Coinductive Predicates and Final Sequences in a Fibration[†]

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Coinductive predicates express persisting "safety" specifications of transition systems. Previous observations by Hermida and Jacobs identify coinductive predicates as suitable final coalgebras in a *fibration*—a categorical abstraction of predicate logic. In this paper we follow the spirit of a seminal work by Worrell and study final sequences in a fibration. Our main contribution is to identify some categorical "size restriction" axioms that guarantee stabilization of final sequences after ω steps. In its course we develop a relevant categorical infrastructure that relates fibrations and locally presentable categories, a combination that does not seem to be studied a lot. The genericity of our fibrational framework can be exploited for: binary relations (i.e. the logic of "binary predicates") for which a coinductive predicate is bisimilarity; constructive logics (where interests are growing in coinductive predicates); and logics for name-passing processes.

1. Introduction

Coinductive predicates postulate properties of state-based dynamic systems that persist after a succession of transitions. In computer science, *safety properties* of nonterminating, reactive systems are examples of paramount importance. This has led to an extensive study of specification languages in the form of fixed point logics and model-checking algorithms.

In this paper we follow (Hermida and Jacobs, 1998; Hermida, 1993)—whose results are further extended in (Fumex et al., 2011; Atkey et al., 2012), see also (Jacobs, 2012, Chap. 6)—and take a categorical view on coinductive predicates. Here *coalgebras* represent transition systems; a *fibration* is a "predicate logic"; and a coinductive predicate is

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identified as a suitable coalgebra in a fibration. Our contribution is the study of *final sequences*—an iterative construction of final coalgebras that is studied notably in (Worrell, 2005; Adámek, 2003)—in such a fibrational setting.

Coalgebras have been successfully used as a categorical abstraction of transition systems (see e.g. (Rutten, 2000; Jacobs, 2012)): by varying base categories and functors, coalgebras bring general results that work for a variety of systems at once. Fixed point logics (or modal logics in general), too, have been actively studied coalgebraically: coalgebraic modal logic is a prolific research field (see (Cîrstea et al., 2011)); their base category is typically **Sets** but works like (Klin, 2007) go beyond and use presheaf categories for processes in name-passing calculi; and literature including (Cîrstea and Sadrzadeh, 2008; Venema, 2006; Cîrstea et al., 2009) studies coalgebraic fixed point logics.

Unlike most of these works, we follow (Hermida and Jacobs, 1998; Hermida, 1993) and parametrize the underlying "predicate logic" too with the categorical notion of *fibration*. The conventional setting of classical logic is represented by the fibration $\begin{smallmatrix}\mathbf{Pred}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ (see Appendix C for an introduction to fibrations).

fibration $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$	$\begin{smallmatrix}\mathbf{Pred}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$	$\begin{smallmatrix}\mathbf{Rel}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$
coalgebra	invariant	bisimulation
final coalgebra	coinductive predicate	bisimilarity

However there are various other "logics" modeled as fibrations, and hence the fibrational language provides a uniform treatment of these different settings. An example is binary relations (instead of unary predicates) that form a fibration $\begin{smallmatrix}\mathbf{Rel}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ (see Appendix C). In this case coinductive predicates are bisimilarity relations (see the above table, and Example 7.2 later).

Another example is predicates in constructive logics. They are modeled by the subobject fibration of a topos. In fact, coinductive predicates in constructive logics are an emerging research topic: coinduction is supported in the theorem prover Coq (based on the constructive *calculus of constructions*), see e.g. (Bertot and Komendantskaya, 2008); and, working in Coq, some interesting differences between classically equivalent (co)inductive predicates have been studied e.g. in (Nakata et al., 2011).

Yet another example is modal logics for processes in various name-passing calculi. They are best modeled by the subobject fibration of a suitable (pre)sheaf category like $\mathbf{Sets^I}$ and $\mathbf{Sets^F}$ (Stark, 1996; Fiore and Turi, 2001; Fiore and Staton, 2006; Miculan, 2008; Staton, 2011).

1.1. *Coinductive Predicates and Their Construction, Conventionally*

In order to illustrate our technical contributions (§3) we here present a special case, with classical logic and Kripke models. We first introduce syntax.

Definition 1.1 (Rudimentary logic $\mathsf{R}\nu$). In this tiny fragment of the μ-calculus, fixed-point operators are limited to the greatest one at the outermost position; and moreover all the formulas are "rank-1," that is, the fixed-point variable u occurs precisely under one modal operator.

$$\mathsf{R}\nu_u \ni \alpha ::= a \mid \overline{a} \mid \Box u \mid \Diamond u \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \; ; \qquad \mathsf{R}\nu \ni \beta ::= \nu u.\,\alpha \; . \qquad (1)$$

Here a belongs to the set AP of *atomic propositions*; $\overline{a}$ stands for the negation of a; and u is the only fixed-point variable (with possibly multiple occurrences).

An $\mathsf{R}\nu$-formula can be thought of as a recursive definition of a coinductive predicate. Later we will model such a "definition" categorically as a predicate lifting. Among specifications expressible in $\mathsf{R}\nu$ is (may-) deadlock freedom ("there is an infinite path"). It is expressed by $\nu u.\,\Diamond u$ and is our recurring example.

An $\mathsf{R}\nu$-formula is interpreted in Kripke models. Let $c = (X, \to, V)$ be a Kripke model, where X is a state space, $\to \,\subseteq X \times X$ is a transition relation and $V \colon X \to \mathcal{P}(\mathsf{AP})$ is a valuation. The conventional interpretation $[\nu u.\alpha]_c$ of $\mathsf{R}\nu$-formulas in the Kripke model c is given as follows (see e.g. (Bradfield and Stirling, 2006)). Firstly, we interpret $\alpha \in \mathsf{R}\nu_u$ as a function $[\alpha]_c \colon \mathcal{P}X \to \mathcal{P}X$. Concretely:

$$[a]_c(P) = \{x \mid a \in V(x)\} \qquad\qquad [\overline{a}]_c(P) = \{x \mid a \notin V(x)\}$$
$$[\Box u]_c(P) = \{x \mid \forall y \in X.\,(x \to y \text{ implies } y \in P)\} \quad [\alpha \wedge \alpha']_c(P) = [\alpha]_c(P) \cap [\alpha']_c(P)$$
$$[\Diamond u]_c(P) = \{x \mid \exists y \in X.\,(x \to y \text{ and } y \in P)\} \qquad [\alpha \vee \alpha']_c(P) = [\alpha]_c(P) \cup [\alpha']_c(P)$$

This function $[\alpha]_c$ is easily seen to be monotone, since u occurs only positively in α. Finally we define $[\nu u.\alpha]_c \subseteq X$ to be the greatest fixed point of the monotone function $[\alpha]_c \colon \mathcal{P}X \to \mathcal{P}X$.

The Knaster-Tarski theorem guarantees the existence of such a greatest fixed point $[\nu u.\alpha]_c$ in a complete lattice $\mathcal{P}X$. However its proof is highly nonconstructive. In contrast, a well-known iterative construction (Cousot and Cousot, 1979) computes $[\nu u.\alpha]_c$ as the limit of the following descending chain (see also (Bradfield and Stirling, 2006)). Here $\top$ denotes the subset $X \subseteq X$.

$$\top \; \geq \; [\alpha]_c\top \; \geq \; [\alpha]_c^2\top \; \geq \; \cdots \qquad (2)$$

An issue now is the length of the chain. If $[\alpha]_c$ preserves limits $\bigwedge$ (which is the case with $\alpha \equiv \Box u$), clearly ω steps are enough and yields $\bigwedge_{i \in \omega}\big([\alpha]_c^i\top\big)$ as the greatest fixed point. This is not the case with $\alpha \equiv \Diamond u$. Indeed, for the Kripke model c_1 below $[\nu u.\,\Diamond u]_{c_1} \neq \bigwedge_{i \in \omega}\big([\Diamond u]_{c_1}^i\top\big)$: there is no infinite path from the root; but it satisfies $[\Diamond u]_{c_1}^i\top$ ("there is a path of length $\geq i$") for each i.

$$(3)$$

Yet the chain (2) eventually stabilizes, bounded by the size of the poset $\mathcal{P}X$: in each step before stabilization, at least one element must be thrown away. Therefore the cal-

culation of $[\nu u.\alpha]_c$ proceeds, in general, via *transfinite* induction. This is what we call a *state space bound* for the chain (2).

Besides a state space bound, another (possibly better and seemingly less known) bound can be obtained from a *behavioral view*. One realizes that not only the size of the state space X but also the *branching degree* can be used to bound the length of the chain (2). This is a result similar to the one in (Hennessy and Milner, 1985, Theorem 2.1); the latter is stated for bisimilarity as a coinductive *relation*, not for a coinductive *predicate*. We formally state (an instance of) the result for the record.

Lemma 1.2 (Behavioral bound). Let $c = (X, \to, V)$ be a finitely branching Kripke model. For $\alpha = \Diamond u$, the chain (2) stabilizes after ω steps and yields $[\nu u.\Diamond u]_c$ as its limit, that is, $\bigwedge_{i \in \omega} \big([\Diamond u]_c^i \top\big) = [\nu u.\Diamond u]_c$.

Proof. The essence of the result lies in the fact that the limit $\bigwedge_{i \in \omega} \big([\Diamond u]_c^i \top\big)$ is a $\Diamond$-invariant, which we shall prove now. Assume that a state x satisfies $\bigwedge_{i \in \omega} \big([\Diamond u]_c^i \top\big)$; we have to show that x satisfies $[\Diamond u]_c \Big(\bigwedge_{i \in \omega} \big([\Diamond u]_c^i \top\big)\Big)$, that is, there is a successor x' of x that satisfies the limit $\bigwedge_{i \in \omega} \big([\Diamond u]_c^i \top\big)$.

Since x satisfies $[\Diamond u]_c^i \top$ ("there is a path of length $\geq i$") for each i, for each $i \geq 1$, there is a successor x_i of x that satisfies $[\Diamond u]_c^{i-1} \top$. By c being finitely branching, the set $\{x_1, x_2, \dots\}$ of such successors turns out to be finite and there exists a successor x' of x such that $x' = x_i$ for infinitely many i. It follows (from $[\Diamond u]_c^i \top \leq [\Diamond u]_c^j \top$ if $j \leq i$) that this x' satisfies $[\Diamond u]_c^i \top$ for all $i \in \omega$, and hence satisfies $\bigwedge_{i \in \omega} \big([\Diamond u]_c^i \top\big)$. This proves that the limit $\bigwedge_{i \in \omega} \big([\Diamond u]_c^i \top\big)$ is an invariant, and hence $\bigwedge_{i \in \omega} \big([\Diamond u]_c^i \top\big) \leq [\nu u.\Diamond u]_c$.

For the last equality claimed in the lemma, the other direction $[\nu u.\Diamond u]_c \leq \bigwedge_{i \in \omega} \big([\Diamond u]_c^i \top\big)$ is easy: $[\nu u.\Diamond u]_c \leq [\Diamond u]_c^i \top$ is easily shown by induction on i. This concludes the proof. $\square$

Note that Lemma 1.2 holds however large the state space X is. Moreover it easily generalizes from $\nu u.\Diamond u$ to an arbitrary Rν-formula $\nu u.\alpha$. Note also that the counterexample c_1 in (3) is not finitely branching and does not contradict with Lemma 1.2.

1.2. *Final Sequences in a Fibration*

This paper is about putting the observations in §1.1 in general categorical terms. Our starting observation is that the chain (2) resembles a *final sequence*, a classic construction of a final coalgebra.

In the theory of coalgebra a *final F-coalgebra* is of prominent importance since it is a fully abstract domain with respect to the *F-behavioral equivalence*. Therefore a natural question is if a final F-coalgebra exists; the well-known Lambek lemma prohibits e.g. a final $\mathcal{P}$-coalgebra for the (full) powerset functor $\mathcal{P}$. What matters is the *size* of F: when it is suitably bounded, it is known that a final coalgebra can be constructed via the following *final F-sequence*.

$$1 \xleftarrow{\;!\;} F1 \xleftarrow{\;F!\;} \cdots \xleftarrow{\;F^{i-1}!\;} F^i 1 \xleftarrow{\;F^i!\;} \cdots \tag{4}$$

Here 1 is a final object in $\mathbb{C}$, and ! is the unique arrow. In particular, if F is *finitary*, a final coalgebra arises as a suitable subobject (or a quotient) of the ω-limit of

the final sequence (4). These constructions in **Sets** are worked out in (Pattinson, 2003; Worrell, 2005); the one in (Worrell, 2005) is further extended to locally presentable categories (those are categories suited for speaking of "size") with additional assumptions in (Adámek, 2003).

Turning back to coinductive predicates, indeed, the fibrational view (Hermida and Jacobs, 1998; Hermida, 1993) identifies coinductive predicates as final coalgebras in a fibration. This leads us to scrutinize final sequences in a fibration. Our main result (Theorem 3.9) is a categorical generalization of the behavioral ω-bound (§1.1)—more precisely we axiomatize categorical "size restrictions" for that bound to hold.

The conditions are formulated in the language of locally presentable categories (see e.g. (Adámek and Rosický, 1994); also Appendix B); and the combination of fibrations and locally presentable categories does not seem to have been studied a lot (an exception is (Makkai and Paré, 1989, §5.3)). We therefore develop a relevant categorical infrastructure (§6). Our results there include a sufficient condition for the total category $\mathrm{Sub}(\mathbb{C})$ of a subobject fibration to be locally (finitely) presentable, and the same for a family fibration $\mathrm{Fam}(\Omega)$. Via these results, in §7 we list some concrete examples of fibrations to which our results in §3 on the behavioral bounds apply. They include: $\begin{smallmatrix}\mathbf{Pred}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ (classical logic); $\begin{smallmatrix}\mathbf{Rel}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ (for bisimulation and bisimilarity); $\begin{smallmatrix}\mathrm{Sub}(\mathbb{C})\\\downarrow\\\mathbb{C}\end{smallmatrix}$ for $\mathbb{C}$ that is locally finitely presentable and locally Cartesian closed (a topos is a special case); and $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ for a well-founded algebraic lattice Ω.

1.3. *Contributions*

To summarize, our contributions are: 1) combination of the mathematical observations in (Hermida, 1993; Hermida and Jacobs, 1998) and (Jacobs, 2012, Chap. 6) for a general formulation of coinductive predicates; 2) categorical behavioral bounds for final sequences that approximate coinductive predicates; and 3) a categorical infrastructure that relates fibrations and locally presentable categories.

Compared to the earlier version (Hasuo et al., 2013) of the current paper, the main differences are as follows. Here we additionally address *inductive* predicates over coinductive datatypes (see §5). We identify them as coinductive predicates in the *fiberwise opposite* $\begin{smallmatrix}\mathbb{P}^{(\mathrm{op})}\\\downarrow p^{(\mathrm{op})}\\\mathbb{C}\end{smallmatrix}$ of the original fibration $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$, so that the difference between inductive and coinductive predicates becomes a matter of categorical duality. The examples in §7 are extended accordingly, studying inductive predicates on top of coinductive ones. Besides, we include all the proofs that were omitted in (Hasuo et al., 2013) for space reasons.

1.4. *Organization of the Paper*

In §2 we identify coinductive predicates as final coalgebras in a fibration, following the ideas of (Hermida, 1993; Hermida and Jacobs, 1998; Jacobs, 2012). The main technical results are in §3, where we axiomatize size restrictions on fibrations and functors for a

final sequence to stabilize after ω steps. These results are reorganized in §4 in a fibration of invariants. We see in §5, which is added to an earlier version of this paper (Hasuo et al., 2013), that the results in §2–4 apply to inductive predicates too. The next two sections are devoted to examples: firstly in §6 we develop a necessary categorical infrastructure; and then in §7 we discuss concrete examples. In §8 we conclude with some directions of future work. In Appendices we present minimal introductions to the theories of coalgebras, locally presentable categories and fibrations—the three categorical disciplines that our technical developments rely on.

2. Coinductive Predicates as Final Coalgebras

In this section we follow the ideas in (Hermida, 1993; Hermida and Jacobs, 1998; Jacobs, 2012) and characterize coinductive predicates in various settings (for different behavior types, and in various underlying logics) in the language of fibrations. An introduction to fibrations is e.g. in (Jacobs, 1999); see also Appendix C. In this paper for simplicity we focus on poset fibrations. It should however not be hard to move to general fibrations.

Convention 2.1 (Fibration). We refer to poset fibrations (where each fiber is a poset rather than a category) simply as *fibrations*.

Definition 2.2 (Predicate lifting). Let $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ be a fibration and F be an endofunctor on $\mathbb{C}$. A *predicate lifting* of F along p is a functor $\varphi\colon \mathbb{P}\to\mathbb{P}$ such that (φ,F) is an endomap of fibrations.

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\ \varphi\ } & \mathbb{P} \\ p\downarrow & & \downarrow p \\ \mathbb{C} & \xrightarrow{\ F\ } & \mathbb{C} \end{array} \qquad (5)$$

This means: that the above diagram commutes; and that φ preserves Cartesian arrows, that is, $\varphi(f^*Q) = (Ff)^*(\varphi Q)$. See below.

$$\begin{array}{cccc} \mathbb{P} & f^*Q \xrightarrow{\ \overline{f}Q\ } Q & \varphi(f^*Q) \xrightarrow{\ \varphi(\overline{f}Q)\ } \varphi Q & \\ \Big\downarrow{\scriptstyle p} & & (Ff)^*(\varphi Q) \xrightarrow[\overline{Ff}(\varphi Q)]{} & (6) \\ \mathbb{C} & X \xrightarrow{\ f\ } Y & FX \xrightarrow{\ Ff\ } FY & \end{array}$$

In the prototype example $\begin{smallmatrix}\mathbf{Pred}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$, the above definition coincides (see (Jacobs, 2012)) with the one used in coalgebraic modal logic (see e.g. (Cîrstea et al., 2011)), the latter being a (monotone) natural transformation $2^{(-)} \overset{\varphi}{\Rightarrow} 2^{F(-)}\colon \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$. In particular: the naturality requirement corresponds to the preservation of Cartesian arrows (6); and monotonicity of φ comes from the functoriality of $\varphi\colon \mathbb{P}\to\mathbb{P}$.

