\alpha$ is preserved by f. (It is; see Prop. 3.11). Here we are comparing the predicate $[\![\nu u.\alpha]\!]_c \subseteq X$ with the predicate $[\![\nu u.\alpha]\!]_d \subseteq Y$ *reindexed* via $f : X \to Y$. The latter is concretely described as the *inverse image*

$$f^{-1}\big(\, [\![\nu u.\alpha]\!]_d \,\big) \;=\; \big\{\, x \in X \,\big|\, f(x) \in [\![\nu u.\alpha]\!]_d \,\big\} \;.$$

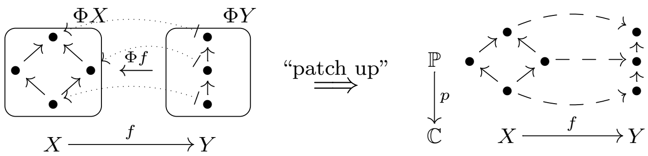
Therefore a reindexing structure is also relevant to predicate logic: a function $f : X \to Y$ induces reindexing $f^{-1} : 2^Y \to 2^X$. Additionally, the map f^{-1} is monotone.

To summarize: 1) predicates on a universe X form a poset; 2) a function $f : X \to Y$ between universes induces a monotone reindexing function from the collection of predicates over X to that over Y. Such a situation is nicely described as a (contravariant) functor

$$\Phi \;:\; \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{Posets} \;, \tag{A.4}$$

where **Posets** is the category of posets and monotone functions. The functor Φ assigns, to each 'universe' $X \in \mathbb{C}$, the poset ΦX of predicates over X. Moreover, $f : X \to Y$ in $\mathbb{C}$ induces a reindexing map $\Phi f : \Phi Y \to \Phi X$. This functor Φ is a special case of an *indexed category* [29, §1.10].

In the current paper, however, we favor an equivalent presentation of such a structure by a *fibration*, since we find the latter to be more amenable to generalization of structures in ordinary category theory (such as limits). The equivalence between index categories and fibrations are well-known; here we sketch the *Grothendieck construction* from the former to the latter. Its idea is to "patch up" the posets $(\Phi X)_{X \in \mathbb{C}}$ and form a big category $\mathbb{P}$, as in the following figure.



On the right we add some arrows (denoted by $\dashrightarrow$) so that we have an arrow $(\Phi f)(Q) \to Q$ in $\mathbb{P}$ for each $Q \in \Phi Y$. (On the left the arrows $\mapsto$ depicts the action of the map Φf.) The above diagram in $\mathbb{P}$ should be understood as a Hasse diagram: those arrows which arise from composition are not depicted.

Formally:

Definition A.8 (The Grothendieck construction) Given $\Phi : \mathbb{C}^{\mathrm{op}} \to \mathbf{Posets}$, we define the category $\mathbb{P}_\Phi$ by

- its object is a pair (X, P) of an object $X \in \mathbb{C}$ and an element P of the poset ΦX; and
- its arrow $f : (X, P) \to (Y, Q)$ is an arrow $f : X \to Y$ in $\mathbb{C}$ such that

$$P \le (\Phi f)(Q) \ .$$

Here $\le$ refers to the order of ΦX.

Thus arises a category $\mathbb{P}_\Phi$ that incorporates: the order structure of each of the posets $(\Phi X)_{X \in \mathbb{C}}$; and the reindexing structure by $(\Phi f)_{f:\ \mathbb{C}\text{-arrow}}$. For fixed $X \in \mathbb{C}$, the objects of the form (X, P) and the arrows id_X between them form a subcategory of $\mathbb{P}$. This is denoted by $\mathbb{P}_X$ and called the *fiber* over X. It is obvious that $\mathbb{P}_X$ is a poset that is isomorphic to ΦX.

Moreover, there is a canonical projection functor $p : \mathbb{P} \to \mathbb{C}$ that carries (X, P) to X.

Formal Definition of (Poset) Fibration

We axiomatize those structures which arise in the way described above.

Definition A.9 ((Poset) fibration) A *(poset) fibration* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ consists of two categories $\mathbb{P}, \mathbb{C}$ and a functor $p : \mathbb{P} \to \mathbb{C}$, that satisfy the following properties.

- Each fiber $\mathbb{P}_X$ is a poset. Here the *fiber* $\mathbb{P}_X$ for $X \in \mathbb{C}$ is the subcategory of $\mathbb{P}$ consisting of objects $P \in \mathbb{P}$ such that $pP = X$ and arrows $f : P \to Q$ such that $pf = \mathrm{id}_X$ (such arrows are said to be *vertical*).
- Given $f : X \to Y$ in $\mathbb{C}$ and $Q \in \mathbb{P}_Y$, there is an object $f^*Q \in \mathbb{P}_X$ and a $\mathbb{P}$-arrow $\overline{f}Q : f^*Q \to Q$ with the following universal property. For any $P \in \mathbb{P}_X$ and $g : P \to Q$ in $\mathbb{P}$, if $pg = f$ then g factors through $\overline{f}(Q)$ uniquely via a vertical arrow. That is, there exists a unique g' such that $g = \overline{f}(Q) \circ g'$ and $pg' = \mathrm{id}_X$.

$$\begin{tikzcd}
\mathbb{P} \arrow[dd, "p"] & & Q & & f^*Q \arrow[r, "\overline{f}(Q)"] & Q \\
& & & \Longrightarrow & P \arrow[u, dotted, "g'"] \arrow[ur, "g"'] & \\
\mathbb{C} & X \arrow[r, "f"] & Y & & X \arrow[r, "f"] & Y
\end{tikzcd}$$

- The correspondences $(\_)^*$ and $\overline{(\_)}$ are functorial:

$$\mathrm{id}_Y^* Q = Q \ , \qquad (g \circ f)^*(Q) = f^*(g^* Q) \ ,$$
$$\overline{\mathrm{id}_Y}(Q) = \mathrm{id}_Q \ , \qquad \overline{g \circ f}(Q) = \overline{g}Q \circ \overline{f}(g^* Q) \ .$$

The last equality can be depicted as follows.

$$\begin{tikzcd}
\mathbb{P} \arrow[dd, "p"] & f^*(g^*Q) \arrow[r, "\overline{f}(g^*Q)"] \arrow[d, equal] & g^*Q \arrow[r, "\overline{g}Q"] & Q \\
& (g \circ f)^*Q \arrow[urr, "\overline{g \circ f}(Q)"'] & & \\
\mathbb{C} & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

The category $\mathbb{P}$ is called the *total category* of the fibration; $\mathbb{C}$ is the *base category*. The arrow $\overline{f}Q : f^*Q \to Q$ is called the *Cartesian lifting* of f and Q. An arrow in $\mathbb{P}$ is *Cartesian* (or *reindexing*) if it coincides with $\overline{f}Q$ for some f and Q.

In the case where $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ is induced by an indexed category $\Phi : \mathbb{C}^{\mathrm{op}} \to \mathbf{Posets}$ via Def. A.8, a Cartesian lifting is obviously given by $f^*(Q) = (\Phi f)(Q)$.

In the current paper we focus on poset fibrations (which we shall simply call *fibrations*). In a (general) fibration a fiber $\mathbb{P}_X$ is not just a poset but a category, and this elicits a lot of technical subtleties. Nevertheless, it should not be hard to generalize the current paper's results to general, not necessarily poset, fibrations (especially to the split ones).

We shall often denote a vertical arrow in $\mathbb{P}$ (i.e. an arrow inside a fiber) by $\leq$.

Examples

Example A.10 (Subobject fibration) Let $\mathbb{C}$ be a (well-powered) category with finite limits. The category $\mathrm{Sub}(\mathbb{C})$ is defined by: its object is a pair (P, X) of $X \in \mathbb{C}$ and its subobject $P \rightarrowtail X$ (we write $(P \rightarrowtail X) \in \mathrm{Sub}(\mathbb{C})$); and its arrow $(P \rightarrowtail X) \xrightarrow{f} (V \rightarrowtail Y)$ is a $\mathbb{C}$-arrow $f : X \to Y$ that restricts to $P \to Q$. That is, given an arrow $f : X \to Y$ in $\mathbb{C}$,

$$\begin{array}{l} f \text{ is an arrow in } \mathrm{Sub}(\mathbb{C}) \\ (P \overset{m}{\rightarrowtail} X) \xrightarrow{f} (Q \overset{n}{\rightarrowtail} Y) \end{array} \iff \exists f' \text{ s.t. } \begin{tikzcd} P \arrow[r, dashed, "f'"] \arrow[d, tail, "m"'] & Q \arrow[d, tail, "n"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} . \tag{A.5}$$

The projection $(P \rightarrowtail X) \mapsto X$ defines a functor; thus arises the *subobject fibration* $\begin{smallmatrix}\mathrm{Sub}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{smallmatrix}$ of $\mathbb{C}$. In particular, given $X \xrightarrow{f} Y$ in $\mathbb{C}$ and $(Q \rightarrowtail Y) \in \mathrm{Sub}(Y)$, the Cartesian lifting f^*Q is defined by a pullback.

$$\begin{tikzcd} f^*Q \arrow[r, dashed, "\overline{f}Q"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner"] & Q \arrow[d, tail, "n"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

A special case is the following most straightforward modeling of predicate logic. It arises from the contravariant powerset functor $2^{(\_)} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Posets}$ via Def. A.8.

Example A.11 ($\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$) The subobject fibration $\begin{smallmatrix}\mathrm{Sub}(\mathbf{Sets})\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ of **Sets** is denoted by $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$. An object of its total category is often denoted by $(U \subseteq X)$. Reindexing is given by inverse images.

More concretely, in the category **Pred**, an object is a pair (P, X) of a set X and its subset $P \subseteq X$; an arrow $(P \subseteq X) \xrightarrow{f} (Q \subseteq Y)$ is a function $X \xrightarrow{f} Y$ that restricts to $P \to Q$ (i.e. $P \subseteq f^{-1}Q$).

Example A.12 (Rel) The fibration $\begin{smallmatrix}\mathbf{Rel}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ can be introduced from $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ via the following change-of-base.

$$\begin{tikzcd}
\mathbf{Rel} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Pred} \arrow[d] \\
\mathbf{Sets} \arrow[r, "X \mapsto X \times X"'] & \mathbf{Sets}
\end{tikzcd}$$

Concretely, an object of **Rel** is a pair (X, R) of a set X and a relation $R \subseteq X \times X$; an arrow $f : (X, R) \to (Y, S)$ is a function $f : X \to Y$ such that xRx' implies $f(x)Sf(x')$. See [29, p. 14].

Example A.13 (Family fibration) The *family fibration* $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ over a poset Ω is introduced as follows. An object in the fiber $\mathrm{Fam}(\Omega)_X$ is a function $f : X \to \Omega$; and an arrow $(X \xrightarrow{f} \Omega) \xrightarrow{k} (Y \xrightarrow{g} \Omega)$ in the total category $\mathrm{Fam}(\Omega)$ is a function $k : X \to Y$ such that $f(x) \le g(k(x))$ for each $x \in X$. See e.g. [29, Def. 1.2.1] for more details.

Structures in a Fibration

In a fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$, a $\mathbb{C}$-arrow $X \xrightarrow{f} Y$ induces a correspondence $\mathbb{P}_Y \xrightarrow{f^*} \mathbb{P}_X$ via reindexing. This is easily seen to be a monotone map (i.e. a functor between posets as categories).

Definition A.14 (Fiberwise (co)limits) A fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ is said to have *fiberwise limits* if:

- each fiber $\mathbb{P}_X$ has, as a category, all limits (meaning it has arbitrary inf's $\bigwedge$); and
- for each $\mathbb{C}$-arrow $X \xrightarrow{f} Y$, the reindexing functor $\mathbb{P}_Y \xrightarrow{f^*} \mathbb{P}_X$ preserves these limits.

In this case each fiber $\mathbb{P}_X$ has a final object (denoted by $\top_X$).

Similarly, a fibration has *fiberwise colimits* if each fiber has them and they are preserved by reindexing.

The following notions must be distinguished from "fiberwise (co)products."

Definition A.15 ((Co)products between fibers) A fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ is said to have *products (between fibers)* if

- each reindexing functor $f^* : \mathbb{P}_Y \to \mathbb{P}_X$ has a right adjoint $f^* \dashv \prod_f$; and
- the functors $(\prod_f)_f$ satisfy the so-called *Beck-Chevalley condition*. See [29, §1.9].

Similarly, a fibration has *coproducts (between fibers)* if each reindexing has a left adjoint $\coprod_f$ and they satisfy the Beck-Chevalley condition.

The prototype example $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ has fiberwise (co)limits: each fiber is a complete lattice; and $\bigwedge$ and $\bigvee$ are preserved by inverse images. It has (co)products $\coprod$ between fibers, too: specifically $\coprod_f$ is given by the *direct image* of the function f. See [29, §1.9].

Throughout the paper we rely on the following result. It follows from [29, Lem. 9.1.2 & Prop. 9.2.1], and extends Lem. 3.5.

Lemma A.16 *Let* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ *be a fibration. Assume that* $\mathbb{C}$ *is complete; then the following are equivalent.*

(i) *The fibration* p *has fiberwise limits.*

(ii) *The total category* $\mathbb{P}$ *is complete and* $p : \mathbb{P} \to \mathbb{C}$ *preserves limits.*

If this is the case, a limit of a small diagram $(P_I)_{I\in\mathbb{I}}$ *in* $\mathbb{P}$ *can be given by*

$$\textstyle\bigwedge_{I\in\mathbb{I}}(\pi_I^* P_I) \qquad \textit{over } \mathrm{Lim}_{I\in\mathbb{I}} X_I .$$

Here $X_I := pP_I$*;* $(\mathrm{Lim}_{I\in\mathbb{I}} X_I \stackrel{\pi_I}{\to} X_I)_{I\in\mathbb{I}}$ *is a limiting cone in* $\mathbb{C}$*; and* $\bigwedge_{I\in\mathbb{I}}$ *denotes the limit computed in the fiber* $\mathbb{P}_{\mathrm{Lim}_I X_I}$.

(Sort of) dually, let $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ *be a fibration with coproducts* $\coprod$ *between fibers, and assume that* $\mathbb{C}$ *is cocomplete. Then* p *has fiberwise colimits if and only if* $\mathbb{P}$ *is cocomplete and* $p : \mathbb{P} \to \mathbb{C}$ *preserves colimits. In this case a colimit of a small diagram* $(P_I)_{I\in\mathbb{I}}$ *in* $\mathbb{P}$ *can be given by*

$$\textstyle\bigvee_{I\in\mathbb{I}}(\coprod_{\kappa_I} P_I) \quad \textit{over } \mathrm{Colim}_I X_I ,$$

where $X_I := pP_I$ *and* $(X_I \stackrel{\kappa_I}{\to} \mathrm{Colim}_I X_I)_{I\in\mathbb{I}}$ *is a colimiting cocone in* $\mathbb{C}$. □

B Appendix: Omitted Proofs

B.1 Proof of Lem. 3.6

Proof We proceed by steps.

a) We observe that, in Fig. 1, the top diagram is carried to the one below by the functor $p : \mathbb{P} \to \mathbb{C}$. This is straightforward: the arrow $\varphi\top_1 \to \top_1$ must be carried to the unique arrow $! : F1 \dashrightarrow 1$; on the mediating arrow b' in $\mathbb{P}$, since pb' is again a mediating arrow in $\mathbb{C}$, it must coincide with b.

b) Before moving on, we observe that Cond. iii) in Def. 3.2 yields a seemingly stronger statement (Cond. iii') below).

Sublemma B.1 *For a finitely determined fibration* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ *the following holds.*

iii') Let $X \in \mathbb{C}$*;* $P, Q \in \mathbb{P}_X$*; and* $(Y_J)_{J\in\mathbb{J}}$ *be an arbitrary filtered diagram in* $\mathbb{C}$ *such that* $\mathrm{Colim}_J Y_J = X$*, with a colimiting cocone* $(Y_J \stackrel{\gamma_J}{\to} X)_{J\in\mathbb{J}}$*. Then* $P \leq Q$ *if and* *only if for each* $J \in \mathbb{J}$, $\gamma_J^* P \le \gamma_J^* Q$ *in* $\mathbb{P}_{Y_J}$.

Proof (Of Sublem. B.1) The only nontrivial statement is the 'if' part of the direction iii) $\Rightarrow$ iii'). It suffices to show that $\gamma_J^* P \le \gamma_J^* Q$ (for each $J \in \mathbb{J}$) implies $\kappa_I^* P \le \kappa_I^* Q$ (for each $I \in \mathbb{I}$), where κ_I and $\mathbb{I}$ are as in Cond. iii).