We think of predicate liftings as (co)recursive definitions of coinductive predicates (see Example 2.4). On top of it, we identify coinductive predicates (and invariants) as coalgebras in a fiber.

Definition 2.3 (Invariant, coinductive predicate). Let φ be a predicate lifting of F along $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$; and $X \overset{c}{\to} FX$ be a coalgebra in $\mathbb{C}$. They together induce an endofunctor (a monotone function) on the fiber $\mathbb{P}_X$, namely $\mathbb{P}_X \overset{\varphi}{\to} \mathbb{P}_{FX} \overset{c^*}{\to} \mathbb{P}_X$, where φ restricts to $\mathbb{P}_X \to \mathbb{P}_{FX}$ because of (5).

1 A φ-*invariant* in c is a $(c^* \circ \varphi)$-coalgebra in $\mathbb{P}_X$, that is, an object $P \in \mathbb{P}_X$ such that $P \le c^*(\varphi P)$ in $\mathbb{P}_X$.

2 The φ-*coinductive predicate* in c is the final $(c^* \circ \varphi)$-coalgebra (if it exists). Its carrier shall be denoted by $[\![\nu\varphi]\!]_c$. It is therefore the largest φ-invariant in c; Lambek's lemma yields that $[\![\nu\varphi]\!]_c = (c^* \circ \varphi)([\![\nu\varphi]\!]_c)$.

Example 2.4 ($\mathsf{R}\nu$). The conventional interpretation $[\nu u.\alpha]_c$ (described in §1.1) of $\mathsf{R}\nu$-formulas is a special case of Definition 2.3. Indeed, let us work in the fibration $\begin{smallmatrix} \mathbf{Pred} \\ \downarrow \\ \mathbf{Sets} \end{smallmatrix}$, and with the endofunctor $F_{\mathrm{K}} = \mathcal{P}(\mathsf{AP}) \times \mathcal{P}(\_)$ on **Sets**. An F_{K}-coalgebra $X \overset{c}{\to} \mathcal{P}(\mathsf{AP}) \times \mathcal{P}X$ is precisely a Kripke model: c combines a valuation $X \to \mathcal{P}(\mathsf{AP})$ and the map $X \to \mathcal{P}X$ that carries a state to the set of its successors. To each formula $\alpha \in \mathsf{R}\nu_u$ we associate a predicate lifting φ_α of F_{K}. This is done inductively as follows.

$$\begin{aligned}
\varphi_a(U \subseteq X) &= \big(\{V \in F_{\mathrm{K}}X \mid a \in \pi_1(V)\} \subseteq F_{\mathrm{K}}X \big) \\
\varphi_{\overline{a}}(U \subseteq X) &= \big(\{V \in F_{\mathrm{K}}X \mid a \notin \pi_1(V)\} \subseteq F_{\mathrm{K}}X \big) \\
\varphi_{\Box u}(U \subseteq X) &= \big(\{V \in F_{\mathrm{K}}X \mid \pi_2(V) \subseteq U\} \subseteq F_{\mathrm{K}}X \big) \\
\varphi_{\Diamond u}(U \subseteq X) &= \big(\{V \in F_{\mathrm{K}}X \mid \pi_2(V) \cap U \ne \emptyset\} \subseteq F_{\mathrm{K}}X \big) \\
\varphi_{\alpha \wedge \alpha'}(U \subseteq X) &= \big((\varphi_\alpha U \cap \varphi_{\alpha'} U) \subseteq F_{\mathrm{K}}X \big) \\
\varphi_{\alpha \vee \alpha'}(U \subseteq X) &= \big((\varphi_\alpha U \cup \varphi_{\alpha'} U) \subseteq F_{\mathrm{K}}X \big)
\end{aligned} \tag{7}$$

In the above, π_1 and π_2 denote the projections from $F_{\mathrm{K}}X = \mathcal{P}(\mathsf{AP}) \times \mathcal{P}X$. Then it is easily seen by induction that $[\![\nu\varphi_\alpha]\!]_c$ in Definition 2.3 coincides with the conventional interpretation $[\nu u.\alpha]_c$ described in §1.1.

In fact, the predicate liftings φ_α in (7) are the ones commonly used in coalgebraic modal logic (where they are presented as natural transformations). We point out that the same definition of φ_α—they are written in the internal language of toposes—works for the subobject fibration $\begin{smallmatrix} \mathrm{Sub}(\mathbb{C}) \\ \downarrow \\ \mathbb{C} \end{smallmatrix}$ of any topos $\mathbb{C}$. Therefore the categorical definition of coinductive predicates (Definition 2.3) allows us to interpret the language $\mathsf{R}\nu$ in constructive underlying logics. Suitable completeness of $\mathbb{C}$ ensures that a final $(c^* \circ \varphi)$-coalgebra in Definition 2.3 exists.

Proposition 2.5. Let φ be a predicate lifting of F along $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$; $X \overset{c}{\to} FX$ be a coalgebra in $\mathbb{C}$; and $P \in \mathbb{P}_X$. We have $P \le [\![\nu\varphi]\!]_c$ if and only if there exists a φ-invariant Q such that $P \le Q$. $\hfill\square$

The proposition is trivial but potentially useful. It says that an invariant can be used as a "witness" for a coinductive predicate. This is how bisimilarity is commonly established

(namely by finding a bisimulation); and it can be used e.g. in (Abramsky and Winschel, 2015, §6) as an alternative to the metric coinduction principle used there.[†]

Remark 2.6. The coalgebraic modal logic literature exploits the fact that there can be many predicate liftings (in the form of natural transformations) of the same functor F. Different predicate liftings correspond to different modalities (such as $\square$ vs. $\lozenge$ for the same functor $\mathcal{P}$). This view of predicate liftings is also the current paper's (see Example 2.4).

In contrast, in fibrational studies like (Hermida, 1993; Hermida and Jacobs, 1998; Fumex et al., 2011; Atkey et al., 2012), use of predicate liftings has focused on the validity of the *(co)induction proof principle*. For such purposes it is necessary to choose a predicate lifting φ that is "comprehensive enough," covering all the possible F-behaviors. In fact, it is common in these studies that "the" predicate lifting, denoted by $\mathrm{Pred}(F)$, is assigned to a functor F. An exception is (Jacobs, 2010).

3. Final Sequences in a Fibration

Here we present our main technical result (Theorem 3.9). It generalizes known behavioral ω-bounds (like (Hennessy and Milner, 1985, Theorem 2.1); see §1.1); and claims that the chain (2) for a coinductive predicate stabilizes after ω steps, assuming that the behavior type functor F and the underlying logic $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ are "finitary" in a suitable sense (but no size restriction on φ).

3.1. *Size Restrictions on a Fibration*

We axiomatize finitariness conditions in the language of locally presentable categories (see Appendix B for a minimal introduction). Singling out these conditions lies at the heart of our technical contribution.

Definition 3.1 (LFP category). A category $\mathbb{C}$ is *locally finitely presentable (LFP)* if it is cocomplete and it has a (small) set $\mathbb{F}$ of finitely presentable (FP) objects such that every object is a filtered colimit of objects in $\mathbb{F}$.

Definition 3.2 (Finitely determined fibration). A (poset) fibration $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ is *finitely determined* if it satisfies the following.

1 $\mathbb{C}$ is LFP, with a set $\mathbb{F}$ of FP objects (as in Definition 3.1).
2 $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ has fiberwise limits and colimits (as in Definition C.9).
3 For arbitrary $X \in \mathbb{C}$, let $(X_I)_{I \in \mathbb{I}}$ be the canonical diagram for X with respect to $\mathbb{F}$ (i.e. $\mathbb{I} = \mathbb{F}/X$, see Lemma B.4), with a colimiting cocone $(X_I \xrightarrow{\kappa_I} X)_{I \in \mathbb{I}}$. Then for any

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[†] To be precise: only if we take PE in (Abramsky and Winschel, 2015)—that is in fact a least fixed-point specification—as an atomic proposition (and that is essentially what is done in the proofs in (Abramsky and Winschel, 2015, §6)). Our future work on nested μ's and ν's will more adequately address the situation.

$P, Q \in \mathbb{P}_X$,

$$P \leq Q \iff \kappa_I^* P \leq \kappa_I^* Q \text{ in } \mathbb{P}_{X_I} \text{ for each } I \in \mathbb{I}.$$

The intuition behind Cond. 3 is that a predicate $P \in \mathbb{P}_X$ (over arbitrary $X \in \mathbb{C}$) is determined by its restrictions $(\kappa_I^* P)_{I \in \mathbb{I}}$ to FP objects X_I. One convenient sufficient condition for Cond. 3 is that the total category $\mathbb{P}$ is itself LFP, with its FP objects residing above the FP objects in $\mathbb{C}$ (Corollary 6.2). We note that Cond. 1 guarantees, since LFP implies completeness, that an (ω^{op}-)limit $F^\omega 1$ of the final F-sequence (4) exists. However this does not mean (nor do we need) that $F^\omega 1$ carries a final F-coalgebra; it fails for $F = \mathcal{P}_\omega$, see (Worrell, 2005).

Definition 3.3 (Well-founded fibration). A *well-founded fibration* is a finitely determined fibration that further satisfies:

4   If $X \in \mathbb{F}$ (hence FP), the fiber $\mathbb{P}_X$ is such that: the category $\mathbb{P}_X^{\mathrm{op}}$ consists solely of FP objects.
    Since $\mathbb{P}_X$ is complete, this is equivalent to: there is no (ω^{op}-)chain $P_0 > P_1 > \cdots$ in $\mathbb{P}_X$ that is strictly descending.

We note that the following stronger variant of the condition

4' For *any* $X \in \mathbb{C}$, there is no strictly descending ω^{op}-chain in $\mathbb{P}_X$

rarely holds (it fails in $\underset{\textbf{Sets}}{\overset{\textbf{Pred}}{\downarrow}}$). The original Cond. 4 holds in many examples (as we will see later in §7) thanks to the restriction that X is FP.

Remark 3.4. Conditions 3–4 mention a fixed set $\mathbb{F}$ of FP objects. It is not hard to see that this is not necessary, and we can take as $\mathbb{F}$ the set of *all* FP objects without loss of generality. (Stating the conditions in terms of $\mathbb{F}$ is an advantage when it comes to checking them, though.)

Let us first note that, by (Adámek and Rosický, 1994, Remark 1.9), any FP object $Y \in \mathbb{C}$ is a split quotient of some $X \in \mathbb{F}$, i.e. there exists $q \colon X \twoheadrightarrow Y$ and $i \colon Y \rightarrowtail X$ with $q \circ i = \mathrm{id}_Y$.

Then we indeed have the following. On Cond. 3, for an FP object Y and $\kappa' \colon Y \to X$, take $X' \in \mathbb{F}$ with a splitting $X' \overset{q}{\twoheadrightarrow} Y \overset{i}{\rightarrowtail} X'$. Then we can take I such that $X_I = X'$ and $\kappa_I = \kappa' \circ q$. Hence $\kappa_I^* P \leq \kappa_I^* Q$ in $\mathbb{P}_{X_I}$ induces $\kappa'^* P \leq \kappa'^* Q$ in $\mathbb{P}_Y$ because $\kappa' = \kappa_I \circ i$. On Cond. 4, for an FP object Y, take $X \in \mathbb{F}$ with a splitting $X \overset{q}{\twoheadrightarrow} Y \overset{i}{\rightarrowtail} X$. Then a strictly decreasing chain $Q_0 > Q_1 > \cdots$ in $\mathbb{P}_Y$ induces a strictly decreasing chain $q^* Q_0 > q^* Q_1 > \cdots$ in $\mathbb{P}_X$. Here the strictness of the latter is by $i^* q^* Q_n = Q_n$.

The following trivial fact is written down for the record.

Lemma 3.5. A finitely determined fibration $\underset{\mathbb{C}}{\overset{\mathbb{P}}{\downarrow}} p$ is well-founded if $\mathbb{P}_X$ is a finite category for each $X \in \mathbb{F}$.   □

3.2. *Final Sequences in a Fibration*

The following result from (Jacobs, 1999, Proposition 9.2.1) is crucial in our development.

Lemma 3.6. Let $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ be a fibration, with $\mathbb{C}$ being complete. Then p has fiberwise limits if and only if $\mathbb{P}$ is complete and $p\colon \mathbb{P} \to \mathbb{C}$ preserves limits. If this is the case, a limit of a small diagram $(P_I)_{I\in\mathbb{I}}$ in $\mathbb{P}$ can be given by

$$\bigwedge_{I\in\mathbb{I}}(\pi_I^* P_I) \qquad \text{over } \operatorname*{Lim}_{I\in\mathbb{I}} X_I.$$

Here $X_I := pP_I$; $(\operatorname*{Lim}_{I\in\mathbb{I}} X_I \overset{\pi_I}{\to} X_I)_{I\in\mathbb{I}}$ is a limiting cone in $\mathbb{C}$; and $\bigwedge_{I\in\mathbb{I}}$ denotes the the inf in the fiber $\mathbb{P}_{\operatorname{Lim}_I X_I}$. Moreover $\bigwedge_{I\in\mathbb{I}}(\pi_I^* P_I)$ is a limit of the diagram of shape $\mathbb{I}$, namely $\pi_I^* P_I \le \pi_J^* P_J$ holds for any $I \to J$ in $\mathbb{I}$. $\qquad\square$

Figure 1 presents two sequences. Here we assume that $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ is finitely determined (Definition 3.2) and that φ is a predicate lifting of F. In the bottom diagram (in $\mathbb{C}$), the
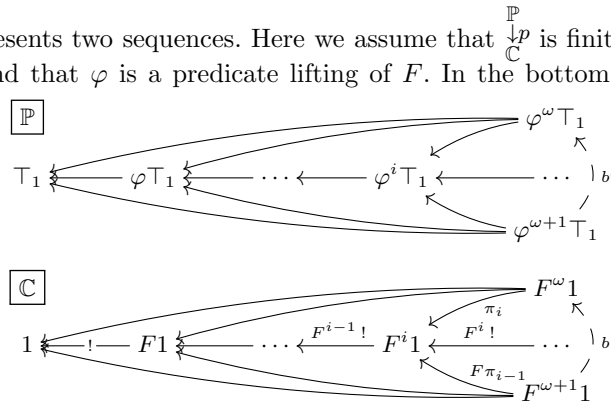

Fig. 1. Final sequences in a fibration

object $1 \in \mathbb{C}$ is a final one (it exists since LFP implies completeness); $F1 \overset{!}{\to} 1$ is the unique map; $F^{\omega+1}1 := F(F^\omega 1)$; and b is a unique mediating arrow to the limit $F^\omega 1$. In the top diagram (in $\mathbb{P}$), the object $\top_1$ is the final object in the fiber $\mathbb{P}_1$; by Lemma 3.6 this is precisely a final object in the total category $\mathbb{P}$. Hence this diagram is nothing but a final sequence for the functor φ in $\mathbb{P}$. A limit $\varphi^\omega\top_1$ of this final sequence exists, again by Lemma 3.6, and moreover it can be chosen above $F^\omega 1$. We define $\varphi^{\omega+1}\top_1 := \varphi(\varphi^\omega\top_1)$.

Lemma 3.7 (Key lemma). Let $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ be a well-founded fibration; $F\colon \mathbb{C} \to \mathbb{C}$ be finitary; and φ be a predicate lifting of F. Then the final φ-sequence "stabilizes" after ω steps (modulo reindexing via b). Precisely: in Figure 1, we have $\varphi^{\omega+1}\top_1 = b^*(\varphi^\omega\top_1)$.

Proof. We proceed by steps.

Step a. We observe that, in Figure 1, the top diagram is carried to the one below by the functor $p\colon \mathbb{P} \to \mathbb{C}$. This is straightforward: the arrow $\varphi\top_1 \to \top_1$ must be carried to the unique arrow $!\colon F1 \dashrightarrow 1$; on the mediating arrow b' in $\mathbb{P}$, since pb' is again a mediating arrow in $\mathbb{C}$, it must coincide with b.

Step b. Before moving on, we observe that Cond. 3 in Definition 3.2 yields a seemingly stronger statement (Cond. 3' below).

Sublemma 3.8. For a finitely determined fibration $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ the following holds.

3' Let $X \in \mathbb{C}$; $P, Q \in \mathbb{P}_X$; and $(Y_J)_{J\in\mathbb{J}}$ be an arbitrary filtered diagram in $\mathbb{C}$ such that

$\text{Colim}_J Y_J = X$, with a colimiting cocone $(Y_J \overset{\gamma_J}{\to} X)_{J \in \mathbb{J}}$. Then $P \leq Q$ if and only if for each $J \in \mathbb{J}$, $\gamma_J^* P \leq \gamma_J^* Q$ in $\mathbb{P}_{Y_J}$.

Proof. (Of Sublemma 3.8) The only nontrivial statement is the "if" part of the direction $3 \Rightarrow 3$'. It suffices to show that $\gamma_J^* P \leq \gamma_J^* Q$ (for each $J \in \mathbb{J}$) implies $\kappa_I^* P \leq \kappa_I^* Q$ (for each $I \in \mathbb{I}$), where κ_I and $\mathbb{I}$ are as in Cond. 3.

Let $I \in \mathbb{I}$. Since X_I is FP, an arrow $\kappa_I \colon X_I \to X$ to a filtered colimit $X = \text{Colim}_J Y_J$ factors through some $Y_{J_I} \overset{\gamma_{J_I}}{\to} X$, as in the diagram below.

$$X_I \xrightarrow{\qquad \kappa_I \qquad} X = \text{Colim}_J Y_J$$
$$\underset{h_I}{\searrow} \ Y_{J_I} \ \underset{\gamma_{J_I}}{\nearrow}$$

Now we have $\kappa_I^* P = h_I^* \gamma_{J_I}^* P \leq h_I^* \gamma_{J_I}^* Q = \kappa_I^* Q$, where the inequality is by the assumption that $\gamma_J^* P \leq \gamma_J^* Q$ for each $J \in \mathbb{J}$. This proves Sublemma 3.8. □

Step c. By Step a we see that $\varphi^{\omega+1} \top_1 \leq b^*(\varphi^\omega \top_1)$ by the universality of a Cartesian arrow. In what follows we shall prove its converse:

$$b^*(\varphi^\omega \top_1) \leq \varphi^{\omega+1} \top_1 \qquad \text{in } \mathbb{P}_{F^{\omega+1}1}. \tag{8}$$

Let us take a filtered diagram $(X_I)_{I \in \mathbb{I}}$ in $\mathbb{C}$ such that $X_I \in \mathbb{F}$ (for each $I \in \mathbb{I}$) and $F^\omega 1 = \text{Colim}_{I \in \mathbb{I}} X_I$, with $(X_I \overset{\kappa_I}{\to} F^\omega 1)_{I \in \mathbb{I}}$ being the colimiting cocone. Then we have

$$F^{\omega+1}1 = F(\underset{I \in \mathbb{I}}{\text{Colim}}\, X_I) = \underset{I \in \mathbb{I}}{\text{Colim}}\, FX_I \ ,$$

by the assumption that F is finitary; moreover $(FX_I \overset{F\kappa_I}{\to} F^{\omega+1}1)_{I \in \mathbb{I}}$ is a colimiting cocone. The diagram $(X_I)_{I \in \mathbb{I}}$ is filtered, and so is the latter diagram $(FX_I)_{I \in \mathbb{I}}$. Thus by Cond. 3' in Sublemma 3.8, showing the following proves (8):

$$(F\kappa_I)^*\big(b^*(\varphi^\omega \top_1)\big) \leq (F\kappa_I)^*(\varphi^{\omega+1} \top_1) \quad \text{for each } I \in \mathbb{I}. \tag{9}$$

Step d. To prove (9) we first prove the following fact: for each $I \in \mathbb{I}$ there exists $i_I \in \omega$ such that

$$\kappa_I^*(\varphi^\omega \top_1) = \kappa_I^*\big(\pi_{i_I}^*(\varphi^{i_I} \top_1)\big) \quad \text{in } \mathbb{P}_{X_I}. \tag{10}$$

That is: the final sequence in $\mathbb{P}$ (Figure 1), when restricted to X_I (that is FP), stabilizes within finitely many steps. Indeed, by Lemma 3.6 the ω^{op}-limit $\varphi^\omega \top_1$ is described as an ω^{op}-limit (i.e. an inf of a descending sequence) in $\mathbb{P}_{F^\omega 1}$:

$$\varphi^\omega \top_1 \ = \ \bigwedge_{i \in \omega} \pi_i^*(\varphi^i \top_1). \tag{11}$$

Therefore we have $\kappa_I^*(\varphi^\omega \top_1) = \bigwedge_{i \in \omega} \kappa_I^* \pi_i^*(\varphi^i \top_1)$ since reindexing κ_I^* preserves fiberwise limits $\bigwedge$. Here the sequence $\big(\kappa_I^* \pi_i^*(\varphi^i \top_1)\big)_{i \in \omega}$ in $\mathbb{P}_{X_I}$ is also descending. Therefore, by p being a well-founded fibration (Definition 3.3) and X_I being FP, there exists $i_I \in \omega$ at which the descending sequence $\big(\kappa_I^* \pi_i^*(\varphi^i \top_1)\big)_{i \in \omega}$ in $\mathbb{P}_{X_I}$ stabilizes, that is,

$$\kappa_I^*\big(\bigwedge_{i \in \omega} \pi_i^*(\varphi^i \top_1)\big) = \bigwedge_{i \in \omega} \kappa_I^* \pi_i^*(\varphi^i \top_1) = \kappa_I^*\big(\pi_{i_I}^*(\varphi^{i_I} \top_1)\big) \quad \text{in } \mathbb{P}_{X_I}.$$

Combined with (11), this proves (10).

Step e. Finally let us prove (9). For each $I \in \mathbb{I}$,

$$(F\kappa_I)^* \big(b^*(\varphi^\omega \top_1) \big) = (F\kappa_I)^* \big(b^*(\bigwedge_{i \in \omega} \pi_i^*(\varphi^i \top_1)) \big) \quad \text{by (11)}$$

$$= \bigwedge_{i \in \omega} (F\kappa_I)^* \big(b^*(\pi_i^*(\varphi^i \top_1)) \big) \quad \text{reindexing preserves } \bigwedge$$

$$\leq \bigwedge_{j \in \omega} (F\kappa_I)^* \big(b^*(\pi_{j+1}^*(\varphi^{j+1} \top_1)) \big) \quad \text{letting } i = j+1 \text{ for } i \geq 1$$

$$= \bigwedge_{j \in \omega} (F\kappa_I)^* \big((F\pi_j)^*(\varphi^{j+1} \top_1) \big) \quad \text{by } \pi_{j+1} \circ b = F\pi_j \text{ (see Figure 1)}$$

$$= \bigwedge_{j \in \omega} \varphi \big(\kappa_I^* \pi_j^*(\varphi^j \top_1) \big) \quad \text{by Definition 2.2}$$

$$\leq \varphi \big(\kappa_I^* \pi_{i_I}^*(\varphi^{i_I} \top_1) \big) \quad \text{letting } j = i_I \text{ on the LHS}$$

$$= \varphi \big(\kappa_I^*(\varphi^\omega \top_1) \big) \quad \text{by (10)}$$

$$= (F\kappa_I)^*(\varphi^{\omega+1} \top_1) \quad \text{by Definition 2.2 and } \varphi^{\omega+1} \top_1 = \varphi(\varphi^\omega \top_1).$$

This proves (9) and concludes the proof of Lemma 3.7. $\qquad\qquad \square$

The object $\varphi^\omega \top_1$ is a "prototype" of φ-coinductive predicates in various coalgebras. This is part of the main theorem below.

It is standard that a coalgebra $X \xrightarrow{c} FX$ in $\mathbb{C}$ induces a cone over the final F-sequence, and hence a mediating arrow $X \to F^\omega 1$ (see below). Concretely, $c_i \colon X \to F^i 1$ is defined inductively by: $X \xrightarrow{c_0} 1$ is !; and c_{i+1} is the composite $X \xrightarrow{c} FX \xrightarrow{Fc_i} F^{i+1}1$. The induced arrow to the limit $F^\omega 1$ is denoted by c_ω.

$$ \tag{12} $$

Note that $F^\omega 1$ does not necessarily carry a final F-coalgebra (see Remark 3.12).

Theorem 3.9 (Main result). Let $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$ be a well-founded fibration; $F \colon \mathbb{C} \to \mathbb{C}$ be a finitary functor; φ be a predicate lifting of F along p; and $X \xrightarrow{c} FX$ be a coalgebra in $\mathbb{C}$.

1 The φ-coinductive predicate $[\![\nu\varphi]\!]_c$ in c (Definition 2.3) exists. It is obtained by the following reindexing of $\varphi^\omega \top_1$, where c_ω is the mediating map in (12).

$$[\![\nu\varphi]\!]_c = c_\omega^*(\varphi^\omega \top_1) \tag{13}$$

2 Moreover, the predicate $[\![\nu\varphi]\!]_c$ is the limit of the following ω^{op}-chain in the fiber $\mathbb{P}_X$

$$\top_X \geq (c^* \circ \varphi)(\top_X) \geq (c^* \circ \varphi)^2(\top_X) \geq \cdots, \tag{14}$$

that stabilizes after ω steps. That is, $[\![\nu\varphi]\!]_c = \bigwedge_{i \in \omega} (c^* \circ \varphi)^i(\top_X)$.

Proof. We proceed by steps.

Step a. We first show that the descriptions of $[\![\nu\varphi]\!]_c$ in the items 1–2 are the same:

$$c_\omega^*(\varphi^\omega \top_1) = \bigwedge_{i \in \omega} (c^* \circ \varphi)^i(\top_X). \tag{15}$$

We have

$$
\begin{aligned}
c_\omega^*(\varphi^\omega \top_1) &= c_\omega^*\Big(\bigwedge_{i \in \omega} \pi_i^*(\varphi^i \top_1)\Big) \quad \text{by Lemma 3.6} \\
&= \bigwedge_{i \in \omega} c_\omega^*\big(\pi_i^*(\varphi^i \top_1)\big) \quad \text{since reindexing preserves } \bigwedge \\
&= \bigwedge_{i \in \omega} c_i^*(\varphi^i \top_1) \quad \text{by the definition of } c_\omega.
\end{aligned}
\tag{16}
$$

Furthermore, $c_i^*(\varphi^i \top_1)$ in the above is seen to be equal to $(c^* \circ \varphi)^i(\top_X)$. This is shown by induction on $i \in \omega$. For $i = 0$ the claim amounts to $!^*(\top_1) = \top_X$, which holds since reindexing preserves $\top$. For the step case,

$$
\begin{aligned}
c_{i+1}^*(\varphi^{i+1} \top_1) &= c^*(Fc_i)^*(\varphi^{i+1} \top_1) \quad \text{by } c_{i+1} = Fc_i \circ c \\
&= c^*\big(\varphi(c_i^*(\varphi^i \top_1))\big) \quad \text{by Definition 2.2} \\
&= (c^* \circ \varphi)\big((c^* \circ \varphi)^i(\top_X)\big) \quad \text{by induction hypothesis.}
\end{aligned}
$$

Therefore the equation (15) holds.

Step b. In order to show that $\bigwedge_{i \in \omega}(c^* \circ \varphi)^i(\top_X)$ is the φ-coinductive predicate in c, we shall exhibit that the chain (14)—the final $(c^* \circ \varphi)$-sequence in $\mathbb{P}_X$—stabilizes after ω steps. By (15), the claim $(c^* \circ \varphi)\big(\bigwedge_{i \in \omega}(c^* \circ \varphi)^i(\top_X)\big) = \bigwedge_{i \in \omega}(c^* \circ \varphi)^i(\top_X)$ reduces to

$$(c^* \circ \varphi)\big(c_\omega^*(\varphi^\omega \top_1)\big) = c_\omega^*(\varphi^\omega \top_1). \tag{17}$$

Step c. Finally we shall prove (17):

$$
\begin{aligned}
c^*\big(\varphi(c_\omega^*(\varphi^\omega \top_1))\big) &= c^*\big((Fc_\omega)^*(\varphi(\varphi^\omega \top_1))\big) \quad \text{by Definition 2.2} \\
&= c^*\big((Fc_\omega)^*(b^*(\varphi^\omega \top_1))\big) \quad \text{by Lemma 3.7} \\
&= (b \circ Fc_\omega \circ c)^*(\varphi^\omega \top_1) \\
&= c_\omega^*(\varphi^\omega \top_1).
\end{aligned}
\tag{18}
$$

For the last equality we used $b \circ Fc_\omega \circ c = c_\omega$, which is proved by showing that $b \circ Fc_\omega \circ c$ is also a mediating map in (12). Indeed, for each $i \geq 1$,

$$
\begin{aligned}
\pi_i \circ b \circ Fc_\omega \circ c &= F\pi_{i-1} \circ Fc_\omega \circ c \quad \text{see Figure 1} \\
&= Fc_{i-1} \circ c \quad \text{by (12)} \\
&= c_i \quad \text{by the definition of } c_i.
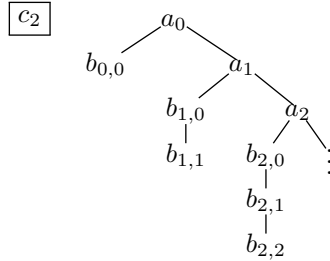\end{aligned}
$$

This concludes the proof. $\qquad\square$

Example 3.10 ($\mathsf{R}\nu$). We shall continue Example 2.4 and derive from Theorem 3.9 the behavioral bound result described in §1.1: the chain (2) stabilizes after ω steps, for each $\alpha \in \mathsf{R}\nu_u$ and each *finitely branching* Kripke model c.

Indeed, the latter is the same thing as a coalgebra $X \xrightarrow{c} F_{\mathrm{fbK}} X$, where $F_{\mathrm{fbK}} = \mathcal{P}(\mathsf{AP}) \times$

$\mathcal{P}_\omega(\_)$. Compared to F_{K} in Example 2.4 the powerset functor is restricted from $\mathcal{P}$ to $\mathcal{P}_\omega$; this makes F_{fbK} a finitary functor. Still the same definition of φ_α defines a predicate lifting of F_{fbK}. Theorem 3.9.2 can then be applied to the fibration $\begin{smallmatrix}\mathbf{Pred}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ (easily seen to be well-founded, Example 7.1), the finitary functor F_{fbK} and the predicate lifting φ_α for each α. It is not hard to see that the function $[\alpha]_c\colon \mathcal{P}X \to \mathcal{P}X$ in §1.1 coincides with $c^* \circ \varphi_\alpha\colon \mathbf{Pred}_X \to \mathbf{Pred}_X$ (note that $\mathbf{Pred}_X \cong 2^X \cong \mathcal{P}X$); thus the chain (2) coincides with (14) that stabilizes after ω steps by Theorem 3.9.

Remark 3.11. The ω-bound of the length of the chain (14) is sharp.

A (counter)example is given in the setting of Example 3.10, by the predicate lifting $\varphi_{\Diamond u}$ and the coalgebra (i.e. Kripke structure) c_2 below. There $b_{i,i}$ has no successors. Indeed, while $[\![\nu\varphi_{\Diamond u}]\!]_{c_2}$ is $\{a_i \mid i \in \omega\}$, its i-th approximant $((c_2)_i^* \circ \varphi_{\Diamond u}^i)(\top_X)$ in (14) contains $b_{i,0}$ too.

Remark 3.12. It is notable that Theorem 3.9 imposes no size restrictions on $\varphi\colon \mathbb{P} \to \mathbb{P}$. Being a predicate lifting is enough. To find an example such that φ is not finitary is future work. Our main theorem would not become trivial even if it turns out that φ is always finitary.

Final F-sequences are commonly used for the construction of a final F-coalgebra. It is not always the case, however, that the limit $F^\omega 1$ is itself the carrier of a final coalgebra (even for finitary F; see (Worrell, 2005, §5)). One obtains a final coalgebra either by: 1) quotienting $F^\omega 1$ by the behavioral equivalence (see e.g. (Pattinson, 2003)); or 2) continuing the final sequence till $\omega + \omega$ steps. The latter construction is worked out in (Worrell, 2005) (in **Sets**) and in (Adámek, 2003) in LFP $\mathbb{C}$ with additional assumptions). Its relevance to the current work is yet to be investigated.

We emphasize that a final φ-sequence "stabilizes" in ω steps *relatively* to the underlying final F-sequence. In fact we can also show that the final φ-sequence *absolutely* stabilizes in $\omega + \omega$ steps for some LFP $\mathbb{C}$ including **Sets**; a proof can be done by observing that the final φ-sequence stabilizes as soon as the final F-sequence stabilizes, once we are beyond ω steps.

To show directly the stabilization of the final φ-sequence in $\omega + \omega$ steps, one may want to prove that $\mathbb{P}$ is *strongly LFP* as in (Adámek, 2003) and that φ is finitary. Neither of these seems easy.

Coalgebra morphisms are compatible with coinductive predicates. This fact, like Proposition 2.5, is potentially useful in establishing coinductive predicates.

Proposition 3.13. Let $f\colon X \to Y$ be a coalgebra morphism from $X \xrightarrow{c} FX$ to $Y \xrightarrow{d} FY$. In the setting of Lemma 3.7 and Theorem 3.9:

1 If $Q \in \mathbb{P}_Y$ is a φ-invariant in d, so is $f^*Q \in \mathbb{P}_X$ in c.
2 We have $[\![\nu\varphi]\!]_c = f^*\big([\![\nu\varphi]\!]_d\big)$.

Proof. For the item 1:

$$
\begin{aligned}
f^*Q \;\le\; & f^*d^*(\varphi Q) \quad Q \text{ is an invariant} \\
=\; & c^*(Ff)^*(\varphi Q) \quad f \text{ is a homomorphism} \\
=\; & (c^* \circ \varphi)(f^*Q) \quad \text{by Definition 2.2.}
\end{aligned}
$$

For the item 2, the coalgebras give rise to mediating arrows $X \xrightarrow{c_\omega} F^\omega 1$ and $Y \xrightarrow{d_\omega} F^\omega 1$, respectively, as in (12). It is easy to see that $c_\omega = d_\omega \circ f$ (using the universality of the limit $F^\omega 1$); using (13) the claim follows. □

Remark 3.14. The current paper focuses on *finitely presentable* objects, *finitary* functors, etc.—i.e. the ω-presentable setting (see (Adámek and Rosický, 1994, §1.B)). This is for the simplicity of presentation: the results, as usual (as e.g. in (Klin, 2007)), can be easily generalized to the λ-presentable setting for an arbitrary regular cardinal λ. In such an extended setting we obtain a behavioral λ-bound.

4. A Fibration of Invariants

We organize the above observations in a more abstract fibered setting. The technical results are mostly standard; see e.g. (Hermida, 1993; Hermida and Jacobs, 1998) and (Jacobs, 2012, Chap.6).

We write $\mathbf{Coalg}(F)$ for the category of F-coalgebras.