Let $I \in \mathbb{I}$. Since X_I is FP, an arrow $\kappa_I : X_I \to X$ to a filtered colimit $X = \mathrm{Colim}_J Y_J$ factors through some $Y_{J_I} \overset{\gamma_{J_I}}{\to} X$, as in the diagram below.

$$\begin{tikzcd} X_I \arrow[rr, "\kappa_I"] \arrow[dr, "h_I"'] & & X = \mathrm{Colim}_J Y_J \\ & Y_{J_I} \arrow[ur, "\gamma_{J_I}"'] & \end{tikzcd}$$

Now we have $\kappa_I^* P = h_I^* \gamma_{J_I}^* P \le h_I^* \gamma_{J_I}^* Q = \kappa_I^* Q$, where the inequality is by the assumption that $\gamma_J^* P \le \gamma_J^* Q$ for each $J \in \mathbb{J}$. This proves Sublem. B.1. $\square$

c) By Step a) we see that $\varphi^{\omega+1}\top_1 \le b^*(\varphi^\omega \top_1)$ by the universality of a Cartesian arrow. In what follows we shall prove its converse:

$$b^*(\varphi^\omega \top_1) \le \varphi^{\omega+1}\top_1 \qquad \text{in } \mathbb{P}_{F^{\omega+1}1}. \tag{B.1}$$

Let us take a directed diagram $(X_I)_{I \in \mathbb{I}}$ in $\mathbb{C}$ such that $X_I \in \mathbb{F}$ (for each $I \in \mathbb{I}$) and $F^\omega 1 = \mathrm{Colim}_{I \in \mathbb{I}} X_I$, with $(X_I \overset{\kappa_I}{\to} F^\omega 1)_{I \in \mathbb{I}}$ being the colimiting cocone. Then we have

$$F^{\omega+1}1 = F(\operatorname*{Colim}_{I \in \mathbb{I}} X_I) = \operatorname*{Colim}_{I \in \mathbb{I}} F X_I \enspace ,$$

by the assumption that F is finitary; moreover $(F X_I \overset{F\kappa_I}{\to} F^{\omega+1}1)_{I \in \mathbb{I}}$ is a colimiting cocone. The diagram $(X_I)_{I \in \mathbb{I}}$ is directed, and so is the latter diagram $(F X_I)_{I \in \mathbb{I}}$. Thus by Cond. iii') in Sublem. B.1, showing the following proves (B.1).

$$(F\kappa_I)^*\Big(b^*(\varphi^\omega \top_1)\Big) \le (F\kappa_I)^*(\varphi^{\omega+1}\top_1) \quad \text{for each } I \in \mathbb{I}. \tag{B.2}$$

d) To prove (B.2) we first prove the following fact: for each $I \in \mathbb{I}$ there exists $i_I \in \omega$ such that

$$\kappa_I^*(\varphi^\omega \top_1) = \kappa_I^*\big(\pi_{i_I}^*(\varphi^{i_I}\top_1)\big) \quad \text{in } \mathbb{P}_{X_I}. \tag{B.3}$$

That is: the final sequence in $\mathbb{P}$ (Fig. 1), when restricted to X_I (that is FP), stabilizes within finitely many steps. Indeed, by Lem. A.16 the limit $\varphi^\omega \top_1$ is described as an inf in $\mathbb{P}_{F^\omega 1}$:

$$\varphi^\omega \top_1 \;=\; \textstyle\bigwedge_{i \in \omega} \pi_i^*(\varphi^i \top_1) \enspace . \tag{B.4}$$

Therefore we have $\kappa_I^*(\varphi^\omega \top_1) = \bigwedge_{i \in \omega} \kappa_I^* \pi_i^*(\varphi^i \top_1)$ since reindexing κ_I^* preserves fiberwise limits $\bigwedge$. Now we claim that the sequence $\big(\kappa_I^* \pi_i^*(\varphi^i \top_1)\big)_{i \in \omega}$ in $\mathbb{P}_{X_I}$ is descending: it follows from the fact that $\big(\pi_i^*(\varphi^i \top_1)\big)_{i \in \omega}$ in $\mathbb{P}_{F^\omega 1}$ is descending, which in turn is shown from the universality of the Cartesian arrow $\overline{\pi_i}(\varphi^i \top_1)$. See below.

$$\begin{tikzcd} \mathbb{P} \arrow[ddd] & & & \pi_i^*(\varphi^i \top_1) \arrow[dddlll, bend right, "\overline{\pi_i}(\varphi^i \top_1)"'] \\ & & & \pi_{i+1}^*(\varphi^{i+1}\top_1) \arrow[u, dashed, "!"'] \arrow[ddl] \\ & & & \\ & \varphi^i \top_1 & \varphi^{i+1}\top_1 \arrow[l] & \\ & & & F^\omega 1 \arrow[dll, bend right, "\pi_i"'] \arrow[dl, "\pi_{i+1}"] \\ \mathbb{C} & F^i 1 & F^{i+1}1 \arrow[l, "F^i\,!"] & \end{tikzcd}$$

Therefore, by p being a well-founded fibration (Def. 3.3), there exists $i_I \in \omega$ at which the descending sequence stabilizes, that is,

$$\textstyle\bigwedge_{i\in\omega} \pi_i^*(\varphi^i \top_1) = \pi_{i_I}^*(\varphi^{i_I} \top_1) \quad \text{in } \mathbb{P}_{F^\omega 1}.$$

Combined with (B.4), this proves (B.3).

e) Finally let us prove (B.2). For each $I \in \mathbb{I}$,

$$\begin{aligned}
&(F\kappa_I)^*\big(\, b^*(\varphi^\omega \top_1)\,\big) \\
&= (F\kappa_I)^*\big(\, b^*\big(\textstyle\bigwedge_{i\in\omega} \pi_i^*(\varphi^i \top_1)\big)\,\big) \quad \text{by (B.4)} \\
&= \textstyle\bigwedge_{i\in\omega} (F\kappa_I)^*\big(\, b^*\big(\pi_i^*(\varphi^i \top_1)\big)\,\big) \quad \text{reindexing preserves } \bigwedge \\
&= \textstyle\bigwedge_{i\in\omega} (F\kappa_I)^*\big(\,(F\pi_{i-1})^*(\varphi^i \top_1)\,\big) \\
&\qquad\qquad\qquad\qquad \text{by } \pi_i \circ b = F\pi_{i-1} \text{ (see Fig. 1)} \\
&= \textstyle\bigwedge_{i\in\omega} \varphi\big(\,(\pi_{i-1} \circ \kappa_I)^*(\varphi^{i-1} \top_1)\,\big) \quad \text{by Def. 2.2} \\
&\le \varphi\big(\,(\pi_{i_I} \circ \kappa_I)^*(\varphi^{i_I} \top_1)\,\big) \quad \text{letting } i = i_I + 1 \text{ on the LHS} \\
&= \varphi\big(\, \kappa_I^* \pi_{i_I}^*(\varphi^{i_I} \top_1)\,\big) = \varphi\big(\, \kappa_I^*(\varphi^\omega \top_1)\,\big) \quad \text{by (B.3)} \\
&= (F\kappa_I)^*(\varphi^{\omega+1} \top_1) \\
&\qquad\qquad \text{by Def. 2.2 and } \varphi^{\omega+1}\top_1 = \varphi(\varphi^\omega \top_1).
\end{aligned}$$

This proves (B.2) and concludes the proof of Lem. 3.6. □

B.2 Proof of Thm. 3.7

Proof We proceed by steps.

ia) We first show that $c_\omega^*(\varphi^\omega \top_1)$ indeed carries a $(c^* \circ \varphi)$-coalgebra.

$$\begin{aligned}
&c^*\big(\varphi(c_\omega^*(\varphi^\omega \top_1))\big) \\
&= c^*\big((Fc_\omega)^*(\varphi(\varphi^\omega \top_1))\big) \quad \text{by Def. 2.2} \\
&= c^*\big((Fc_\omega)^*(b^*(\varphi^\omega \top_1))\big) \quad \text{by Lem. 3.6} \\
&= (b \circ Fc_\omega \circ c)^*(\varphi^\omega \top_1) \\
&= c_\omega^*(\varphi^\omega \top_1) \ .
\end{aligned}$$

For the last equality we used $b \circ Fc_\omega \circ c = c_\omega$, which is proved by showing that $b \circ Fc_\omega \circ c$ is also a mediating map in (7). Indeed, for each $i \in \omega$,

$$\begin{aligned}
&\pi_i \circ b \circ Fc_\omega \circ c \\
&= F\pi_{i-1} \circ Fc_\omega \circ c \quad \text{see Fig. 1} \\
&= Fc_{i-1} \circ c \quad \text{by (7)} \\
&= c_i \quad \text{by def. of } c_i.
\end{aligned}$$

ib) We show that the coalgebra obtained in Step a) is final. Let $U \leq c^*(\varphi U)$ be an arbitrary $(c^* \circ \varphi)$-coalgebra (i.e. a φ-invariant in c), where $U \in \mathbb{P}_X$. We aim to establish the following diagram in $\mathbb{P}$ and see that it is above the one in (7).