Proposition 4.1. Let φ be a predicate lifting of F along $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$. Then the fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ is lifted to a fibration $\begin{smallmatrix}\mathbf{Coalg}(\varphi)\\ \downarrow \bar{p}\\ \mathbf{Coalg}(F)\end{smallmatrix}$, with two forgetful functors forming a map of fibrations from the latter to the former.

Proof. It is easy to check each fiber $\mathbf{Coalg}(\varphi)_{X \xrightarrow{c} FX}$ is a poset. Let $(X \xrightarrow{c} FX) \xrightarrow{f} (Y \xrightarrow{d} FY)$ be an arrow in $\mathbf{Coalg}(F)$, and $P \xrightarrow{s} \varphi P$ be above $Y \xrightarrow{d} FY$. A Cartesian lifting of f is obtained as in the following diagram.

$$
\boxed{\mathbb{P}} \qquad
\begin{array}{ccc}
\varphi f^*P & \xrightarrow{\;\varphi\overline{f}(P)\;} & \varphi P \\
{\scriptstyle t}\big\uparrow & & \big\uparrow{\scriptstyle s} \\
f^*P & \xrightarrow[\;\overline{f}(P)\;]{} & P
\end{array}
$$

$$
\boxed{\mathbb{C}} \qquad
\begin{array}{ccc}
FX & \xrightarrow{\;Ff\;} & FY \\
{\scriptstyle c}\big\uparrow & & \big\uparrow{\scriptstyle d} \\
X & \xrightarrow[\;f\;]{} & Y
\end{array}
$$

Here we used the universality of the Cartesian lifting $\varphi\overline{f}(P)$ (see Definition 2.2).

The two forgetful functors constitute a map of fibrations: the commutativity (5) is obvious, and Cartesian liftings in $\begin{smallmatrix}\mathbf{Coalg}(\varphi)\\\downarrow\overline{p}\\\mathbf{Coalg}(F)\end{smallmatrix}$ (which we constructed above) are based on the Cartesian liftings in $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$. $\qquad\square$

The next observation explains the current section's title.

Proposition 4.2. Let $\begin{smallmatrix}\mathbf{Coalg}(\varphi)\\\downarrow\overline{p}\\\mathbf{Coalg}(F)\end{smallmatrix}$ be the lifted fibration in Proposition 4.1. For each coalgebra $X \xrightarrow{c} FX$, the fiber over c coincides with the poset of φ-invariants in c. That is:

$$\mathbf{Coalg}(\varphi)_{X\xrightarrow{c}FX} \xrightarrow{\;\cong\;} \mathbf{Coalg}(c^* \circ \varphi) \atop \searrow \; \mathbb{P}_X \; \swarrow \quad .$$

Proof. Given a φ-coalgebra $P \xrightarrow{s} \varphi P$ above $X \xrightarrow{c} FX$, we use the universality of the Cartesian lifting of c to obtain a $(c^* \circ \varphi)$-coalgebra as in the following diagram.

$$c^*(\varphi P) \xrightarrow{\overline{c}(\varphi P)} \varphi P \atop \begin{array}{c} \uparrow \\ \vdots \\ P \end{array} \qquad \nearrow s$$

Conversely, given a $(c^* \circ \varphi)$-coalgebra $Q \xrightarrow{t} c^*(\varphi Q)$, we obtain a φ-coalgebra above $X \xrightarrow{c} FX$ as the following composite.

$$c^*(\varphi Q) \xrightarrow{\overline{c}(\varphi Q)} \varphi Q \atop \begin{array}{c} t\uparrow \\ Q \end{array}$$

Then it is straightforward to see that the mappings are monotone and inverse to each other. The mappings commute with the forgetful functors since they do not change the carriers. $\qquad\square$

Therefore Theorem 3.9.1 and Proposition 3.13.2 state the fibration $\begin{smallmatrix}\mathbf{Coalg}(\varphi)\\\downarrow\overline{p}\\\mathbf{Coalg}(F)\end{smallmatrix}$ has fiberwise final objects. (At least part of) this statement itself is shown quite easily using the Knaster-Tarski theorem (each fiber is a complete lattice). Our contribution is their concrete construction as ω^{op}-limits (Theorem 3.9.2).

The following lemma is essentially a special case of Lemma 3.6, but see also (Jacobs, 1999, Proposition 9.2.1 and Exercise 9.2.4).

Lemma 4.3. Let $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ be a fibration; and assume that $\mathbb{C}$ has a final object. Then $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ has a fiberwise final object if and only if $\mathbb{P}$ has a final object that is above the final object of $\mathbb{C}$. $\qquad\square$

By applying the lemma to $\begin{smallmatrix} \mathbf{Coalg}(\varphi) \\ \downarrow \overline{p} \\ \mathbf{Coalg}(F) \end{smallmatrix}$, we obtain a basic relationship between coinductive predicates and final coalgebras.

Corollary 4.4. Let φ be a predicate lifting of F along $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$; and assume that a final F-coalgebra exists. The following are equivalent.

1. The coinductive predicate $[\![\nu\varphi]\!]_c$ exists for each coalgebra $c\colon X \to FX$. Moreover they are preserved by reindexing (along coalgebra morphisms).
2. There exists a final φ-coalgebra that is above the final F-coalgebra. $\qquad\square$

As noted in Remark 3.12, however, our concrete construction of coinductive predicates does not rely on a final F-coalgebra.

5. Inductive predicates over coinductive datatypes

The central topic of the current paper is coinductive predicates over coinductive datatypes, the latter identified as coalgebras in the base category $\mathbb{C}$ of a fibration $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$. Some variations are possible, namely: *inductive/coinductive* predicates over *inductive/coinductive* datatypes. For example, (Hermida and Jacobs, 1998) focus on: inductive predicates over inductive datatypes (the latter identified as algebras); and coinductive predicates over coinductive datatypes (as we have done in the previous sections).

It turns out that, among these four variations, inductive predicates over coinductive datatypes allow a straightforward adaptation of our current categorical framework by taking the *fiberwise opposite* $\begin{smallmatrix} \mathbb{P}^{(\mathrm{op})} \\ \downarrow p^{(\mathrm{op})} \\ \mathbb{C} \end{smallmatrix}$ of the fibration $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$ we are interested in. We present these results in the current section. The study of the other two variations—inductive predicates over inductive datatypes, and coinductive predicates over inductive datatypes—is left as future work. In fact we have preliminary observations that under certain assumptions these two variations coincide. Their details will be presented in another venue.

The following is the definition of an inductive predicate (on a coinductive datatype). It is not hard to see that the definition generalizes e.g. the semantics of the μ operator of the modal μ-calculus in a Kripke model. Later in Lemma 5.4 we will identify it as a coinductive predicate in the fiberwise opposite.

Definition 5.1 (Inductive predicate). Let φ be a predicate lifting along a fibration $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$; and $X \xrightarrow{c} FX$ be a coalgebra in $\mathbb{C}$. The φ-*inductive predicate* in c is the initial $(c^* \circ \varphi)$-algebra (if it exists). We denote its carrier by $[\![\mu\varphi]\!]_c$. Hence, it is the smallest predicate $P \in \mathbb{P}_X$ such that $P \geq c^*(\varphi P)$ in $\mathbb{P}_X$.

In what follows we utilize the notion of *fiberwise opposite* $\begin{smallmatrix} \mathbb{P}^{(\mathrm{op})} \\ \downarrow p^{(\mathrm{op})} \\ \mathbb{C} \end{smallmatrix}$ of a fibration $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$ ((Bénabou, 1975); see also (Jacobs, 1999, Definition 1.10.11)). Intuitively, the fiberwise opposite $p^{(\mathrm{op})}$ is obtained by opposing the order in each fiber $\mathbb{P}_X$ but leaving the base category $\mathbb{C}$, as well as the reindexing structure, as in the original fibration p. The precise

definition is best stated via indexed categories and the Grothendieck construction. It is left to the appendix (Lemma C.13).

Some remarks are in order. Firstly, the total category $\mathbb{P}^{(\mathrm{op})}$ of the fiberwise opposite $\mathbb{P}^{(\mathrm{op})}_{\downarrow p^{(\mathrm{op})}} \mathbb{C}$ is in general different from the opposite category $\mathbb{P}^{\mathrm{op}}$ (in the usual sense) of $\mathbb{P}$. The same applies to the functor $p^{(\mathrm{op})}$, that is different from the opposite functor p^{op}. We emphasize that in the fiberwise opposite $\mathbb{P}^{(\mathrm{op})}_{\downarrow p^{(\mathrm{op})}} \mathbb{C}$, the base category $\mathbb{C}$ stays the same. We also note that $\mathbb{P}^{(\mathrm{op})}_{\downarrow p^{(\mathrm{op})}} \mathbb{C}$ is a fibration, unlike the opposite functor $\mathbb{P}^{\mathrm{op}}_{\downarrow p^{\mathrm{op}}} \mathbb{C}^{\mathrm{op}}$ of p that is canonically an opfibration.

Notation 5.2. For distinction, we denote reindexing functors in fibrations $\mathbb{P}_{\downarrow p} \mathbb{C}$ and $\mathbb{P}^{(\mathrm{op})}_{\downarrow p^{(\mathrm{op})}} \mathbb{C}$ by f^* and $f^\#$, respectively. They are in fact the same monotone functions between fibers as posets:

$$
\begin{array}{ccc}
(\mathbb{P}^{(\mathrm{op})})_Y & \xrightarrow{\ f^\#\ } & (\mathbb{P}^{(\mathrm{op})})_X \\
\| & & \| \\
(\mathbb{P}_Y)^{\mathrm{op}} & \xrightarrow{\ (f^*)^{\mathrm{op}}\ } & (\mathbb{P}_X)^{\mathrm{op}}
\end{array}
\qquad \text{for } f \colon X \to Y.
$$

The following result, although straightforward, is essential for the subsequent technical development.

Lemma 5.3. Let $\mathbb{P}_{\downarrow p} \mathbb{C}$ be a fibration and F be an endofunctor on $\mathbb{C}$. For a predicate lifting $\varphi \colon \mathbb{P} \to \mathbb{P}$ of F along p, there exists a canonical predicate lifting $\varphi^{(\mathrm{op})} \colon \mathbb{P}^{(\mathrm{op})} \to \mathbb{P}^{(\mathrm{op})}$, which we call the *fiberwise opposite* of φ, of F along the fibration $\mathbb{P}^{(\mathrm{op})}_{\downarrow p^{(\mathrm{op})}} \mathbb{C}$.

Proof. We give an explicit construction here, although the statement is almost trivial when stated in terms of indexed categories.

On objects, we define $\varphi^{(\mathrm{op})} P = \varphi P$. For the action on arrows, we first note that an arrow $P \to Q$ in $\mathbb{P}^{(\mathrm{op})}$ above $f \colon X \to Y$ exists if and only if $P \le f^\# Q$ in $(\mathbb{P}^{(\mathrm{op})})_X = (\mathbb{P}_X)^{\mathrm{op}}$. Exploiting this fact, $\varphi^{(\mathrm{op})}$'s action on the arrow $P \to Q$ is defined to be the unique arrow $\varphi^{(\mathrm{op})} P \to \varphi^{(\mathrm{op})} Q$ above $Ff \colon FX \to FY$. The last (unique) arrow exists, indeed: we have $\varphi^{(\mathrm{op})} P \le (Ff)^\# \varphi^{(\mathrm{op})} Q$ in $(\mathbb{P}^{(\mathrm{op})})_{FX}$ by $\varphi P \ge \varphi f^* Q = (Ff)^* \varphi Q$ in $\mathbb{P}_{FX}$. Here the last equality is because φ is a predicate lifting. $\square$

Lemma 5.4. Let P be a predicate over $X \in \mathbb{C}$.

1 The object $P \in \mathbb{P}_X$ carries a $(c^* \circ \varphi)$-algebra if and only if $P \in (\mathbb{P}_X)^{\mathrm{op}} = (\mathbb{P}^{(\mathrm{op})})_X$ is a $\varphi^{(\mathrm{op})}$-invariant in c.

2 The φ-inductive predicate in c is the $\varphi^{(\mathrm{op})}$-coinductive predicate in c. That is, $[\![\mu\varphi]\!]_c = [\![\nu(\varphi^{(\mathrm{op})})]\!]_c$ as objects in $\mathbb{P}_X$.

Proof. The category of $(c^* \circ \varphi)$-algebras in $\mathbb{P}_X$ is dually equivalent to the category of

$(c^{\#} \circ \varphi^{(\mathrm{op})})$-coalgebras in $(\mathbb{P}^{(\mathrm{op})})_X$, since the following diagram (in **Posets**) commutes.

$$
\begin{array}{ccccc}
(\mathbb{P}^{(\mathrm{op})})_X & \xrightarrow{(\varphi^{(\mathrm{op})})_X} & (\mathbb{P}^{(\mathrm{op})})_{FX} & \xrightarrow{c^{\#}} & (\mathbb{P}^{(\mathrm{op})})_X \\
\| & & \| & & \| \\
(\mathbb{P}_X)^{\mathrm{op}} & \xrightarrow{\varphi_X^{\mathrm{op}}} & (\mathbb{P}_{FX})^{\mathrm{op}} & \xrightarrow{(c^*)^{\mathrm{op}}} & (\mathbb{P}_X)^{\mathrm{op}}
\end{array}
$$

$\square$

Thanks to the previous characterization—inductive predicates in $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ as coinductive ones in $\begin{smallmatrix}\mathbb{P}^{(\mathrm{op})}\\\downarrow p^{(\mathrm{op})}\\\mathbb{C}\end{smallmatrix}$—we can apply all the results that we have obtained so far to inductive predicates. Notice again that the base category $\mathbb{C}$ has remained the same. The characterization in Lemma 5.4 can be seen as a generalization of the duality $\mu u.\,\varphi(u) = \neg\nu u.\,\neg\varphi(\neg u)$ between least and greatest fixed points in classical logics—the latter is a special case where fibers are self-dual, i.e. $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix} \cong \begin{smallmatrix}\mathbb{P}^{(\mathrm{op})}\\\downarrow p^{(\mathrm{op})}\\\mathbb{C}\end{smallmatrix}$.

Via the last characterization, our main result (Theorem 3.9) can also be used to show the stabilization of the ω-chain when calculating inductive predicates (see Corollary 5.8). The inductive predicate on $F^{\omega}1$ is not a limit nor a colimit in $\mathbb{P}$, but it is a limit in $\mathbb{P}^{(\mathrm{op})}$ (see Definition 5.7).

Definition 5.5 (Co-well-founded fibration). A *co-well-founded fibration* is a finitely determined fibration that further satisfies:

$\overline{4}$ If $X \in \mathbb{F}$ (hence FP), the fiber $\mathbb{P}_X$ is such that: the category $\mathbb{P}_X$ consists solely of FP objects.

Since $\mathbb{P}_X$ is cocomplete, this is equivalent to: there is no $(\omega\text{-})$chain $P_0 < P_1 < \cdots$ in $\mathbb{P}_X$ that is strictly ascending.

Lemma 5.6. For a finitely determined fibration $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$, its fiberwise opposite $\begin{smallmatrix}\mathbb{P}^{(\mathrm{op})}\\\downarrow p^{(\mathrm{op})}\\\mathbb{C}\end{smallmatrix}$ is also finitely determined. Moreover, $p^{(\mathrm{op})}$ is well-founded if and only if the fibration p is co-well-founded.

Proof. It is trivial that the fibration $p^{(\mathrm{op})}$ satisfies the condition 1 (of Definition 3.2) if and only if p satisfies it. For the condition 2, $p^{(\mathrm{op})}$ has fiberwise limits and colimits, because p has fiberwise colimits and limits, respectively. The condition 3 for $p^{(\mathrm{op})}$ is obviously equivalent to the one for p since reindexing functors $\kappa_I^*, \kappa_I^{\#}$ are the same as functions. By $(\mathbb{P}^{(\mathrm{op})})_X = (\mathbb{P}_X)^{\mathrm{op}}$, $p^{(\mathrm{op})}$ satisfies the condition 4 if and only if p satisfies $\overline{4}$. $\square$

Definition 5.7. Let $\perp_1$ be the least element of the fiber $\mathbb{P}_1$ (hence the greatest in $(\mathbb{P}^{(\mathrm{op})})_1$). We denote by $\varphi^{\omega}\perp_1 \in \mathbb{P}_{F^{\omega}1}$ the limit of the following diagram in $\mathbb{P}^{(\mathrm{op})}$. It is easily seen to reside above the final F-sequence in $\mathbb{C}$.

$$
\perp_1 \longleftarrow \varphi\perp_1 \longleftarrow \cdots \longleftarrow \varphi^i\perp_1 \longleftarrow \cdots \qquad \text{in } \mathbb{P}^{(\mathrm{op})}
$$

Note here that $\perp_1$ is the final object in $\mathbb{P}^{(\mathrm{op})}$, and the object $\varphi\perp_1$ is the functor $\varphi^{(\mathrm{op})}$ applied to $\perp_1$. Therefore the above diagram is the final $\varphi^{(\mathrm{op})}$-sequence in $\mathbb{P}^{(\mathrm{op})}$.

Using Lemma 3.6, it is not hard to see that $\varphi^\omega \bot_1 = \bigvee_{i \in \omega} \pi_i^*(\varphi^i \bot_1)$ in the fibration $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$, where $(\pi_i \colon F^\omega 1 \to F^i 1)_{i \in \omega}$ is the limiting cone for the final F-sequence in $\mathbb{C}$.

The following is our main result adapted to inductive predicates. In particular it states that an inductive predicate is computed as a supremum of an ω-chain.

Corollary 5.8. Let $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$ be a co-well-founded fibration; $F \colon \mathbb{C} \to \mathbb{C}$ be a finitary functor; φ be a predicate lifting of F along p; and $X \xrightarrow{c} FX$ be a coalgebra in $\mathbb{C}$.