$$
\begin{tikzcd}
 & & & & \varphi^\omega \top_1 \arrow[llll, bend right] \arrow[lll, bend right] \arrow[dl, bend right] & \\
\top_1 & \varphi \top_1 \arrow[l] & \cdots \arrow[l] & \varphi^i \top_1 \arrow[l] & \cdots \arrow[l] & \\
 & & & & U \arrow[llll, bend left] \arrow[lll, bend left] \arrow[ul, bend left] \arrow[uu, dashed, bend right, "c'_\omega"'] &
\end{tikzcd}
\tag{B.5}
$$

We first show that

$$
U \leq c_i^*(\varphi^i \top_1) \qquad \text{for each } i \in \omega. \tag{B.6}
$$

The proof is by induction. The base case $i = 0$ is obvious since reindexing c_i^* preserves $\top$. For the step case:

$$
\begin{aligned}
& U \leq c^*(\varphi U) \quad U \text{ carries a } (c^* \circ \varphi)\text{-coalgebra} \\
& \leq c^*\big(\varphi\big(c_i^*(\varphi^i \top_1)\big)\big) \quad \text{by induction hypothesis} \\
& = c^*\big((Fc_i)^*(\varphi^{i+1} \top_1)\big) \quad \text{by Def. 2.2} \\
& = (c_{i+1})^*(\varphi^{i+1} \top_1) \quad \text{by def. of } c_{i+1}.
\end{aligned}
$$

This proves (B.6) and establishes the arrows $U \to \varphi^i \top_1$ in (B.5), for each i. Therefore we obtain a mediating map $c'_\omega : U \dashrightarrow \varphi^\omega \top_1$ to the limit $\varphi^\omega \top_1$, too. The arrow c'_ω is easily shown to be above c_ω (much like b' in Fig. 1 is shown to be above b); this means $U \leq c_\omega^*(\varphi^\omega \top_1)$. Since $\mathbb{P}_X$ is a poset, this arrow $\leq$ is necessarily a coalgebra morphism from U to $c_\omega^*(\varphi^\omega \top_1)$; moreover it is a unique such. This proves i).

ii) We have

$$
\begin{aligned}
& [\![\nu\varphi]\!]_c \\
& = c_\omega^*(\varphi^\omega \top_1) \quad \text{by i)} \\
& = c_\omega^*\big(\textstyle\bigwedge_{i\in\omega} \pi_i^*(\varphi^i \top_1)\big) \quad \text{by Lem. A.16} \\
& = \textstyle\bigwedge_{i\in\omega} c_\omega^*\big(\pi_i^*(\varphi^i \top_1)\big) \quad \text{since reindexing preserves } \bigwedge \\
& = \textstyle\bigwedge_{i\in\omega} c_i^*(\varphi^i \top_1) \quad \text{by def. of } c_\omega.
\end{aligned}
\tag{B.7}
$$

Furthermore, $c_i^*(\varphi^i \top_1)$ in the above is seen to be equal to $(c^* \circ \varphi)^i(\top_X)$. This is shown by induction on $i \in \omega$. For $i = 0$ the claim amounts to $!^*(\top_1) = \top_X$, which holds since reindexing preserves $\top$. For the step case,

$$
\begin{aligned}
& c_{i+1}^*(\varphi^{i+1} \top_1) \\
& = c^*(Fc_i)^*(\varphi^{i+1} \top_1) \quad \text{by } c_{i+1} = Fc_i \circ c \\
& = c^*\varphi\big(c_i^*(\varphi^i \top_1)\big) \quad \text{by Def. 2.2} \\
& = (c^* \circ \varphi)\big((c^* \circ \varphi)^i(\top_X)\big) \quad \text{by induction hypothesis.}
\end{aligned}
$$

Finally let us check that the chain (9) stabilizes after ω steps.

$$\begin{aligned}
&(c^* \circ \varphi)\big(\textstyle\bigwedge_{i\in\omega}(c^* \circ \varphi)^i \top_X\big) \\
&= (c^* \circ \varphi)\big(\textstyle\bigwedge_{i\in\omega} c_i^*(\varphi^i \top_1)\big) \quad \text{by the previous paragraph} \\
&= (c^* \circ \varphi)\big(c_\omega^*(\varphi^\omega \top_1)\big) \quad \text{by (B.7)} \\
&= c^*(Fc_\omega)^*\big(\varphi(\varphi^\omega \top_1)\big) \quad \text{by Def. 2.2} \\
&= c^*(Fc_\omega)^*\big(b^*(\varphi^\omega \top_1)\big) \quad \text{by Lem. 3.6} \\
&= c_\omega^*(\varphi^\omega \top_1) \quad \text{by } b \circ Fc_\omega \circ c = c_\omega\text{, see Step 1a)} \\
&= c_\omega^*(\textstyle\bigwedge_{i\in\omega} \pi_i^*(\varphi^i \top_1)) \quad \text{by (B.4)} \\
&= \textstyle\bigwedge_{i\in\omega} c_\omega^* \pi_i^*(\varphi^i \top_1) \\
&= \textstyle\bigwedge_{i\in\omega} c_i^*(\varphi^i \top_1) \\
&= \textstyle\bigwedge_{i\in\omega}(c^* \circ \varphi)^i \top_X \quad \text{by the previous paragraph.}
\end{aligned}$$

This concludes the proof. □

B.3 Proof of Prop. 3.11

Proof 1)

$$\begin{aligned}
f^*Q \le f^*d^*(\varphi Q) &\quad Q \text{ is an invariant} \\
= c^*(Ff)^*(\varphi Q) &\quad f \text{ is a homomorphism} \\
= (c^* \circ \varphi)(f^*Q) &\quad \text{by Def. 2.2.}
\end{aligned}$$

2) The coalgebras give rise to mediating arrows $X \overset{c_\omega}{\to} F^\omega 1$ and $Y \overset{d_\omega}{\to} F^\omega 1$, respectively, as in (7). It is easy to see that $c_\omega = d_\omega \circ f$ (using the universality of the limit $F^\omega 1$); using (8) the claim follows. □

B.4 Proof of Prop. 4.1

Proof It is easy to check each fiber $\mathbf{Coalg}(\varphi)_{X \overset{c}{\to} FX}$ is a poset. Let $(X \overset{c}{\to} FX) \overset{f}{\to} (Y \overset{d}{\to} FY)$ be an arrow in $\mathbf{Coalg}(F)$, and $P \overset{s}{\to} \varphi P$ be above $Y \overset{d}{\to} FY$. A Cartesian lifting of f are obtained as in the following diagram.

$$\boxed{\mathbb{P}} \qquad \begin{tikzcd}
\varphi f^* P \arrow[r, "\varphi \overline{f}(P)"] & \varphi P \\
f^* P \arrow[u, dashed, "t"] \arrow[r, "\overline{f}(P)"'] & P \arrow[u, "s"']
\end{tikzcd}$$

$$\boxed{\mathbb{C}} \qquad \begin{tikzcd}
FX \arrow[r, "Ff"] & FY \\
X \arrow[u, "c"] \arrow[r, "f"'] & Y \arrow[u, "d"']
\end{tikzcd}$$

Here we used the universality of the Cartesian lifting $\varphi \overline{f}(P)$ (see Def. 2.2).

The two forgetful functors constitute a map of fibrations: the commutativity (4) is obvious, and Cartesian liftings in $\begin{smallmatrix}\mathbf{Coalg}(\varphi)\\ \downarrow\overline{p}\\ \mathbf{Coalg}(F)\end{smallmatrix}$ (which we constructed above) are based on the Cartesian liftings in $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$. □

B.5 Proof of Prop. 4.2

Proof Given a φ-coalgebra $P \xrightarrow{s} \varphi P$ above $X \xrightarrow{c} FX$, we use the universality of the Cartesian lifting of c to obtain a $(c^* \circ \varphi)$-coalgebra as in the following diagram.

$$\begin{tikzcd} c^*\varphi P \arrow[r, "\overline{c}(\varphi P)"] & \varphi P \\ P \arrow[u] \arrow[ur, "s"'] & \end{tikzcd}$$

Conversely, given a $(c^* \circ \varphi)$-coalgebra $Q \xrightarrow{t} c^*(\varphi Q)$, we obtain a φ-coalgebra above $X \xrightarrow{c} FX$ as the following composite.

$$\begin{tikzcd} c^*\varphi Q \arrow[r, "\overline{c}(\varphi Q)"] & \varphi Q \\ Q \arrow[u, "t"] & \end{tikzcd}$$

Then it is straightforward to see that the mappings are monotone and inverse to each other. The mappings commute with the forgetful functors since they do not change the carriers. □

B.6 Proof of Lem. 5.4

Proof The proof is by steps.

a) First we show that $\mathrm{Sub}(\mathbb{C})$ is complete and cocomplete. We rely on Lem. A.16.

We start with fiberwise limits in $\begin{smallmatrix}\mathrm{Sub}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{smallmatrix}$; the proof is like in [29, Example 1.8.3(iii)]. By Lem. A.4 an LFP category $\mathbb{C}$ is complete. This equips each fiber $\mathrm{Sub}(X)$ with arbitrary inf's $\bigwedge$ computed as wide pullbacks. A reindexing functor (by pullbacks) preserves these inf's since limits commute. Therefore by Lem. A.16 the total category $\mathrm{Sub}(\mathbb{C})$ is complete.

By the assumption that $\mathbb{C}$ is an LCCC, $\begin{smallmatrix}\mathrm{Sub}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{smallmatrix}$ has products $\prod_f \vdash f^*$ between fibers [29, Cor. 1.9.9].

Next we show that $\begin{smallmatrix}\mathrm{Sub}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{smallmatrix}$ has fiberwise colimits. Each fiber (which is a poset) has arbitrary inf's; hence it is a complete lattice and arbitrary sup's also exist. These sup's (i.e. colimits in a fiber) are preserved by reindexing f^* since the latter is a left adjoint $f^* \dashv \prod_f$.

We further show that $\begin{smallmatrix}\mathrm{Sub}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{smallmatrix}$ has coproducts $\coprod$ between fibers. An abstract proof can be given by Freyd's adjoint functor theorem (note that each fiber $\mathrm{Sub}(X)$

is a complete lattice, and that reindexing f^* preserves inf's). Instead we explicitly introduce $\coprod$ exploiting a factorization structure of LFP $\mathbb{C}$ (Lem. A.4.ii). Namely, given $(P \overset{m}{\rightarrowtail} X) \in \mathrm{Sub}(X)$ and $f : X \to Y$, the coproduct $\coprod_f P$ is defined by the (StrongEpi, Mono)-factorization of $f \circ m$, as below.

$$\begin{tikzcd} P \arrow[r, two heads] \arrow[d, tail, "m"'] & \coprod_f P \arrow[d, tail] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{B.8}$$

The fact that $\coprod_f P \leq Q$ if and only if $P \leq f^* Q$ is easily proved using the diagonalization property of the factorization structure. This establishes $\coprod_f$ as a left adjoint to reindexing f^*. These coproducts $\coprod$ satisfy the Bech-Chevalley condition since the products $\prod$ do [29, Lem. 1.9.7]. Using Lem. A.16 we conclude that $\mathrm{Sub}(\mathbb{C})$ is cocomplete.

b) First we prove that, if P and X are both FP in $\mathbb{C}$, then $(P \overset{m}{\rightarrowtail} X)$ is FP in $\mathrm{Sub}(\mathbb{C})$. Let $(Q_I \overset{n_I}{\rightarrowtail} Y_I)_{I \in \mathbb{I}}$ be a filtered diagram in $\mathrm{Sub}(\mathbb{C})$; $(Q \overset{n}{\rightarrowtail} Y)$ its colimit; and $g : (P \overset{m}{\rightarrowtail} X) \to (Q \overset{n}{\rightarrowtail} Y)$ an arrow in $\mathrm{Sub}(\mathbb{C})$. By Lem. A.16 the colimit $(Q \rightarrowtail Y)$ can be explicitly described as

$$Y = \operatorname*{Colim}_{I \in \mathbb{I}} Y_I \ , \qquad Q = \bigvee_{I \in \mathbb{I}} \textstyle\coprod_{\kappa_I} Q_I \ , \tag{B.9}$$

where $(Y_I \overset{\kappa_I}{\to} Y)_{I \in \mathbb{I}}$ is a colimiting cocone.

Sublemma B.2 *The object $Q \in \mathbb{C}$ is a colimit $\mathrm{Colim}_{I \in \mathbb{I}} Q_I$ computed in $\mathbb{C}$.*

Proof (Of the sublemma) Both $(Q_I)_{I \in \mathbb{I}}$ and $(Y_I)_{I \in \mathbb{I}}$ are $\mathbb{I}$-shaped diagrams in $\mathbb{C}$ with a monotransformation $(Q_I \overset{n_I}{\rightarrowtail} Y_I)_I$. Therefore by [4, Cor. 1.60], the induced arrow $\mathrm{Colim}_I Q_I \to \mathrm{Colim}_I Y_I$ is monic, establishing $\mathrm{Colim}_I Q_I \in \mathrm{Sub}(Y)$. It suffices to find arrows a, b in the diagram below.

$$\begin{tikzcd} \mathrm{Colim}_I Q_I \arrow[rr, shift left, "a"] \arrow[dr, tail] & & Q \arrow[ll, shift left, "b"] \arrow[dl, tail, "n"] \\ & Y & \end{tikzcd} \tag{B.10}$$

The arrow a is obtained in the following way. Since κ_I is an arrow $(Q_I \rightarrowtail Y_I) \to (Q \rightarrowtail Y)$ in $\mathrm{Sub}(\mathbb{C})$, by (A.5) we have an arrow $Q_I \to Q$ in $\mathbb{C}$, for each $I \in \mathbb{I}$. These arrows induce a as a mediating arrow.

To obtain b in (B.10), since $Q = \bigvee_{I \in \mathbb{I}} \coprod_{\kappa_I} Q_I$ (see (B.9)), it suffices to find b_I below for each $I \in \mathbb{I}$.

$$\begin{tikzcd} \coprod_{\kappa_I} Q_I \arrow[rr, "b_I"] \arrow[dr, tail] & & \mathrm{Colim}_I Q_I \arrow[dl, tail] \\ & Y & \end{tikzcd}$$

This is obtained as the following diagonal fill-in. Recall that $Y = \mathrm{Colim}_I Y_I$.

$$\begin{tikzcd} Q_I \arrow[r, two heads] \arrow[d] \arrow[rr, bend left, tail, "n_I"] & \coprod_{\kappa_I} Q_I \arrow[d, tail] \arrow[dl, dashed, "b_I"'] & Y_I \arrow[dl, "\kappa_I"] \\ \mathrm{Colim}_I Q_I \arrow[r, tail, "{[\kappa_I \circ n_I]_I}"'] & \mathrm{Colim}_I Y_I & \end{tikzcd}$$

This proves Sublem. B.2. $\square$

We are back in Step b). Since $g : (P \overset{m}{\rightarrowtail} X) \to (Q \overset{n}{\rightarrowtail} Y)$ is an arrow in $\mathrm{Sub}(\mathbb{C})$, we also have an arrow $g' : P \to Q$, such that $n \circ g' = g \circ m$, by (A.5). Now Q and Y are filtered colimits of $(Q_I)_{I \in \mathbb{I}}$ and $(Y_I)_{I \in \mathbb{I}}$, respectively (the former is by Sublem. B.2). Since P and X are FP, $I_0 \in \mathbb{I}$ can be chosen such that g factors through $Y_{I_0} \to Y$ and g' factors through $Q_{I_0} \to Q$. That is,

$$\begin{tikzcd}
P \arrow[r, "h'"] \arrow[d, tail, "m"'] \arrow[rr, bend left, "g'"] & Q_{I_0} \arrow[r] \arrow[d, tail, "n_{I_0}"] & Q \arrow[d, tail, "n"] \\
X \arrow[r, "h"'] \arrow[rr, bend right, "g"'] & Y_{I_0} \arrow[r, "\kappa_{I_0}"'] & Y
\end{tikzcd} \quad .$$

It is not (yet) necessarily the case that the square on the left commutes, i.e. $n_{I_0} \circ h' = h \circ m$. The two arrows give factorizations of the arrow $n \circ g' = g \circ m : P \to Y$ via $Y_{I_0} \overset{\kappa_{I_0}}{\to} Y$; since P is FP, there exists $I_1 \in \mathbb{I}$ with $i : I_0 \to I_1$ such that

$$(Yi) \circ n_{I_0} \circ h' = (Yi) \circ h \circ m$$

(essential uniqueness of factorization, [4, Def. 1.1]). It is clear that, for such I_1, the arrow g in $\mathrm{Sub}(\mathbb{C})$ factors through $(Q_{I_1} \rightarrowtail Y_{I_1}) \overset{\kappa_{I_1}}{\to} (Q \rightarrowtail Y)$. This concludes Step b) that $(P \rightarrowtail X)$ is FP in $\mathrm{Sub}(\mathbb{C})$.

c) Recall that

$$\mathbb{F}_{\mathrm{Sub}(\mathbb{C})} \;:=\; \{\, (P \rightarrowtail X) \mid P, X \in \mathbb{F} \} \ . \tag{B.11}$$

The set $\mathbb{F}_{\mathrm{Sub}(\mathbb{C})}$ in (B.11) is small, since $\mathbb{F}$ is small and $\mathbb{C}$ is well-powered [4, Rem. 1.56].