1 The φ-inductive predicate $[\![\mu\varphi]\!]_c$ in c exists. It is obtained by the following reindexing of $\varphi^\omega \bot_1$, where c_ω is the mediating map in (12).

$$[\![\mu\varphi]\!]_c = c_\omega^*(\varphi^\omega \bot_1)$$

2 Moreover, the predicate $[\![\mu\varphi]\!]_c$ is the colimit of the following ω-chain in the fiber $\mathbb{P}_X$

$$\bot_X \leq (c^* \circ \varphi)(\bot_X) \leq (c^* \circ \varphi)^2(\bot_X) \leq \cdots ,$$

that stabilizes after ω steps. That is, $[\![\mu\varphi]\!]_c = \bigvee_{i \in \omega} (c^* \circ \varphi)^i(\bot_X)$.

Proof. By Lemma 5.4, Lemma 5.6, and Theorem 3.9. □

Corollary 5.9. Let φ be a predicate lifting of F along $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$; and $\begin{smallmatrix} (\mathbf{Coalg}(\varphi^{(\mathrm{op})}))^{(\mathrm{op})} \\ \downarrow \overline{p^{(\mathrm{op})}}^{(\mathrm{op})} \\ \mathbf{Coalg}(F) \end{smallmatrix}$ be the fiberwise opposite of the lift of the fibration $\begin{smallmatrix} \mathbb{P}^{(\mathrm{op})} \\ \downarrow p^{(\mathrm{op})} \\ \mathbb{C} \end{smallmatrix}$ (see Proposition 4.1 and Lemma 5.3). For each coalgebra $X \xrightarrow{c} FX$, the following diagram commutes.

$$\left(\left(\mathbf{Coalg}(\varphi^{(\mathrm{op})}) \right)^{(\mathrm{op})} \right)_{X \xrightarrow{c} FX} \xrightarrow{\ \cong\ } \mathbf{Alg}(c^* \circ \varphi)$$
$$((\mathbb{P}_X)^{\mathrm{op}})^{\mathrm{op}} = \mathbb{P}_X$$

Proof. Apply Proposition 4.2 for the predicate lifting $\varphi^{(\mathrm{op})}$ along $\begin{smallmatrix} \mathbb{P}^{(\mathrm{op})} \\ \downarrow p^{(\mathrm{op})} \\ \mathbb{C} \end{smallmatrix}$, we obtain

$$\mathbf{Coalg}(\varphi^{(\mathrm{op})})_{X \xrightarrow{c} FX} \xrightarrow{\ \cong\ } \mathbf{Coalg}(c^\# \circ \varphi^{(\mathrm{op})}) ,$$
$$(\mathbb{P}^{(\mathrm{op})})_X$$

whose opposite categories are the ones in the diagram we want to prove. □

Coalgebra morphisms are compatible with inductive predicates just as in Proposition 3.13. Therefore the inductive predicates $[\![\mu\varphi]\!]_c$ form a fiberwise initial object $\bot = [\![\mu\varphi]\!]$ of the fibration $\overline{p^{(\mathrm{op})}}^{(\mathrm{op})}$.

6. Examples at Large

Here are several results that ensure a fibration to be finitely determined or well-founded, and hence enable us to apply Theorem 3.9. Some of them are well-known; others—

especially those which relate fibrations and locally (finitely) presentable categories, including Lemma 6.3 and Lemma 6.7—seem to be new.

The following results provide sufficient conditions for a fibration to be finitely determined (Definition 3.2). Recall that a full subcategory $\mathbb{F}$ of $\mathbb{P}$ is said to be *dense* if each object $P \in \mathbb{P}$ is a colimit of the canonical diagram $\mathbb{F}/P \xrightarrow{\pi} \mathbb{F} \hookrightarrow \mathbb{P}$.

Lemma 6.1. Let $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$ be a fibration with fiberwise limits and colimits and coproducts $\coprod$ between fibers. Assume further that $\mathbb{C}$ is LFP with a set $\mathbb{F}_\mathbb{C}$ of FP objects (as in Definition 3.1). If the total category $\mathbb{P}$ has a dense subcategory $\mathbb{F}_\mathbb{P}$ such that every $R \in \mathbb{F}_\mathbb{P}$ is above $\mathbb{F}_\mathbb{C}$ (i.e. $pR \in \mathbb{F}_\mathbb{C}$), then p is finitely determined.

Proof. The only nontrivial part is the $\Leftarrow$ direction of Cond. 3. For that it suffices to show that arbitrary $P \in \mathbb{P}$ is a colimit of the diagram $(\kappa_I^* P)_{I \in \mathbb{I}}$. Here $\mathbb{I}$ and κ_I are as in Cond. 3.

By Lemma C.11 the colimit $\mathrm{Colim}_{I \in \mathbb{I}} \kappa_I^* P$ is described as $\bigvee_{I \in \mathbb{I}} \coprod_{\kappa_I} \kappa_I^* P$ using a sup $\bigvee$ in $\mathbb{P}_X$, since $(X_I \xrightarrow{\kappa_I} X)_{I \in \mathbb{I}}$ is colimiting. We have $\coprod_{\kappa_I} \kappa_I^* P \leq P$ as a counit of an adjunction; therefore $\mathrm{Colim}_{I \in \mathbb{I}} \kappa_I^* P \leq P$.

Thus it suffices to show that $P \leq \mathrm{Colim}_{I \in \mathbb{I}} \kappa_I^* P$ in $\mathbb{P}_X$. Let $(P_J)_{J \in \mathbb{J}}$ be a diagram in $\mathbb{P}$ such that $P_J \in \mathbb{F}_\mathbb{P}$ and there is a colimiting cocone $(P_J \xrightarrow{g_J} P)_{J \in \mathbb{J}}$. Such a diagram exists since $\mathbb{F}_\mathbb{P}$ is dense.

By the assumption, for each J the object $P_J \in \mathbb{F}_\mathbb{P}$ lies above an object in $\mathbb{F}_\mathbb{C}$. Therefore the arrow $pg_J \colon pP_J \to pP = X$ is an object of $\mathbb{F}_\mathbb{C}/X$; since $\mathbb{I} = \mathbb{F}_\mathbb{C}/X$, we can choose $I_J \in \mathbb{I}$ such that $\kappa_{I_J} = pg_J$. Now an arrow $P_J \xrightarrow{g_J} P$ in $\mathbb{P}$ induces

$$P_J \leq (pg_J)^* P = \kappa_{I_J}^* P \tag{19}$$

by the universality of Cartesian arrows. We proceed as follows.

$$P = \mathrm{Colim}_{J \in \mathbb{J}} P_J \stackrel{(*)}{=} \bigvee_{J \in \mathbb{J}} \coprod_{pg_J} P_J \stackrel{(\dagger)}{\leq} \bigvee_{J \in \mathbb{J}} \coprod_{\kappa_{I_J}} \kappa_{I_J}^* P \leq \bigvee_{I \in \mathbb{I}} \coprod_{\kappa_I} \kappa_I^* P \stackrel{(*)}{=} \mathrm{Colim}_{I \in \mathbb{I}} \kappa_I^* P \ .$$

For $(*)$ we used Lemma C.11; $(\dagger)$ holds since I_J is chosen so that $\kappa_{I_J} = pg_J$ and (19) hold. This concludes the proof. $\qquad\square$

Corollary 6.2. Let $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$ be a fibration with fiberwise limits and colimits and coproducts $\coprod$ between fibers, where $\mathbb{C}$ is LFP with a set $\mathbb{F}_\mathbb{C}$ of FP objects (in Definition 3.1). If the total category $\mathbb{P}$ is also LFP, with a set $\mathbb{F}_\mathbb{P}$ of FP objects (as in Definition 3.1) chosen so that every $R \in \mathbb{F}_\mathbb{P}$ is above $\mathbb{F}_\mathbb{C}$, then p is finitely determined. $\qquad\square$

6.1. *Subobject Fibrations*

The following is one of the results that are nontrivial.

Lemma 6.3. Let $\mathbb{C}$ be an LFP category with $\mathbb{F}$ being a set of FP objects (as in Definition 3.1). Then the total category $\mathrm{Sub}(\mathbb{C})$ of the subobject fibration is LFP: the set

$$\mathbb{F}_{\mathrm{Sub}(\mathbb{C})} := \{(P \rightarrowtail X) \mid X \in \mathbb{F}, \text{ and there exists a strong epi } Z \twoheadrightarrow P \text{ such that } Z \in \mathbb{F}\}$$

consists of FP objects in $\mathrm{Sub}(\mathbb{C})$; and every object $(Q \rightarrowtail Y) \in \mathrm{Sub}(\mathbb{C})$ is a colimit of a filtered diagram in $\mathbb{F}_{\mathrm{Sub}(\mathbb{C})}$.

Proof. The proof is by steps.

Step a. First we show that $\mathrm{Sub}(\mathbb{C})$ is complete and cocomplete. We rely on Lemma C.11.

We start with fiberwise limits in $\begin{smallmatrix}\mathrm{Sub}(\mathbb{C})\\\downarrow\\\mathbb{C}\end{smallmatrix}$; the proof is like in (Jacobs, 1999, Example 1.8.3(iii)). By Lemma B.6 an LFP category $\mathbb{C}$ is complete. This equips each fiber $\mathrm{Sub}(X)$ with arbitrary inf's $\bigwedge$ computed as wide pullbacks. A reindexing functor (by pullbacks) preserves these inf's since limits commute. Therefore by Lemma C.11 the total category $\mathrm{Sub}(\mathbb{C})$ is complete.

Each fiber (which is a poset) has arbitrary inf's; hence it is a complete lattice and arbitrary sup's also exist.

Next we show that $\begin{smallmatrix}\mathrm{Sub}(\mathbb{C})\\\downarrow\\\mathbb{C}\end{smallmatrix}$ is a bifibration (Definition C.3). An abstract proof can be given by Freyd's adjoint functor theorem (note that each fiber $\mathrm{Sub}(X)$ is a complete lattice, and that reindexing f^* preserves inf's). Instead we explicitly introduce $\coprod$ exploiting a factorization structure of LFP $\mathbb{C}$ (Lemma B.6.2). Namely, given $(P \overset{m}{\rightarrowtail} X) \in \mathrm{Sub}(X)$ and $f \colon X \to Y$, the opreindexing $\coprod_f P$ is defined by the $(\mathrm{StrongEpi}, \mathrm{Mono})$-factorization of $f \circ m$, as below.[‡]

$$
\begin{array}{ccc}
P & \twoheadrightarrow & \coprod_f P \\
\scriptstyle{m}\downarrow\rotatebox{90}{$\scriptstyle\prec$} & & \downarrow\rotatebox{90}{$\scriptstyle\prec$} \\
X & \underset{f}{\longrightarrow} & Y
\end{array}
\tag{20}
$$

The fact that $\coprod_f P \le Q$ if and only if $P \le f^*Q$ is easily proved using the diagonalization property of the factorization structure. This establishes $\coprod_f$ as a left adjoint to reindexing f^*. Using Lemma C.11 we conclude that $\mathrm{Sub}(\mathbb{C})$ is cocomplete.

Step b. Let $\mathrm{Im} \colon \mathbb{C}/Y \to \mathrm{Sub}(Y)$ be the image functor defined by the $(\mathrm{StrongEpi}, \mathrm{Mono})$-factorization (i.e. $\mathrm{Im}\, f = \coprod_f X$ for $f \colon X \to Y$). In the notation in Lemma B.11.2, we have

$$
\begin{aligned}
\mathbb{F}_{\mathrm{Sub}(\mathbb{C})} &= \{\mathrm{Im}\, f \mid X \in \mathbb{F},\ f \in \mathbb{F}/X\} \\
&= \{(P \rightarrowtail X) \in \mathrm{Sub}(\mathbb{C}) \mid X \in \mathbb{F},\ P \in \mathbb{F}_{\mathrm{Sub}(X)}\}.
\end{aligned}
$$

The set $\mathbb{F}_{\mathrm{Sub}(\mathbb{C})}$ is small, since $\mathbb{F}$ is small and $\mathbb{F}_{\mathrm{Sub}(X)}$ is small for each $X \in \mathbb{F}$.

Step c. First we prove that $(P \overset{m}{\rightarrowtail} X) \in \mathbb{F}_{\mathrm{Sub}(\mathbb{C})}$ is FP in $\mathrm{Sub}(\mathbb{C})$.

Sublemma 6.4. *Let $(Q_I \overset{n_I}{\rightarrowtail} Y_I)_{I \in \mathbb{I}}$ be a filtered diagram in $\mathrm{Sub}(\mathbb{C})$. Then pointwise colimits $\mathrm{Colim}_{I \in \mathbb{I}} Y_I$ and $\mathrm{Colim}_{I \in \mathbb{I}} Q_I$ in $\mathbb{C}$ form a colimit of the diagram in $\mathrm{Sub}(\mathbb{C})$:*

$$
\underset{I \in \mathbb{I}}{\mathrm{Colim}}(Q_I \overset{n_I}{\rightarrowtail} Y_I) = \Big(\underset{I \in \mathbb{I}}{\mathrm{Colim}}\, Q_I \to \underset{I \in \mathbb{I}}{\mathrm{Colim}}\, Y_I \Big).
$$

Proof. (Of the sublemma) On the one hand, by Lemma C.11 the colimit $(Q \overset{n}{\rightarrowtail} Y) =$

[‡] Opreindexings $\coprod_f \dashv f^*$ do not have to satisfy the Beck-Chevalley condition.

$\mathrm{Colim}_{I \in \mathbb{I}}(Q_I \rightarrowtail Y_I)$ can be explicitly described as

$$Y = \operatorname*{Colim}_{I \in \mathbb{I}} Y_I \ , \qquad Q = \bigvee_{I \in \mathbb{I}} \coprod_{\kappa_I} Q_I \ , \tag{21}$$

where $(Y_I \xrightarrow{\kappa_I} Y)_{I \in \mathbb{I}}$ is a colimiting cocone. On the other hand, both $(Q_I)_{I \in \mathbb{I}}$ and $(Y_I)_{I \in \mathbb{I}}$ are $\mathbb{I}$-shaped diagrams in $\mathbb{C}$ with a monotransformation $(Q_I \xrightarrow{n_I} Y_I)_I$. Therefore by (Adámek and Rosický, 1994, Corollary 1.60), the induced arrow $n' \colon \mathrm{Colim}_I Q_I \to \mathrm{Colim}_I Y_I$ is monic. In $\mathrm{Sub}(Y)$ we have

$$\begin{aligned} Q &= \bigvee_{I \in \mathbb{I}} \mathrm{Im}\big(Q_I \xrightarrow{n_I} Y_I \xrightarrow{\kappa_I} Y\big) \\ &= \mathrm{Im}\Big(\big(\operatorname*{Colim}_{I \in \mathbb{I}} Q_I\big) \xrightarrow{n'} Y\Big) \quad \text{by Lemma B.7} \\ &= \operatorname*{Colim}_{I \in \mathbb{I}} Q_I \quad \text{the arrow } n' \text{ is already monic.} \end{aligned}$$

As we have $Y = \mathrm{Colim}_I Y_I$ in $\mathbb{C}$, this concludes the proof of the sublemma. $\qquad\square$

Let $(Q_I \xrightarrow{n_I} Y_I)_{I \in \mathbb{I}}$ be a filtered diagram in $\mathrm{Sub}(\mathbb{C})$; $(Q \xrightarrow{n} Y)$ be its colimit; and $g \colon (P \xrightarrow{m} X) \to (Q \xrightarrow{n} Y)$ be an arrow in $\mathrm{Sub}(\mathbb{C})$.

There exists an FP object $Z \in \mathbb{F}$ and a strong epimorphism $p \colon Z \twoheadrightarrow P$ by the definition of $\mathbb{F}_{\mathrm{Sub}(\mathbb{C})}$ and $\mathbb{F}_{\mathrm{Sub}(X)}$. The preservation of filtered colimits is shown as follows.

$$\operatorname*{Colim}_{I \in \mathbb{I}} \Big(\mathrm{Sub}(\mathbb{C})\big((P \xrightarrow{m} X), (Q_I \xrightarrow{n_I} Y_I)\big) \Big)$$

$$\stackrel{(*)}{\cong} \operatorname*{Colim}_{I \in \mathbb{I}} \{(f_I \colon X \to Y_I, g_I \colon Z \to Q_I) \mid f_I \circ m \circ p = n \circ g_I\}$$

$$= \operatorname*{Colim}_{I \in \mathbb{I}} \Big(\mathbb{C}(X, Y_I) \times_{\mathbb{C}(Z,Y_I)} \mathbb{C}(Z, Q_I) \Big)$$

$$\qquad \text{where } \mathbb{C}(X, Y_I) \times_{\mathbb{C}(Z,Y_I)} \mathbb{C}(Z, Q_I) \text{ is a suitable pullback}$$

$$\cong \Big(\operatorname*{Colim}_{I \in \mathbb{I}} \mathbb{C}(X, Y_I) \Big) \times_{\mathrm{Colim}_{I \in \mathbb{I}} \mathbb{C}(Z,Y_I)} \Big(\operatorname*{Colim}_{I \in \mathbb{I}} \mathbb{C}(Z, Q_I) \Big)$$

$$\qquad \textbf{Sets} \text{ is LFP and hence filtered colimits and finite limits commute}$$

$$\cong \mathbb{C}(X, \operatorname*{Colim}_{I \in \mathbb{I}} Y_I) \times_{\mathbb{C}(Z, \mathrm{Colim}_{I \in \mathbb{I}} Y_I)} \mathbb{C}(Z, \operatorname*{Colim}_{I \in \mathbb{I}} Q_I) \quad X, Z \text{ are FP in } \mathbb{C}$$

$$= \mathbb{C}(X, Y) \times_{\mathbb{C}(Z,Y)} \mathbb{C}(Z, Q) \quad \text{by Sublemma 6.4}$$

$$= \{(f \colon X \to Y, g \colon Z \to Q) \mid f \circ m \circ p = n \circ g\}$$

$$\stackrel{(\dagger)}{\cong} \mathrm{Sub}(\mathbb{C})\big((P \xrightarrow{m} X), (Q \xrightarrow{n} Y)\big)$$

where the bijection $(*)$ is by the diagonal fill-in

and (†) follows similarly.

Step d. The following observation on canonical diagrams with respect to $\mathbb{F} \subseteq \mathbb{C}$ and $\mathbb{F}_{\mathrm{Sub}(\mathbb{C})} \subseteq \mathrm{Sub}(\mathbb{C})$ is useful.

Sublemma 6.5. The forgetful functor $\begin{smallmatrix} \mathbb{F}_{\mathrm{Sub}(\mathbb{C})}/n \\ \downarrow \\ \mathbb{F}/Y \end{smallmatrix}$ is an opfibration.

Proof. Recall that $\begin{smallmatrix} \mathrm{Sub}(\mathbb{C}) \\ \downarrow \\ \mathbb{C} \end{smallmatrix}$ is a bifibration. Then $\begin{smallmatrix} \mathbb{F}_{\mathrm{Sub}(\mathbb{C})} \\ \downarrow \\ \mathbb{F} \end{smallmatrix}$ is an opfibration, because the full subcategory $\mathbb{F}_{\mathrm{Sub}(\mathbb{C})} \subseteq \mathrm{Sub}(\mathbb{C})$ is closed under opreindexing as depicted in the diagram

$$
\begin{array}{ccccc}
Z & \longrightarrow\!\!\!\!\!\rightarrow & P & \longrightarrow\!\!\!\!\!\rightarrow & \coprod_f P \\
& & \rotatebox{90}{$\subset$}\downarrow & & \rotatebox{90}{$\subset$}\downarrow \\
& & X & \xrightarrow{\ f\ } & X' \ .
\end{array}
$$

By the diagonal fill-in

$$
\begin{array}{ccccc}
P & \longrightarrow\!\!\!\!\!\rightarrow & \coprod_f P & \dashrightarrow & Q \\
\rotatebox{90}{$\subset$}\downarrow & & \rotatebox{90}{$\subset$}\downarrow & & \rotatebox{90}{$\subset$}\downarrow n \\
X & \xrightarrow{\ f\ } & X' & \longrightarrow & Y \ ,
\end{array}
$$

the opreindexing in $\begin{smallmatrix} \mathbb{F}_{\mathrm{Sub}(\mathbb{C})} \\ \downarrow \\ \mathbb{F} \end{smallmatrix}$ lifts to an opreindexing in $\begin{smallmatrix} \mathbb{F}_{\mathrm{Sub}(\mathbb{C})}/n \\ \downarrow \\ \mathbb{F}/Y \end{smallmatrix}$. $\qquad\square$

Step e. In the remainder of the proof we show that every object $(Q \overset{n}{\rightarrowtail} Y) \in \mathrm{Sub}(\mathbb{C})$ is a colimit of a filtered diagram in $\mathbb{F}_{\mathrm{Sub}(\mathbb{C})}$. Let us take a filtered diagram $(Y_I)_{I \in \mathbb{I}}$ such that $Y = \mathrm{Colim}_{I \in \mathbb{I}} Y_I$ in $\mathbb{C}$ and $Y_I \in \mathbb{F}$ (for each $I \in \mathbb{I}$).

We shall define a diagram $(Q_J \overset{n_J}{\rightarrowtail} Y_{qJ})_{J \in \mathbb{J}}$ in $\mathbb{F}_{\mathrm{Sub}(\mathbb{C})}$ and a functor $q \colon \mathbb{J} \to \mathbb{I}$. The (colimiting) cocone $(Y_I \overset{\kappa_I}{\to} Y)$ induces a functor $\mathbb{I} \to \mathbb{F}/Y$, and we obtain an opfibration $\begin{smallmatrix} \mathbb{J} \\ \downarrow q \\ \mathbb{I} \end{smallmatrix}$ by change-of-base (Jacobs, 1999, Lemma 1.5.1):

$$
\begin{array}{ccccc}
\mathbb{J} & \dashrightarrow & \mathbb{F}_{\mathrm{Sub}(\mathbb{C})}/n & \xrightarrow{\ \pi\ } & \mathbb{F}_{\mathrm{Sub}(\mathbb{C})} \\
q \downarrow & \lrcorner & \downarrow & & \downarrow \\
\mathbb{I} & \longrightarrow & \mathbb{F}/Y & \xrightarrow{\ \pi\ } & \mathbb{F} \ ,
\end{array}
$$

in particular, $\mathbb{J}_I \cong (\mathbb{F}_{\mathrm{Sub}(\mathbb{C})}/n)_{\kappa_I} \cong \mathbb{F}_{\mathrm{Sub}(Y_I)}/\kappa_I^* Q$:

$$
\begin{array}{ccc}
J & \quad & Q_J \longrightarrow Q \\
\Big| & & \rotatebox{90}{$\subset$}\ \quad\ \rotatebox{90}{$\subset$} \\
q\Big| & & n_J \downarrow \qquad \downarrow n \\
\downarrow & & Q_J \\
I \ , & & Y_I \xrightarrow{\ \kappa_I\ } Y \ ,
\end{array}
\qquad
\begin{array}{c}
Q_J \\
\rotatebox{90}{$\subset$}\downarrow \searrow \\
\kappa_I^* Q \longrightarrow Q \\
\rotatebox{90}{$\subset$}\downarrow \ \lrcorner \ \ \downarrow n \\
Y_I \xrightarrow{\ \kappa_I\ } Y \ .
\end{array}
$$

Therefore by Lemma B.11.2, we have a filtered colimit

$$\bigvee_{J \in \mathbb{J}_I} Q_J = \kappa_I^* Q \qquad \text{in Sub}(Y_I). \tag{22}$$

Moreover, the filtered colimit (22) in $\text{Sub}(X)$ forms a filtered colimit

$$\operatorname*{Colim}_{J \in \mathbb{J}_I}(Q_J \rightarrowtail Y_I) = (\kappa_I^* Q \rightarrowtail Y_I) \qquad \text{in Sub}(\mathbb{C}) \tag{23}$$

because $\text{Sub}(Y_I) \subseteq \text{Sub}(\mathbb{C})$ is closed under filtered colimits. Consequently,

$$
\begin{aligned}
\operatorname*{Colim}_{J \in \mathbb{J}}(Q_J \xrightarrow{n_J} Y_{qJ}) &\cong \operatorname*{Colim}_{I \in \mathbb{I}} \operatorname*{Colim}_{J \in \mathbb{J}_I}(Q_J \rightarrowtail Y_I) \quad \text{by Lemma C.12} \\
&= \operatorname*{Colim}_{I \in \mathbb{I}}(\kappa_I^* Q \rightarrowtail Y_I) \quad \text{by (23)} \\
&= \left(\left(\operatorname*{Colim}_{I \in \mathbb{I}} \kappa_I^* Q \right) \rightarrowtail \left(\operatorname*{Colim}_{I \in \mathbb{I}} Y_I \right) \right) \quad \text{by Sublemma 6.4} \\
&\cong (Q \xrightarrow{n} Y) \quad \text{by Lemma B.8}
\end{aligned}
$$

Step f. Recall that $\begin{smallmatrix} \mathbb{J} \\ \downarrow q \\ \mathbb{I} \end{smallmatrix}$ is an opfibration such that the base category $\mathbb{I}$ and each fiber $\mathbb{J}_I$ are filtered. It is straightforward to show the total category $\mathbb{J}$ is also filtered. $\qquad\square$

It follows from Lemma 6.3 and Corollary 6.2 that the internal logic of a topos that is LFP is finitely determined. Note that an (elementary) topos is necessarily a locally Cartesian closed category (LCCC) (see e.g. (Jacobs, 1999, Proposition 5.4.7)).

Corollary 6.6. Let $\mathbb{C}$ be LFP and at the same time a topos (or more generally an LCCC). Then the subobject fibration $\begin{smallmatrix} \text{Sub}(\mathbb{C}) \\ \downarrow \\ \mathbb{C} \end{smallmatrix}$ is finitely determined.

Proof. By the assumption that $\mathbb{C}$ is an LCCC, $\begin{smallmatrix} \text{Sub}(\mathbb{C}) \\ \downarrow \\ \mathbb{C} \end{smallmatrix}$ has products $\prod_f \vdash f^*$ between fibers (Jacobs, 1999, Corollary 1.9.9). We already proved that each fiber is a complete lattice. These sup's (i.e. colimits in a fiber) are preserved by reindexing f^* since the latter is a left adjoint $f^* \dashv \prod_f$. Namely, the fibration $\begin{smallmatrix} \text{Sub}(\mathbb{C}) \\ \downarrow \\ \mathbb{C} \end{smallmatrix}$ has fiberwise colimits. Opreindexings $\coprod$ satisfy the Beck-Chevalley condition since the products $\prod$ do (Jacobs, 1999, Lemma 1.9.7). Namely, the fibration $\begin{smallmatrix} \text{Sub}(\mathbb{C}) \\ \downarrow \\ \mathbb{C} \end{smallmatrix}$ has coproducts. $\qquad\square$

6.2. *Family Fibrations*

We turn to the family fibration $\begin{smallmatrix} \text{Fam}(\Omega) \\ \downarrow \\ \mathbf{Sets} \end{smallmatrix}$ over a poset Ω (see Appendix C).

Lemma 6.7. Let Ω be an *algebraic lattice*, i.e. a complete lattice in which each element is a join of compact elements. (Equivalently, Ω is LFP when thought of as a category.) Then the total category $\text{Fam}(\Omega)$ is LFP: the set

$$\mathbb{F}_{\text{Fam}(\Omega)} := \left\{\, f \colon X \to \Omega \mid X \text{ is finite, and for each } x \in X,\ f(x) \text{ is compact in } \Omega \,\right\} \tag{24}$$

consists of FP objects in $\mathrm{Fam}(\Omega)$; and every object $(Y, g) \in \mathrm{Fam}(\Omega)$ is a colimit of a filtered diagram in $\mathbb{F}_{\mathrm{Fam}(\Omega)}$. Noting that $f \in \mathbb{F}_{\mathrm{Fam}(\Omega)}$ is above a finite set X, by Lemma 6.1, $\begin{smallmatrix} \mathrm{Fam}(\Omega) \\ \downarrow \\ \mathbf{Sets} \end{smallmatrix}$ is finitely determined.

Proof. **Step a.** Let us first see that the fibration $\begin{smallmatrix} \mathrm{Fam}(\Omega) \\ \downarrow \\ \mathbf{Sets} \end{smallmatrix}$ has fiberwise limits and colimits and coproducts $\coprod$ between fibers. The former follows from Ω being a complete lattice; the latter is shown from (Jacobs, 1999, Lemma 1.9.5). In view of Lemma C.11, it follows that the total category $\mathrm{Fam}(\Omega)$ is cocomplete.

Step b. Before going on we prove the following.

Sublemma 6.8. Let $(Y_I)_{I \in \mathbb{I}}$ be a filtered diagram in **Sets**, and $\mathbb{J} = \int Y_{(\_)}$ be its category of elements, i.e. $\mathbb{J}$ has objects $\{(I, y') \mid I \in \mathbb{I}, y' \in Y_I\}$ and arrows $\mathbb{J}\big((I_1, y'_1), (I_2, y'_2)\big) = \{i \in \mathbb{I}(I_1, I_2) \mid Y_i(y'_1) = y'_2\}$. Let $(Y_I \xrightarrow{\kappa_I} Y)_I$ be a colimiting cocone. For each $y \in Y$, the following full subcategory of $\mathbb{J}$ is filtered:

$$\mathbb{J}_y = \{(I, y') \mid I \in \mathbb{I},\, y' \in Y_I,\, \kappa_I(y') = y\}.$$

Moreover, the category $\mathbb{J}$ is a disjoint sum of the full subcategories:

$$\mathbb{J} = \coprod_{y \in \mathrm{Colim}_{I \in \mathbb{I}} Y_I} \mathbb{J}_y \;. \tag{25}$$

Proof. By $Y = \{(I, y') \mid I \in \mathbb{I},\, y' \in Y_I\}/\sim$ where $(I_1, y'_1) \sim (I_2, y'_2)$ if and only if there exist $I \in \mathbb{I}$, $i_1 \colon I_1 \to I$, and $i_2 \colon I_2 \to I$ such that $Y_{i_1}(y'_1) = Y_{i_2}(y'_2)$ (in Y_I). $\qquad \square$

Step c. We prove that each $(X \xrightarrow{f} \Omega) \in \mathbb{F}_{\mathrm{Fam}(\Omega)}$ is FP in $\mathrm{Fam}(\Omega)$. Let $\big((Y_I \xrightarrow{g_I} \Omega) \xrightarrow{\kappa_I} (Y \xrightarrow{g} \Omega)\big)_{I \in \mathbb{I}}$ be a colimiting cocone in $\mathrm{Fam}(\Omega)$ over a filtered diagram $\mathbb{I}$.

By Lemma C.11 we obtain that $Y = \mathrm{Colim}_{I \in \mathbb{I}} Y_I$; and that

$$
\begin{aligned}
g(y) &= \Big(\bigvee_{I \in \mathbb{I}} \coprod\nolimits_{\kappa_I} g_I \Big)(y) = \bigvee_{I \in \mathbb{I}} \big((\coprod\nolimits_{\kappa_I} g_I)(y) \big) \\
&= \bigvee_{I \in \mathbb{I}} \Big(\bigvee_{y' \in \kappa_I^{-1}(\{y\})} g_I(y') \Big) = \bigvee_{(I, y') \in \mathbb{J}_y} g_I(y') \quad \text{for each } y \in Y.
\end{aligned}
\tag{26}
$$

The first equality is by Lemma C.11; the second is because the order in the fiber $\mathrm{Fam}(\Omega)_Y = \Omega^Y$ is pointwise; and the third is by the concrete description (Jacobs, 1999, Lemma 1.9.5) of $\coprod$ in $\begin{smallmatrix} \mathrm{Fam}(\Omega) \\ \downarrow \\ \mathbf{Sets} \end{smallmatrix}$.

Let $\mathbb{J}$ and $\mathbb{J}_y$ be categories as in Sublemma 6.8. Note that $\begin{smallmatrix} \mathbb{J} \\ \downarrow \\ \mathbb{I} \end{smallmatrix}$ is an opfibration with fibers $\mathbb{J}_I = Y_I$ that are discrete.

$$
\begin{aligned}
&\mathrm{Colim}_{I \in \mathbb{I}}\, \mathrm{Fam}(\Omega)\big((X \xrightarrow{f} \Omega), (Y_I \xrightarrow{g_I} \Omega) \big) \\
&\cong \mathrm{Colim}_{I \in \mathbb{I}} \prod_{x \in X} \coprod_{y' \in Y_I} \big(f(x) \leq_\Omega g_I(y') \big) \quad \text{by the definition of arrows in } \mathrm{Fam}(\Omega) \\
&\cong \prod_{x \in X} \mathrm{Colim}_{I \in \mathbb{I}} \coprod_{y' \in Y_I} \big(f(x) \leq_\Omega g_I(y') \big) \quad \mathbb{I} \text{ is filtered and } X \text{ is finite}
\end{aligned}
$$

$$\cong \prod_{x \in X} \operatorname*{Colim}_{(I,y') \in \mathbb{J}} \big(f(x) \leq_\Omega g_I(y')\big) \quad \text{by Lemma C.12}$$

$$\cong \prod_{x \in X} \coprod_{y \in \operatorname{Colim}_{I \in \mathbb{I}} Y_I} \operatorname*{Colim}_{(I,y') \in \mathbb{J}_y} \big(f(x) \leq_\Omega g_I(y')\big) \quad \text{by (25)}$$

$$\cong \prod_{x \in X} \coprod_{y \in \operatorname{Colim}_{I \in \mathbb{I}} Y_I} \Big(f(x) \leq_\Omega \bigvee_{(I,y') \in \mathbb{J}_y} g_I(y')\Big) \quad \mathbb{J}_y \text{ is filtered and } f(x) \in \Omega \text{ is compact}$$

$$\cong \operatorname{Fam}(\Omega)\big((X \xrightarrow{f} \Omega), (Y \xrightarrow{g} \Omega)\big) \quad \text{by (26)}$$

where $(\_ \leq_\Omega \_)$ denotes the homset $\Omega(\_,\_)$, which has at most one element, in the lattice Ω thought of as a category.

Step d. The collection $\mathbb{F}_{\operatorname{Fam}(\Omega)}$ is obviously small.

Step e. We are done if we prove that every object $P \in \operatorname{Fam}(\Omega)$ is a filtered colimit of its subobjects from $\mathbb{F}_{\operatorname{Fam}(\Omega)}$. This easily follows from the fact that the same is true in **Sets** (obvious) and in Ω (being an algebraic lattice). $\qquad\square$

Remark 6.9. It is worth mentioning that the fibrations $\begin{smallmatrix}\operatorname{Sub}(\mathbb{C})\\\downarrow\\\mathbb{C}\end{smallmatrix}$ (in Lemma 6.3) and $\begin{smallmatrix}\operatorname{Fam}(\Omega)\\\downarrow\\\textbf{Sets}\end{smallmatrix}$ (in Lemma 6.7) are *fiberwise algebraic lattices*, in the following sense: each fiber is an algebraic lattice; and each reindexing f^* between fibers is a "homomorphism" of algebraic lattices, which we define to be a monotone map that preserves arbitrary meets and directed joins. In other words, each reindexing f^* is a finitary right adjoint functor. We have essentially shown this fact in the proofs for these examples (Lemmas 6.3 and 6.7). Indeed, through the Gabriel-Ulmer duality (Gabriel and Ulmer, 1971), a finitary right adjoint functor $f^* \colon \mathbb{P}_Y \to \mathbb{P}_X$ between LFP categories corresponds to a functor $\coprod_f \colon (\mathbb{P}_X)_{\mathrm{FP}} \to (\mathbb{P}_Y)_{\mathrm{FP}}$ that preserves finite colimits, where $(\_)_{\mathrm{FP}}$ denotes the full subcategory consisting of all the FP objects. All this indicates that the preservation of compact elements under the coproduct $\coprod$ is crucial in our developments.