d) In the remainder of the proof we show that every object $(Q \overset{n}{\rightarrowtail} Y) \in \mathrm{Sub}(\mathbb{C})$ is a colimit of the canonical diagram with respect to $\mathbb{F}_{\mathrm{Sub}(\mathbb{C})}$ from (B.11). Let $(Q_J \overset{n_J}{\rightarrowtail} Y_J)_{J \in \mathbb{J}}$ be the canonical diagram (i.e. $\mathbb{J} = (\mathbb{F}_{\mathrm{Sub}(\mathbb{C})} {\downarrow} n)$), with the canonical cocone

$$\bigl(\, (Q_J \overset{n_J}{\rightarrowtail} Y_J) \xrightarrow{f_J} (Q \overset{n}{\rightarrowtail} Y) \,\bigr)_{J \in \mathbb{J}} \ . \tag{B.12}$$

Let us denote the canonical diagram for $Y \in \mathbb{C}$ with respect to $\mathbb{F}$ by $(Y'_I)_{I \in \mathbb{I}}$ (i.e. $\mathbb{I} = (\mathbb{F} {\downarrow} Y)$), with a canonical cocone $(Y'_I \overset{\kappa_I}{\to} Y)_{I \in \mathbb{I}}$. The cocone is colimiting ($Y = \mathrm{Colim}_{I \in \mathbb{I}} Y'_I$) since $\mathbb{C}$ is LFP. In this Step d) we show $\mathrm{Colim}_{J \in \mathbb{J}} Y_J \cong \mathrm{Colim}_{I \in \mathbb{I}} Y'_I = Y$. A cocone $(Y_J \overset{\kappa_{I_J}}{\to} \mathrm{Colim}_{I \in \mathbb{I}} Y'_I)_{J \in \mathbb{J}}$ can be defined by finding (unique) $I_J \in \mathbb{I}$ such that $Y'_{I_J} \overset{f'_{I_J}}{\to} Y$ is equal to $Y_J \overset{f_J}{\to} Y$. In order to see that this cocone is colimiting, let $(Y_J \overset{g_J}{\to} Z)_{J \in \mathbb{J}}$ be another cocone (recall that $\mathbb{J} = (\mathbb{F}_{\mathrm{Sub}(\mathbb{C})} {\downarrow} n)$).

Sublemma B.3 *If the indices $J, J' \in \mathbb{J}$ satisfy $Y_J = Y_{J'}$ and $f_J = f_{J'}$ (cf. (B.12)), then $g_J = g_{J'}$.*

Proof (Of the sublemma) Let 0 be an initial object in $\mathbb{C}$. We first prove that 0 is a subobject of any object of $\mathbb{C}$. Indeed, since $\mathbb{C}$ is an LCCC (hence a CCC), its initial object 0 is *strict*, meaning that if $U \to 0$ exists then U is also initial (a standard result; see e.g. [33, Lem. 1.5.12]). This makes any arrow $0 \to V$ in $\mathbb{C}$ a mono.

Therefore we have an object $(0 \rightarrowtail Y_J)$ in $\mathrm{Sub}(\mathbb{C})$ induced by initiality. In view of (A.5), an arrow f_J in $\mathbb{C}$ induces an arrow

$$(0 \rightarrowtail Y_J) \xrightarrow{f_J = f_{J'}} (Q \overset{n}{\rightarrowtail} Y) \quad \text{in } \mathsf{Sub}(\mathbb{C});$$

therefore there exists an index $J'' \in \mathbb{J} = (\mathbb{F}_{\mathsf{Sub}(\mathbb{C})} \downarrow n)$ such that $(0 \rightarrowtail Y_J) = (Q_{J''} \overset{n_{J''}}{\rightarrowtail} Y_{J''})$, and $f_{J''} = f_J = f_{J'}$.

This gives us the following diagram in $\mathbb{J} = (\mathbb{F}_{\mathsf{Sub}(\mathbb{C})} \downarrow n)$.

$$(Q_J \overset{n_J}{\rightarrowtail} Y_J) \xleftarrow{\mathrm{id}_{Y_J}} (Q_{J''} \overset{n_{J''}}{\rightarrowtail} Y_{J''}) \xrightarrow{\mathrm{id}_{Y_{J'}}} (Q_{J'} \overset{n_{J'}}{\rightarrowtail} Y_{J'})$$

Therefore, since $(Y_J \overset{g_J}{\to} Z)_{J \in \mathbb{J}}$ is a cocone, the following diagram in $\mathbb{C}$ must commute.

$$\begin{tikzcd} Y_J \arrow[dr, bend right, "g_J"'] & Y_{J''} \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] \arrow[d, "g_{J''}"'] & Y_{J'} \arrow[dl, bend left, "g_{J'}"] \\ & Z & \end{tikzcd}$$

This proves $g_J = g_{J'} = g_{J''}$, as required in Sublem. B.3. $\square$

We are back in Step d). For each $Y'_I \overset{f'_I}{\to} Y$ in $(\mathbb{F} \downarrow Y)$, there exists $J_I \in \mathbb{J}$ such that $Y_{J_I} = Y'_I$ and $f_{J_I} = f'_I$ (one can take the initial object 0, which is FP, as Q_{J_I}). Using such J_I we obtain an arrow

$$[g_{J_I}]_{I \in \mathbb{I}} \ : \ \operatorname*{Colim}_{I \in \mathbb{I}} Y'_I \longrightarrow Z \ .$$

That this is a mediating arrow, i.e. that the diagram

$$\begin{tikzcd} Y_J \arrow[rr, "g_J"] \arrow[dr, "\kappa_{I_J}"'] & & Z \\ & \mathrm{Colim}_{I \in \mathbb{I}} Y'_I \arrow[ur, "{[g_{J_I}]_{I \in \mathbb{I}}}"'] & \end{tikzcd}$$

commutes, is precisely the content of Sublem. B.3. Uniqueness of a mediating arrow is easy, too. This proves $\mathrm{Colim}_{J \in \mathbb{J}} Y_J \cong \mathrm{Colim}_{I \in \mathbb{I}} Y'_I$.

e) By Step d) we obtain $Y = \mathrm{Colim}_{J \in \mathbb{J}} Y_J$. In view of Lem. A.16, we are done if we show that $Q = \bigvee_{J \in \mathbb{J}} \coprod_{f_J} Q_J$.

One direction $Q \geq \bigvee_{J \in \mathbb{J}} \coprod_{f_J} Q_J$ is easy: since f_J in (B.12) is an arrow in $\mathsf{Sub}(\mathbb{C})$ we have $Q_J \leq f_J^* Q$, that is, $\coprod_{f_J} Q_J \leq Q$, for each $J \in \mathbb{J}$.

To prove the other direction ($Q \leq \bigvee_{J \in \mathbb{J}} \coprod_{f_J} Q_J$), let $(Q_K)_{K \in \mathbb{K}}$ be the canonical diagram for Q with respect to $\mathbb{F}$ (i.e. $\mathbb{K} = (\mathbb{F} \downarrow Q)$), with the canonical cocone $(Q_K \overset{c_K}{\to} Q)_{K \in \mathbb{K}}$. Then $Q = \mathrm{Colim}_{K \in \mathbb{K}} Q_K$ since $\mathbb{C}$ is LFP; furthermore, much like the proof of Sublem. B.2, we can show that $\mathrm{Colim}_{K \in \mathbb{K}} Q_K \cong \bigvee_{K \in \mathbb{K}} \coprod_{c_K} Q_K$. Hence it suffices to show

$$\coprod_{c_K} Q_K \leq \bigvee_{J \in \mathbb{J}} \coprod_{f_J} Q_J \quad \text{in } \mathsf{Sub}(Y), \text{ for each } K \in \mathbb{K}. \tag{B.13}$$

It is easy to see (using (A.5)) that $n \circ c_K$ is an arrow

$$(Q_K \overset{\mathrm{id}}{\rightarrowtail} Q_K) \xrightarrow{n \circ c_K} (Q \overset{n}{\rightarrowtail} Y)$$

in $\mathrm{Sub}(\mathbb{C})$. Since $Q_K \in \mathbb{F}$, this arrow $n \circ c_K$ is an object of the index category $\mathbb{J} = (\mathbb{F}_{\mathrm{Sub}(\mathbb{C})} {\downarrow} n)$. This yields

$$\textstyle\coprod_{n \circ c_K} Q_K \leq \bigvee_{J \in \mathbb{J}} \coprod_{f_J} Q_J \ . \tag{B.14}$$

Now the following diagram shows that $\coprod_{n \circ c_K} Q_K = \coprod_{c_K} Q_K$ as a subobject of Y, via the uniqueness of factorization.

$$\begin{tikzcd}
Q_K \arrow[r, two heads] \arrow[rr, bend left, "c_K"] \arrow[dr, two heads] & \coprod_{c_K} Q_K \arrow[r, tail] & Q \arrow[r, tail, "n"] & Y \\
& \coprod_{n \circ c_K} Q_K \arrow[u, "\cong"] \arrow[urr, tail] & &
\end{tikzcd}$$

Therefore (B.14) proves (B.13). This concludes the proof. □

B.7 Proof of Lem. 5.2

Proof The only nontrivial part is the $\Leftarrow$ direction of Cond. iii). For that it suffices to show that arbitrary $P \in \mathbb{P}$ is a colimit of the diagram $(\kappa_I^* P)_{I \in \mathbb{I}}$. Here $\mathbb{I}$ and κ_I are as in Cond. iii).

By Lem. A.16 the colimit $\mathrm{Colim}_{I \in \mathbb{I}} \kappa_I^* P$ is described as $\bigvee_{I \in \mathbb{I}} \coprod_{\kappa_I} \kappa_I^* P$ using a sup $\bigvee$ in $\mathbb{P}_X$, since $(X_I \stackrel{\kappa_I}{\to} X)_{I \in \mathbb{I}}$ is colimiting. We have $\coprod_{\kappa_I} \kappa_I^* P \leq P$ as a counit of an adjunction; therefore $\mathrm{Colim}_{I \in \mathbb{I}} \kappa_I^* P \leq P$.

Thus it suffices to show that $P \leq \mathrm{Colim}_{I \in \mathbb{I}} \kappa_I^* P$ in $\mathbb{P}_X$. Let $(P_J)_{J \in \mathbb{J}}$ be a diagram in $\mathbb{P}$ such that $P_J \in \mathbb{F}_{\mathbb{P}}$ and there is a colimiting cocone $(P_J \stackrel{g_J}{\to} P)_{J \in \mathbb{J}}$. Such a diagram exists since $\mathbb{F}_{\mathbb{P}}$ is dense.

By the assumption, for each J the object $P_J \in \mathbb{F}_{\mathbb{P}}$ lies above an object in $\mathbb{F}_{\mathbb{C}}$. Therefore the arrow $pg_J : pP_J \to pP = X$ is an object of $(\mathbb{F}_{\mathbb{C}} \downarrow X)$; since $\mathbb{I} = (\mathbb{F}_{\mathbb{C}} {\downarrow} X)$, we can choose $I_J \in \mathbb{I}$ such that $\kappa_{I_J} = pg_J$. Now an arrow $P_J \stackrel{g_J}{\to} P$ in $\mathbb{P}$ induces

$$P_J \leq (pg_J)^* P = \kappa_{I_J}^* P \tag{B.15}$$

by the universality of Cartesian arrows. We proceed as follows.

$$\begin{aligned}
P = \mathrm{Colim}_{J \in \mathbb{J}} P_J \stackrel{(*)}{=} \textstyle\bigvee_{J \in \mathbb{J}} \coprod_{pg_J} P_J \stackrel{(\dagger)}{\leq} \bigvee_{J \in \mathbb{J}} \coprod_{\kappa_{I_J}} \kappa_{I_J}^* P \\
\leq \bigvee_{I \in \mathbb{I}} \textstyle\coprod_{\kappa_I} \kappa_I^* P \stackrel{(*)}{=} \mathrm{Colim}_{I \in \mathbb{I}} \kappa_I^* P \ .
\end{aligned}$$

For $(*)$ we used Lem. A.16; $(\dagger)$ holds since I_J is chosen so that $\kappa_{I_J} = pg_J$ and (B.15) hold. This concludes the proof. □

B.8 Proof of Lem. 5.6

Proof 1a) Let us first see that $\mathrm{Fam}(\Omega)$ is cocomplete. In view of Lem. A.16, it suffices to show that $\begin{array}{c}\mathrm{Fam}(\Omega)\\ \downarrow \\ \mathbf{Sets}\end{array}$ has fiberwise colimits and coproducts $\coprod$ between fibers (the base category **Sets** is cocomplete). The former follows from Ω being a complete lattice; the latter is shown from [29, Lem. 1.9.5].

1b) Before going on we prove the following.

Sublemma B.4 *An arrow in* $\mathrm{Fam}(\Omega)$ *is a mono if and only if its underlying function is a mono in* **Sets**.

Proof (Of Sublem. B.4) The 'if' part is obvious. For the 'only if' part, let $(X \xrightarrow{f} \Omega) \overset{m}{\rightarrowtail} (Y \xrightarrow{g} \Omega)$ be a monic arrow in $\mathrm{Fam}(\Omega)$, and $k, l : U \to X$ be arrows in **Sets** such that $m \circ k = m \circ l$. This induces the following situation in $\mathrm{Fam}(\Omega)$:

$$(\bot : U \to \Omega) \overset{k}{\underset{l}{\rightrightarrows}} (f : X \to \Omega) \overset{m}{\rightarrowtail} (g : Y \to \Omega) \quad ,$$

where $\bot : U \to \Omega$ is the constant function to the least element $\bot \in \Omega$. Therefore $k = l$; this proves Sublem. B.4. $\square$

1c) We prove that each $(X \xrightarrow{f} \Omega) \in \mathbb{F}_{\mathrm{Fam}(\Omega)}$ is FG (Def. A.7) in $\mathrm{Fam}(\Omega)$. Let $\big((Y_I \xrightarrow{g_I} \Omega) \xrightarrow{h_I} (Y \xrightarrow{g} \Omega)\big)_{I \in \mathbb{I}}$ be a colimiting cocone from a directed diagram $\mathbb{I}$ whose arrows are all monos; and $(X \xrightarrow{f} \Omega) \xrightarrow{k} (Y \xrightarrow{g} \Omega)$ be an arrow in $\mathrm{Fam}(\Omega)$. We aim at showing that k factors through some h_I.

By Lem. A.16 we obtain that $Y = \mathrm{Colim}_{I \in \mathbb{I}} Y_I$; and that

$$\begin{aligned} g(y) &= \big(\textstyle\bigvee_{I \in \mathbb{I}} \coprod_{h_I} g_I\big)(y) = \bigvee_{I \in \mathbb{I}} \big((\coprod_{h_I} g_I)(y)\big) \\ &= \textstyle\bigvee_{I \in \mathbb{I}} \big(\bigvee_{y' \in h_I^{-1}(y)} g_I(y')\big) \quad \text{for each } y \in Y. \end{aligned} \tag{B.16}$$

The first equality is by Lem. A.16; the second is because the order in the fiber $\mathrm{Fam}(\Omega)_Y = \Omega^Y$ is pointwise; and the third is by the concrete description [29, Lem. 1.9.5] of $\coprod$ in $\begin{array}{c}\mathrm{Fam}(\Omega)\\ \downarrow \\ \mathbf{Sets}\end{array}$.

We observe that each h_I is a mono in $\mathrm{Fam}(\Omega)$. To see it, $(Y_I \xrightarrow{h_I} Y)_{I \in \mathbb{I}}$ is a colimiting cocone in **Sets** from a directed diagram of monos; since **Sets** is LFP, we can use [4, Prop. 1.62]; and then we use Sublem. B.4.

Now we have

$$f(x) \le g(k(x)) = \textstyle\bigvee_{I \in \mathbb{I}} \big(\bigvee_{y' \in h_I^{-1}(k(x))} g_I(y')\big) \;;$$

here the first inequality is because k is an $\mathrm{Fam}(\Omega)$-arrow; and the second equality is from (B.16). By the assumptions that $f(x)$ is compact and that X is finite, there exists $I_0 \in \mathbb{I}$ such that $f(x) \le \bigvee_{y' \in h_{I_0}^{-1}(k(x))} g_{I_0}(y')$ for each $x \in X$ (recall that $\mathbb{I}$ is filtered). Furthermore, since $X \xrightarrow{k} Y = \mathrm{Colim}_{I \in \mathbb{I}} Y_I$ is an arrow from an FP object (in **Sets**) to a directed colimit, it factors through some h_{I_1}:

$$\begin{tikzcd} X \arrow[r, "l_{I_1}"] \arrow[dr, "k"'] & Y_{I_1} \arrow[d, tail, "h_{I_1}"] \\ & Y \end{tikzcd} \quad .$$

By choosing I_2 such that $I_0, I_1 \le I_2$, we have

$$f(x) \le \textstyle\bigvee_{y' \in h_{I_2}^{-1}(k(x))} g_{I_2}(y') = g_{I_2}(l_{I_2}(x)) \quad \text{for each } x \in X;$$

here the last equality holds since h_I is an injection and $h_I(l_I(x)) = k(x)$. This proves that l_{I_2} is a $\mathrm{Fam}(\Omega)$-arrow $(X \xrightarrow{f} \Omega) \to (Y_{I_2} \xrightarrow{g_{I_2}} \Omega)$, hence $k = h_{I_2} \circ l_{I_2}$ in $\mathrm{Fam}(\Omega)$. This concludes Step 1c).