We shall, however, assume Cond. 2, the stronger condition that reindexing arrows f^* preserve arbitrary joins, too. This simplifies definitions and emphasizes duality as in Lemma 5.6.

6.3. *Presheaf Categories*

Presheaf categories are well-known examples of LFP categories. See (Adámek and Rosický, 1994).

Example 6.10 (Presheaf categories). Let $\mathbb{A}$ be small. The presheaf category $\textbf{Sets}^{\mathbb{A}}$ is LFP: the set $\mathbb{F}$ of finite colimits of representable presheaves $\mathbf{y}A$, where $\mathbf{y}A = \mathbb{A}(A, \_)$, satisfies the conditions of Definition 3.1. Indeed, any presheaf X is a filtered colimit of objects in $\mathbb{F}$ since X is a colimit (that is not necessarily filtered) of representable presheaves (Lemma 6.14).

For the subobject fibration of a presheaf category $\textbf{Sets}^{\mathbb{A}}$, Cond. 4 and $\overline{4}$ in Definition 3.3 (for $X \in \mathbb{F}$) reduce to the representable case $X = \mathbf{y}A$.

Lemma 6.11. The subobject fibration $\begin{smallmatrix} \mathrm{Sub}(\mathbf{Sets}^{\mathbb{A}}) \\ \downarrow \\ \mathbf{Sets}^{\mathbb{A}} \end{smallmatrix}$ is well-founded if and only if for all $A \in \mathbb{A}$ the poset $\mathrm{Sub}(\mathbf{y}A)$ has no strictly descending chain. The subobject fibration is co-well-founded if and only if for all $A \in \mathbb{A}$ the poset $\mathrm{Sub}(\mathbf{y}A)$ has no strictly ascending chain.

Sublemma 6.12. Let $(X_I)_I$ be a finite diagram in $\mathbf{Sets}^{\mathbb{A}}$. If for each I the poset $\mathrm{Sub}(X_I)$ has no strictly descending chain, then so does $\mathrm{Sub}(\mathrm{Colim}_I X_I)$. If for each I the poset $\mathrm{Sub}(X_I)$ has no strictly ascending chain, then so does $\mathrm{Sub}(\mathrm{Colim}_I X_I)$.

Proof. (Of Sublemma 6.12) We rely on a presentation of colimits by coproducts and coequalizers. In a topos (hence a regular category) $\mathbf{Sets}^{\mathbb{A}}$ coproducts are disjoint (see e.g. (Jacobs, 1999, Exercise 4.5.1)); thus we have an isomorphism of posets

$$\mathrm{Sub}(X_1 + \cdots + X_n) \cong \mathrm{Sub}(X_1) \times \cdots \times \mathrm{Sub}(X_n).$$

Let $X \rightrightarrows Y \xrightarrow{e} Z$ be a coequalizer in $\mathbf{Sets}^{\mathbb{A}}$. The correspondence $e^* \colon \mathrm{Sub}(Z) \to \mathrm{Sub}(Y)$ is easily seen to be injective. Indeed, assume $P \not\cong P'$ in $\mathrm{Sub}(Z)$; then $PA \not\cong P'A$ in $\mathbf{Sets}$ for some $A \in \mathbb{A}$, and since $e_A \colon YA \to ZA$ is surjective, we have

$$(e^*P)A = e_A^{-1}(PA) \not\cong e_A^{-1}(P'A) = (e^*P)A \ .$$

Therefore if $\mathrm{Sub}(Z)$ has a strictly descending or ascending chain, $\mathrm{Sub}(Y)$ has a strictly descending or ascending chain respectively. This concludes the proof of the sublemma. $\square$

Proof. (Of Lemma 6.11) By Example 6.10, Corollary 6.6, and Sublemma 6.12. $\square$

The previous lemma reduces the size problem of the fibration $\begin{smallmatrix} \mathrm{Sub}(\mathbf{Sets}^{\mathbb{A}}) \\ \downarrow \\ \mathbf{Sets}^{\mathbb{A}} \end{smallmatrix}$ to that of $\mathrm{Sub}(\mathbf{y}A)$. In calculating $\mathrm{Sub}(\mathbf{y}A)$, we will be using the following well-known characterization of presheaves as colimits of representables.

Definition 6.13. Let $\mathbb{A}$ be a small category and $P \colon \mathbb{A} \to \mathbf{Sets}$ be a functor. The *category of elements* of P, which is denoted by $\int P$, consists of objects that are pairs $\big(A \in \mathbb{A}, p \in PA\big)$ and arrows

$$\big(\textstyle\int P\big)\big((A,p),(B,q)\big) = \{f \colon A \to B \mid P(f)(p) = q\} \ .$$

Lemma 6.14. Any presheaf $P \in \mathbf{Sets}^{\mathbb{A}}$ is canonically isomorphic to the colimit of representable functors indexed by the category of elements: $P \cong \mathrm{Colim}_{(A,p)\in\int P} \mathbf{y}A$.

Proof. For each object $(A,p) \in \int P$, an arrow $\mathbf{y}A \to P$ is induced by $(\mathbf{y}A)B = \mathbb{A}(A,B) \ni g \mapsto P(g)(p) \in PB$. It is not difficult to see that these arrows $\mathbf{y}A \to P$ are natural in $(A,p) \in \int P$ and form a colimiting cocone. See e.g. (Adámek and Rosický, 1994, Proposition 1.45) for details. $\square$

Proposition 6.16 (presented later) will be our principal tool for calculating $\mathrm{Sub}(\mathbf{y}A)$. The proposition is inspired by the following cocompletion results (Lemma 6.15), which will not be themselves used in our subsequent technical developments.

Lemma 6.15. Let $\mathbb{A}$ be a small category.

1 The category $\mathbf{Sets}^{\mathbb{A}}$ of presheaves is a free cocompletion of the category $\mathbb{A}^{\mathrm{op}}$ (with the unit $\mathbf{y}\colon \mathbb{A}^{\mathrm{op}} \to \mathbf{Sets}^{\mathbb{A}}$), that is, for a functor $F\colon \mathbb{A}^{\mathrm{op}} \to \mathbb{C}$ to a cocomplete category $\mathbb{C}$ there uniquely (up to natural isomorphisms) exists a cocontinuous functor $G\colon \mathbf{Sets}^{\mathbb{A}} \to \mathbb{C}$ such that $F \cong G \circ \mathbf{y}$.

2 Let $P \in \mathbf{Sets}^{\mathbb{A}}$ be a presheaf. There exists an equivalence of categories $\mathbf{Sets}^{\mathbb{A}}/P \cong \mathbf{Sets}^{\int P}$. Hence, the slice category $\mathbf{Sets}^{\mathbb{A}}/P$ is a free cocompletion of the category $(\int P)^{\mathrm{op}}$.

3 Let $A \in \mathbb{A}$ be an object. The category $\mathbf{Sets}^{\mathbb{A}}/(\mathbf{y}A)$ is equivalent to the category $\mathbf{Sets}^{A/\mathbb{A}}$. Hence, the slice category $\mathbf{Sets}^{\mathbb{A}}/(\mathbf{y}A)$ is a free cocompletion of the category $(A/\mathbb{A})^{\mathrm{op}} = \mathbb{A}^{\mathrm{op}}/A$,

Proof. The item 1 is well-known: the functor G is given by $GP = \mathrm{Colim}_{(A,p)\in\int P} FA$. In particular, when we take $\mathbf{y}\colon \mathbb{A}^{\mathrm{op}} \to \mathbf{Sets}^{\mathbb{A}}$ as F, we obtain G that is naturally isomorphic to $\mathrm{id}\colon \mathbf{Sets}^{\mathbb{A}} \to \mathbf{Sets}^{\mathbb{A}}$. This generalizes Lemma 6.14.

The item 2—with a strong fibrational flavor, via the Grothendieck construction—is found e.g. in (Mac Lane and Moerdijk, 1992, Exercise III.8.(a)). The equivalence is given explicitly by

$$\mathbf{Sets}^{\mathbb{A}}/P \xrightarrow{\hspace{2.5cm}} \mathbf{Sets}^{\int P} \qquad\qquad \mathbf{Sets}^{\int P} \xrightarrow{\hspace{2.5cm}} \mathbf{Sets}^{\mathbb{A}}/P$$

$$(Q \xrightarrow{\alpha} P) \longmapsto \big[\, (A,p) \mapsto (\alpha_A)^{-1}(\{p\}) \,\big] \qquad R \longmapsto \big[\, A \mapsto \coprod_{p\in PA} R(A,p) \,\big]$$

where, in the last entry, we only presented a presheaf in $\mathbf{Sets}^{\mathbb{A}}$ (an arrow to P is given obviously by a projection).

The item 3 is obtained from the item 2 and the fact that $\int(\mathbf{y}A) = A/\mathbb{A}$ (an easy observation). $\qquad\square$

Proposition 6.16.

1 Let $\mathbb{A}$ be small. For any $A \in \mathbb{A}$, the subset

$$\{\mathrm{Im}(\mathbf{y}B \xrightarrow{\mathbf{y}f} \mathbf{y}A) \mid B \in \mathbb{A},\ f\colon A \to B\} \subseteq \mathrm{Sub}(\mathbf{y}A)$$

is dense as a full subcategory, that is, for any subpresheaf $Q \rightarrowtail \mathbf{y}A$ there canonically exists a family $(f_I\colon A \to B_I)_I$ such that $Q = \bigvee_I \mathrm{Im}(\mathbf{y}f_I)$. Here $\mathrm{Im}(\alpha)$ denotes the image of an arrow α.

2 Furthermore, assume that every arrow f with domain $A \in \mathbb{A}$ factors as $f = m \circ e$ with an epi e and a split mono m. Then (the image of) the canonical embedding $\mathrm{Quot}(A) \rightarrowtail \mathrm{Sub}(\mathbf{y}A)$ is dense. Here $\mathrm{Quot}(A)$ denotes the poset of quotient objects of A.

Proof. A detailed proof is given in Appendix D. $\qquad\square$

Corollary 6.17. If the following condition 1 holds for each $A \in \mathbb{A}$, then the fibration
$$\begin{array}{c} \mathrm{Sub}(\mathbf{Sets}^{\mathbb{A}}) \\ \downarrow \\ \mathbf{Sets}^{\mathbb{A}} \end{array}$$
is both well-founded and co-well-founded.

1 The subset $\{\mathrm{Im}(\mathbf{y}f) \mid B \in \mathbb{A},\ f\colon B \to A\} \subseteq \mathrm{Sub}(\mathbf{y}A)$ is finite.

Furthermore, for each $A \in \mathbb{A}$, the following condition 2 implies the condition 1 above.

2 Any arrow f with domain A factors as $f = m \circ e$ with an epi e and a split mono m, and moreover, $\mathrm{Quot}(A)$ is a finite set.

Proof. By Lemma 6.11, it is enough to show that for each $A \in \mathbb{A}$ the poset $\mathrm{Sub}(\mathbf{y}A)$ is finite.

Assume that $A \in \mathbb{A}$ satisfies the condition 1: the subset $\{\mathrm{Im}(\mathbf{y}f) \mid B \in \mathbb{A},\, f\colon B \to A\} \subseteq \mathrm{Sub}(\mathbf{y}A)$ is finite. By Proposition 6.16.1, we have $\mathrm{Sub}(\mathbf{y}A) = \{\bigvee_I \mathrm{Im}(\mathbf{y}f_I) \mid (B_I \in \mathbb{A},\, f_I\colon B \to A)_I\}$, which is also finite.

That the condition 2 implies 1 follows from Proposition 6.16.2. $\square$

To determine whether $\mathrm{Im}(\mathbf{y}f) = \mathrm{Im}(\mathbf{y}g)$ holds for arrows f and g with the same domain, the following lemma is useful.

Lemma 6.18. The inclusion relation $\leq$ on $\{\mathrm{Im}(\mathbf{y}f) \in \mathrm{Sub}(\mathbf{y}A) \mid B \in \mathbb{A},\, f\colon A \to B\}$ is the partial order induced by the preorder $\lesssim$ on $\{f \mid B \in \mathbb{A},\, f\colon A \to B\}$. The latter is defined by:

$$(f\colon A \to B) \lesssim (g\colon A \to C) \quad \text{if and only if} \quad f = h \circ g \text{ for some } h\colon C \to B.$$

Proof. Let $f\colon A \to B$, $g\colon A \to C$ be arrows in $\mathbb{A}$. We first observe that

$$
\begin{aligned}
\big(\mathrm{Im}(\mathbf{y}f)\big)D &= \mathrm{Im}\big((\mathbf{y}B)D \xrightarrow{(\mathbf{y}f)_D} (\mathbf{y}A)D\big) \\
&= \{(\mathbf{y}f)_D(k) \mid k \in (\mathbf{y}B)D\} \\
&= \{k \circ f\colon A \to D \mid k\colon B \to D\}
\end{aligned}
\tag{27}
$$

for $D \in \mathbb{A}$.

Assume that $\mathrm{Im}(\mathbf{y}f) \leq \mathrm{Im}(\mathbf{y}g)$ in $\mathrm{Sub}(\mathbf{y}A)$. In particular, it holds $\big(\mathrm{Im}(\mathbf{y}f)\big)B \subseteq \big(\mathrm{Im}(\mathbf{y}g)\big)B$ as subsets of $(\mathbf{y}A)B = \mathbb{A}(A,B)$. We have $f = \mathrm{id}_B \circ f \in \big(\mathrm{Im}(\mathbf{y}f)\big)B$ by (27), hence $f \in \big(\mathrm{Im}(\mathbf{y}g)\big)B$. Thus, there exists $h\colon C \to B$ such that $f = h \circ g$, which is the definition of $f \lesssim g$.

Conversely, assume that $f = h \circ g$ for some $h\colon C \to B$. For any $D \in \mathbb{A}$, we have

$$
\begin{aligned}
\big(\mathrm{Im}(\mathbf{y}f)\big)D &= \{k \circ h \circ g\colon A \to D \mid k\colon B \to D\} \quad \text{by (27)} \\
&\subseteq \{k' \circ g\colon A \to D \mid k'\colon C \to D\} \\
&= \big(\mathrm{Im}(\mathbf{y}g)\big)D \quad \text{by (27)}
\end{aligned}
$$

as subsets of $(\mathbf{y}A)D$. Therefore $\mathrm{Im}(\mathbf{y}f) \leq \mathrm{Im}(\mathbf{y}g)$. $\square$

7. Concrete Examples

Example 7.1 (Pred). The fibration $\begin{smallmatrix}\mathbf{Pred}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ for the conventional setting of classical logic is easily seen to be well-founded and co-well-founded. In particular, $\mathbf{Pred}_X \cong \mathcal{P}X$ is finite if X is FP (i.e. finite). Therefore to any finitary F and any predicate lifting φ, the results in §3 apply.

The (interpretations of the) formulas in $\mathsf{R}\nu$ (see Example 3.10) are examples of coinductive predicates in $\begin{smallmatrix}\mathbf{Pred}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$. Besides them, the study of coalgebraic modal logic has identified

many predicate liftings for many functors F (probabilistic systems, neighborhood frames, strategy frames, weighted systems, etc.; see e.g. (Cîrstea et al., 2011) and the references therein). These "modalities" all define coinductive predicates, to which the results in §3 may apply.

Example 7.2 (Rel). The fibration $\begin{smallmatrix}\mathbf{Rel}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ can be introduced from $\begin{smallmatrix}\mathbf{Pred}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ via change-of-base; concretely, an object of **Rel** is a pair (X, R) of a set X and a relation $R \subseteq X \times X$; an arrow $f \colon (X, R) \to (Y, S)$ is a function $f \colon X \to Y$ such that xRx' implies $f(x)Sf(x')$. See (Jacobs, 1999, p. 14).

This fibration, similarly to $\begin{smallmatrix}\mathbf{Pred}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$, is easily seen to be well-founded and co-well-founded; therefore to any finitary F the results in §3 apply. A predicate lifting φ along $\begin{smallmatrix}\mathbf{Rel}\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ is more commonly called a *relation lifting* (Hermida and Jacobs, 1998); by choosing suitable φ for given F (a "sufficiently comprehensive" one) like in (Hermida and Jacobs, 1998), a φ-invariant is precisely an F-bisimulation relation (in the coalgebraic sense), and the φ-coinductive predicate is F-bisimilarity. We expect that the ω-behavioral bound in Theorem 3.9 can be used to bound execution of bisimilarity checking algorithms by partition refinement (for many different functors F).

In the following example, one can think of Ω as a Heyting algebra, and then the underlying logic becomes constructive.

Example 7.3 (Fam(Ω)). Let Ω be an algebraic lattice that has no strictly descending $(\omega^{\mathrm{op}}\text{-})$chains. Then the family fibration $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ is well-founded (see Lemma 6.7). Therefore to any finitary F the results in §3 apply. It is not hard to interpret the language $\mathsf{R}\nu$ in this setting, by defining predicate liftings similar to (7). This gives examples of coinductive predicates in $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$.

Similarly, fibrations $\begin{smallmatrix}\mathrm{Fam}(\Omega^{\mathrm{op}})\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$ are co-well-founded for algebraic lattices Ω by Lemma 5.6, because the fibrations are fiberwise opposite of well-founded fibrations $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\\downarrow\\\mathbf{Sets}\end{smallmatrix}$.