1d) The collection $\mathbb{F}_{\mathrm{Fam}(\Omega)}$ is obviously small.

1e) We are done if we prove that every object $P \in \mathrm{Fam}(\Omega)$ is a directed colimit of its subobjects from $\mathbb{F}_{\mathrm{Fam}(\Omega)}$. This easily follows from the fact that the same is true in **Sets** (obvious) and in Ω (being an algebraic lattice). □

B.9 Proof of Lem. 5.9

Proof Any presheaf $P \in \mathbf{Sets}^{\mathbb{A}}$ has a canonical isomorphism $\mathrm{Colim}_{(A,p)\in\int P} \mathbf{y}A \cong P$ induced by $(\mathbf{y}A)(B) = \mathbb{A}(A,B) \ni g \mapsto P(g)(p) \in P(A)$ for $A \in \mathbb{A}$ and $p \in P(A)$, where $\int P$ is the category of elements of P. (Remark: The category of elements of covariant functor $P\colon \mathbb{A} \to \mathbf{Sets}$ consists of objects (A,p) in the above and arrows $h\colon (A,p) \to (B,q)$ for all arrows $h\colon B \to A$ in $\mathbb{A}$ such that $P(h)(q) = p$.) In the situation, we assume that P is a subpresheaf of $\mathbf{y}X$. Then $P(g) = (\mathbf{y}X)(g) = (g \circ \_)$ shows that arrows $\{(\_ \circ f) = \mathbf{y}f\colon \mathbf{y}A \to \mathbf{y}X\}_{(A,f)}$ induce the composition $(\mathrm{Colim}_{(A,f)\in\int P} \mathbf{y}A) \cong P \hookrightarrow \mathbf{y}X$. Regarding P as the image $\mathrm{Im}\big((\mathrm{Colim}_{(A,f)\in\int P} \mathbf{y}A) \rightarrowtail \mathbf{y}X\big)$, the following component-wise calculation on objects $B \in \mathbb{A}$ shows $P = \bigcup_{(A,f)\in\int P} \mathrm{Im}\,\mathbf{y}f$:

$$
\begin{aligned}
&\big(\mathrm{Im}\big((\operatorname*{Colim}_{(A,f)\in\int P} \mathbf{y}A) \to \mathbf{y}X\big)\big)(B)\\
&\underset{(*)}{=} \mathrm{Im}\big((\operatorname*{Colim}_{(A,f)\in\int P} \mathbf{y}A)(B) \to \mathbf{y}X(B)\big)\\
&\underset{(\dagger)}{=} \mathrm{Im}\Big(\big(\operatorname*{Colim}_{(A,f)\in\int P} (\mathbf{y}A(B))\big) \to \mathbf{y}X(B)\Big)\\
&= \mathrm{Im}\Big(\big(\textstyle\coprod_{(A,f)\in\int P} (\mathbf{y}A(B))\big)/{\sim} \to \mathbf{y}X(A)\Big)\\
&= \mathrm{Im}\Big(\big(\textstyle\coprod(\mathbf{y}A(B))\big) \twoheadrightarrow \big(\textstyle\coprod(\mathbf{y}A(B))\big)/{\sim} \to \mathbf{y}X(A)\Big)\\
&= \textstyle\bigcup_{(A,f)\in\int P} \mathrm{Im}\big(\mathbf{y}A(B) \to \mathbf{y}X(B)\big)\\
&\underset{(*)}{=} \textstyle\bigcup_{(A,f)\in\int P} \big((\mathrm{Im}\,\mathbf{y}f)(B)\big) \underset{(\dagger)}{=} \big(\textstyle\bigcup_{(A,f)\in\int P} \mathrm{Im}\,\mathbf{y}f\big)(B),
\end{aligned}
$$

where $\sim$ is a suitable equivalence relation in the explicit formula of colimits in **Sets**. Note that Im in the first line and the last line are the images in $\mathbf{Sets}^{\mathbb{A}}$ while they denote the images in **Sets** elsewhere; and that $(*)$ and $(\dagger)$ holds because limits and colimits are component-wise, with a fact for $(*)$ that an image is an equalizer of a cokernel pair in both **Sets** and $\mathbf{Sets}^{\mathbb{A}}$. Therefore, there are only finitely many subpresheaves P of $\mathbf{y}X$ if $\{\mathrm{Im}\,\mathbf{y}f \mid A \in \mathbb{A}, f\colon X \to A\}$ is finite.

For the special case in the second half, we first prove the following.

Sublemma B.5 *The inclusion relation on* $\{\mathrm{Im}\,\mathbf{y}f \mid A \in \mathbb{A}, f\colon X \to A\}$ *is derived from a preorder* $\lesssim$ *on* $\{f \mid A \in \mathbb{A}, f\colon X \to A\}$ *such that* $(f\colon X \to A) \lesssim (g\colon X \to B)$ *iff* $f = h \circ g$ *for some* $h\colon B \to A$.

Proof (Of Sublem. B.5) If $\mathrm{Im}\,\mathbf{y}f \subseteq \mathrm{Im}\,\mathbf{y}g$, then $f = (\mathbf{y}f)_A(\mathrm{id}_A) \in \mathrm{Im}(\mathbf{y}f)(A) \subseteq \mathrm{Im}(\mathbf{y}g)(A) = \{h \circ g \mid h\colon B \to A\}$. Conversely, for $f = h \circ g$, any arrow $k \circ f =$

$(\mathbf{y}f)_C(k) \in \mathrm{Im}(\mathbf{y}f)(C)$ is in $\mathrm{Im}(\mathbf{y}g)(C)$ because $k \circ f = k \circ h \circ g = (\mathbf{y}g)_C(k \circ h)$. □

It is enough to show that $\mathrm{Quot}(X) \ni Y \mapsto \mathrm{Im}(\mathbf{y}Y \rightarrowtail \mathbf{y}X) \in \{\mathrm{Im}\,\mathbf{y}f \mid A \in \mathbb{A}, f\colon A \to X\}$ is a bijection. It is obviously injective because epis $e\colon X \twoheadrightarrow Y$ and $e'\colon X \twoheadrightarrow Y'$ factor through each other if and only if e and e' are the same objects in $\mathrm{Quot}(X)$. We shall prove the mapping is surjective. Let $f\colon X \to A$ be an arbitrary arrow and $f = m \circ e$ be its factorization. Then, $\mathrm{Im}\,\mathbf{y}f \subseteq \mathrm{Im}\,\mathbf{y}e$ and conversely, we also have $\mathrm{Im}\,\mathbf{y}e \subseteq \mathrm{Im}\,\mathbf{y}f$ by $e = r \circ f$ for a retraction r of m. Therefore, $\mathrm{Im}\,\mathbf{y}f = \mathrm{Im}\,\mathbf{y}e$ is a image of the mapping. □

B.10 Proof of Cor. 5.10

Sublemma B.6 *Let $(X_I)_I$ be a finite diagram in $\mathbf{Sets}^{\mathbb{A}}$. If $\mathrm{Sub}(X_I)$ is finite for each I, then so is $\mathrm{Sub}(\mathrm{Colim}_I\, X_I)$.*

Proof (Of Sublem. B.6) In a topos (hence a regular category) $\mathbf{Sets}^{\mathbb{A}}$ coproducts are disjoint (see e.g. [29]); thus we have

$$\mathrm{Sub}(X_1 + \cdots + X_n) \;\cong\; \mathrm{Sub}(X_1) \times \cdots \times \mathrm{Sub}(X_n) \;\; .$$

Let $X \rightrightarrows Y \overset{e}{\twoheadrightarrow} Z$ be a coequalizer in $\mathbf{Sets}^{A}$. The correspondence $e^* : \mathrm{Sub}(Z) \to \mathrm{Sub}(Y)$ is easily seen to be injective. Indeed, assume $P \ncong P'$ in $\mathrm{Sub}(Z)$; then $PA \ncong P'A$ for some $A \in \mathbb{A}$ in $\mathbf{Sets}$, and since e_A is surjective, we have

$$(e^*P)A = e_A^{-1}(PA) \;\ncong\; e_A^{-1}(P'A) = (e^*P)A \;\; .$$

Therefore if $\mathrm{Sub}(Y)$ is finite, so is $\mathrm{Sub}(Z)$. This concludes the proof of the sublemma. □

Proof (Of Cor. 5.10) By Example 5.8, Lem. 5.1, Sublem. B.6, and Cor. 5.5. □