7.1. *Presheaf Examples*

Let **F** be the category of natural numbers as finite sets (i.e. $n = \{0, 1, \ldots, n-1\}$) and all functions between them; $\mathbf{F}_+$ be its full subcategory of nonzero natural numbers; and **I** be the category of natural numbers and injective functions. Coalgebras in the presheaf categories $\mathbf{Sets}^{\mathbf{F}}$, $\mathbf{Sets}^{\mathbf{F}_+}$ and $\mathbf{Sets}^{\mathbf{I}}$ are commonly used for modeling processes in various name-passing calculi. For the π-calculus $\mathbf{Sets}^{\mathbf{I}}$ has been found appropriate (see e.g. (Stark, 1996; Fiore and Turi, 2001; Fiore and Staton, 2006)); while for the fusion calculus we do need non-injective functions in **F** or $\mathbf{F}_+$ (see (Miculan, 2008; Staton, 2011)).

Inspired by (Klin, 2007), we are interested in coinductive predicates for such processes. They are naturally modeled in the subobject fibration of a presheaf category. Here we

find a distinction: the subobject fibrations of both $\mathbf{Sets^F}$ and $\mathbf{Sets^{F+}}$ are well-founded and co-well-founded; but that of $\mathbf{Sets^I}$ is not well-founded (it is co-well-founded). In view of Lemma 6.11, the only condition to check is Cond. 4 or $\overline{4}$ for $X = \mathbf{y}A$.

Example 7.4 ($\mathrm{Sub}(\mathbf{Sets^F})$, $\mathrm{Sub}(\mathbf{Sets^{F+}})$). The subobject fibration $\begin{smallmatrix} \mathrm{Sub}(\mathbf{Sets^{F+}}) \\ \downarrow \\ \mathbf{Sets^{F+}} \end{smallmatrix}$ is well-founded and co-well-founded: this is shown by that the second condition of Corollary 6.17 holds for any $A \in \mathbf{F}_+$. An important fact here is that in $\mathbf{F}$ (or in $\mathbf{Sets}$) a mono with a nonempty domain splits, and thus every mono in $\mathbf{F}_+$ is a split mono.

The subobject fibration $\begin{smallmatrix} \mathrm{Sub}(\mathbf{Sets^F}) \\ \downarrow \\ \mathbf{Sets^F} \end{smallmatrix}$ is well-founded and co-well-founded, too. To show that $\mathrm{Sub}(\mathbf{y}0)$ is finite, we appeal directly to the first condition of Corollary 6.17: we observe by Lemma 6.18 that the set $\{\mathrm{Im}(\mathbf{y}f) \mid n \in \mathbf{F},\ f\colon 0 \to n\}$ is equal to the two-element set $\{\, \mathrm{Im}(\mathbf{y}(0 \overset{\mathrm{id}_0}{\to} 0)),\ \mathrm{Im}(\mathbf{y}(0 \overset{!}{\to} 1)) \,\}$ since $0 \overset{!}{\to} n$ and $0 \overset{!}{\to} m$ factor through each other, for each $n, m \geq 1$.

We turn to functors F and φ. In modeling processes of name-passing calculi as coalgebras in these categories, one typically uses endofunctors F that are constructed from the following building blocks. Let $\mathbf{N} \in \{\mathbf{F}, \mathbf{F}_+, \mathbf{I}\}$.

— Constant functors, binary sum $+$, binary product $\times$, and exponentials $(\_)^X$. These are much like for polynomial functors on $\mathbf{Sets}$. An important example of the first is the *name* presheaf $\mathcal{N} = \mathrm{Hom}(1, \_) \in \mathbf{Sets^N}$.
— The *abstraction* functor $\delta\colon \mathbf{Sets^N} \to \mathbf{Sets^N}$ given by $\delta X = X(\_ + 1)$.
— The free semilattice functor $\mathcal{P}_{\mathrm{f}}$ for finite branching. This captures Kuratowski finiteness and suitable in $\mathbf{Sets^I}$. See e.g. (Fiore and Turi, 2001; Staton, 2011).
— In $\mathbf{Sets^F}$ and $\mathbf{Sets^{F+}}$, another choice of a "finite powerset functor" $\widetilde{K}$ is more appropriate. See (Miculan, 2008); also (Staton, 2011, p. 4).

All such functors are known to be finitary (see e.g. (Miculan, 2008)).

Coinductive predicates in this setting can be introduced much like $\mathrm{R}\nu$ in Example 2.4 (note that $\mathbf{Sets^N}$ is a topos for $\mathbf{N} \in \{\mathbf{F}, \mathbf{F}_+, \mathbf{I}\}$), for properties like deadlock freedom. Such a language can be extended further through the modalities proposed in (Klin, 2007): they correspond to constructions specific to presheaves and include the modality $\langle \overline{a}(b) \rangle$ for a binding "input" operation. More examples will be worked out in our future paper.

Example 7.5 ($\mathrm{Sub}(\mathbf{Sets^\omega})$, $\mathrm{Sub}(\mathbf{Sets^I})$). Consider the presheaf category $\mathbf{Sets^\omega}$ over the ordinal ω as a poset. The fibration $\begin{smallmatrix} \mathrm{Sub}(\mathbf{Sets^\omega}) \\ \downarrow \\ \mathbf{Sets^\omega} \end{smallmatrix}$ is finitely determined but not well-founded. It fails to satisfy Cond. 4 in Definition 3.3: let $P_n\colon \omega \to \mathbf{Sets}$ be the family of presheaves defined by

$$P_n(m) := \quad \left(\ 0 \text{ if } m < n;\quad 1 \text{ if } n \leq m\ \right)$$

for each $n \in \omega$. Then $P_0 > P_1 > \cdots$ is a strictly descending chain in $\mathrm{Sub}(\mathbf{y}0)$. The same counterexample works for $\mathrm{Sub}(\mathbf{Sets^I})$.

In contrast, the fibrations $\begin{smallmatrix} \mathrm{Sub}(\mathbf{Sets^\omega}) \\ \downarrow \\ \mathbf{Sets^\omega} \end{smallmatrix}$ and $\begin{smallmatrix} \mathrm{Sub}(\mathbf{Sets^I}) \\ \downarrow \\ \mathbf{Sets^I} \end{smallmatrix}$ are co-well-founded, by Lemma 6.11 and the following lemma.

Lemma 7.6. For $\mathbb{A} \in \{\omega, \mathbf{I}\}$ and for any $n \in \mathbb{A}$, the poset $\mathrm{Sub}(\mathbf{y}n)$ is isomorphic to the opposite of the ordinal $\omega + 1 = \omega \cup \{\omega\}$. Hence $\mathrm{Sub}(\mathbf{y}n)$ has no strictly increasing chain.

Proof. Firstly we shall invoke Lemma 6.18. Let $f: n \to m$ be an arrow in $\mathbb{A}$. Note that the existence of the arrow f induces $m \geq n$ as natural numbers. For an arrow $g: n \to m'$ in $\mathbb{A}$, it is easy to see that f factors through g if and only if $m' \leq m$. In particular, arrows $f, f': n \rightrightarrows m$ factor through each other; therefore we may denote by $\mathrm{Im}(\mathbf{y}m)$ the image $\mathrm{Im}(\mathbf{y}m \xrightarrow{\mathbf{y}f} \mathbf{y}n) \in \mathrm{Sub}(\mathbf{y}n)$. Moreover by Lemma 6.18, we have

$$\mathrm{Im}(\mathbf{y}m) \leq \mathrm{Im}(\mathbf{y}m') \quad \text{if and only if} \quad m \geq m' \ .$$

Therefore there exists an isomorphism of posets

$$I : \ \omega^{\mathrm{op}} \xrightarrow{\cong} \{\mathrm{Im}(\mathbf{y}m) \mid m \geq n\} = \{\mathrm{Im}(\mathbf{y}m \xrightarrow{\mathbf{y}f} \mathbf{y}n) \in \mathrm{Sub}(\mathbf{y}n) \mid m \in \mathbb{A}, \ f: n \to m\}$$

defined by $I(k) = \mathrm{Im}\big(\mathbf{y}(n+k)\big)$.

We shall induce an isomorphism (the monotone function J below) between the "co-completion" of both-hand sides of the isomorphism I. Let $\mathrm{DSub}(\omega^{\mathrm{op}})$ be the poset of downward closed subsets of ω^{op} ordered by inclusion, and $\omega + 1$ be the ordinal. Let $h: (\omega + 1)^{\mathrm{op}} \to \mathrm{DSub}(\omega^{\mathrm{op}})$ be a function such that

$$h(k') = \quad \big(\downarrow k \ \text{if} \ k' = k \in \omega; \quad \emptyset \ \text{if} \ k' = \omega \ \big)$$

for $k' \in \omega + 1$, where $\downarrow k = \{k, k+1, \ldots\}$ is the downward closure of $\{k\} \subseteq \omega^{\mathrm{op}}$. It is easy to see that h becomes an isomorphism of posets. Since the poset $\mathrm{Sub}(\mathbf{y}n)$ is cocomplete, the isomorphism I induces the diagram

$$
\begin{array}{ccc}
\omega^{\mathrm{op}} & \xrightarrow[\cong]{\quad I \quad} & \{\mathrm{Im}(\mathbf{y}m \xrightarrow{\mathbf{y}f} \mathbf{y}n) \mid m \in \mathbb{A}, \ f: n \to m\} \\
{\scriptstyle \downarrow(\_)}\Big\downarrow & & \Big\downarrow{\scriptstyle i} \\
\mathrm{DSub}(\omega^{\mathrm{op}}) & \xrightarrow[\quad J \quad]{} & \mathrm{Sub}(\mathbf{y}n)
\end{array}
$$

in **Posets**, where i is the canonical inclusion, and $J(S) = \bigvee_{k \in S} I(k)$ is the sup of the images under the isomorphism. It is enough to show that J is also an isomorphism of posets.

On the one hand, the inclusion i is dense as a full subcategory by Proposition 6.16.1, that is, the monotone function J is surjective. On the other hand, the nullary sup $J(\emptyset) = 0$ in $\mathrm{Sub}(\mathbf{y}n)$ is strictly less than any other image $J(\downarrow k) = I(k) = \mathrm{Im}\big(\mathbf{y}(n+k)\big)$ for $k \in \omega$. Hence the monotone function $J: \mathrm{DSub}(\omega^{\mathrm{op}}) \to \mathrm{Sub}(\mathbf{y}n)$ is an embedding (i.e. a monotone injection that reflects the order) that extends the embedding $i \circ I: \omega^{\mathrm{op}} \to \mathrm{Sub}(\mathbf{y}n)$. Therefore, the monotone function J is a surjective embedding, that is, an isomorphism of posets. $\square$

In contrast to $\mathbf{Sets}^{\omega}$, the subobject fibration for $\mathbf{Sets}^{\omega^{\mathrm{op}}}$ is well-founded and co-well-founded by Corollary 6.17. Indeed, arrows $f: n \to m$ in ω^{op} has an (Epi, SplitMono)-factorization $n \twoheadrightarrow m \rightarrowtail m$, and $\mathrm{Quot}_{\omega^{\mathrm{op}}}(n) = \{n, n-1, \ldots, 0\}$ is a finite set.

Remark 7.7. Well-foundedness fails in $\mathrm{Sub}(\mathbf{Sets}^{\omega})$, $\mathrm{Sub}(\mathbf{Sets}^{\mathbf{I}})$, and in $\mathrm{Fam}(\Omega)$ for Ω that does have a strictly descending ω^{op}-chain. This means the logics modeled by the

fibrations are inherently "big." Still, extensions of our results in §3 are possible from finitary (i.e. ω-presentable) to the λ-presentable setting for bigger λ, so that they apply to the (current) nonexamples.

8. Conclusions and Future Work

We have investigated a mathematical theory of coinductive (and inductive) predicates over coinductive datatypes, formalized categorically using coalgebras and fibrations. Our technical results are about iterative constructions of coinductive predicates; they are stated also in abstract categorical terms, using the language of locally presentable categories.

In this paper we focused on purely coinductive predicates and purely inductive ones. However in system verification their combination is very commonly used. Such mixture of induction and coinduction is studied fibrationally in (Hensel and Jacobs, 1997), but over mixed inductive and coinductive data types, and not over a coalgebra. We believe a recent lattice-theoretic characterization of nested/alternating least and greatest fixed points (Hasuo et al., 2016) will provide a handle for suitably extending the current work.

Search for useful coinduction proof principles is an active research topic (see e.g. (Bonchi and Pous, 2013; Hur et al., 2013)). We are interested in the questions of whether these principles are sound in a general fibrational setting, and what novel proof principles a fibrational view can lead to. In fact the well-known technique of *coinduction up-to* has been formulated in fibrational terms (Bonchi et al., 2014) and revealed exciting new applications like nominal automata.

Coalgebraic modal logic is more and more often introduced based on a Stone-like duality (see e.g. (Klin, 2007)). Fibrational presentation of such dualities will combine the benefits of duality-based modal logics and the current results. We are also interested in the relationship to *coalgebraic infinite traces* (Jacobs, 2004; Cîrstea, 2011).

Kozen's *metric coinduction* (Kozen and Ruozzi, 2009) is a construction of coinductive predicates by the Banach fixed point theorem and is an alternative to the current paper's order-theoretic one. Its fibrational formulation is an interesting future topic.

Practical applications of our categorical behavioral bounds shall be pursued, too. Our results' precursor—the bounds for the final sequences in **Sets** (Worrell, 2005; Adámek, 2003)—have been used to bound execution of some algorithms e.g. for state minimization (Adámek et al., 2012; Ferrari et al., 2002; Ferrari et al., 2005). We aim at similar use. Finally, *games* are an extremely useful tool in fixed point logics (also in their coalgebraic generalization, see (Venema, 2006; Cîrstea and Sadrzadeh, 2008; Cîrstea et al., 2009); also (Kupke, 2007)). We plan to investigate the use of games in the current (even more general) fibrational setting.

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Appendix A. Theory of Coalgebra

Given a category $\mathbb{C}$ and an endofunctor $F\colon \mathbb{C} \to \mathbb{C}$, an *F-coalgebra* is a pair of $X \in \mathbb{C}$ and an arrow $c\colon X \to FX$ (we shall denote a coalgebra simply by $X \xrightarrow{c} FX$). The notion has turned out to be a useful categorical abstraction of state-based dynamic systems. In an *F*-coalgebra $X \xrightarrow{c} FX$, the *carrier object* $X \in \mathbb{C}$ is understood as a state space; the functor F specifies the *behavior type*; and the arrow c represents actual dynamics. In the most common setting of $\mathbb{C} = \mathbf{Sets}$, examples of functors F (and the corresponding behavior types) are:

— $A \times (\_)$ for A-stream automata;
— $\mathcal{P}(\mathsf{AP}) \times \mathcal{P}(\_)$ for Kripke models;
— $\mathcal{P}(\mathsf{AP}) \times \mathcal{P}_\omega(\_)$ for finitely branching Kripke models, with where $\mathcal{P}_\omega$ is the finite powerset functor;
— $\mathcal{P}(A \times \_)$ for labeled transition systems;
— $\mathcal{D}(A \times \_)$ for generative probabilistic systems;

and so on. See (Rutten, 2000; Jacobs, 2012) for detailed introduction.

In the theory of coalgebra as a categorical theory of (state-based dynamical) systems, the notion of *final coalgebra* plays a prominent role. A final *F*-coalgebra $Z \xrightarrow{\zeta} FZ$ is one such that, for any *F*-coalgebra $X \xrightarrow{c} FX$, there is a unique morphism of coalgebras from c to ζ.

$$
\begin{array}{ccc}
FX & \dashrightarrow^{F\bar{c}} & FZ \\
c\uparrow & & \text{final}\uparrow\zeta \\
X & \dashrightarrow_{\bar{c}} & Z
\end{array}
\tag{28}
$$

Its system-theoretic significance is that: 1 Z is often the collection of "all possible F-behaviors"; and 2 the induced arrow $\bar{c}$ assigns, to each state in X, its behavior. The "behaviors" here follow a black-box view on systems (it ignores internal states) and often captures the natural notion of "F-bisimilarity."

Therefore a question arises if a final *F*-coalgebra exists. The well-known Lambek lemma (that ζ is necessarily an iso) prohibits e.g. a final $\mathcal{P}$-coalgebra. What matters here is the *size* of F: when it is suitably bounded, a concrete construction of a final coalgebra is known. It obtains a final coalgebra via a *final F-sequence* (Here 1 is a final object in $\mathbb{C}$).

$$
1 \xleftarrow{\;!\;} F1 \longleftarrow \cdots \xleftarrow{F^{i-1}!} F^i 1 \xleftarrow{F^i!} \cdots
\tag{29}
$$

In particular, if F is *finitary* (a size restriction described later), a final coalgebra arises as a suitable quotient of the limit of the final sequence (4). This construction in **Sets** is worked out in (Worrell, 2005); it is further extended to locally presentable categories (those are categories suited for speaking of "size") with additional assumptions in (Adámek, 2003). The current paper's goal is to apply this construction also to coinductive predicates.

Appendix B. Locally Finitely Presentable Categories

The theory of coalgebra has been mainly developed in the base category $\mathbb{C} = \mathbf{Sets}$. Exceptions include the category of nominal sets or (pre)sheaf categories (e.g. (Fiore and

Staton, 2006; Fiore and Staton, 2009)) for name-passing calculi, and Kleisli categories (e.g. (Hasuo et al., 2007; Hasuo, 2010)) for trace semantics and simulation. The current paper follows (Adámek, 2003; Klin, 2007) and finds locally finitely presentable categories a convenient abstract setting. Here we follow (Adámek and Rosický, 1994) and list a minimal set of definitions and results on locally finitely presentable categories.

The following is a categorical formalization of "finiteness" of objects. Examples are finite sets (in **Sets**), and algebras presented by finitely many generators and finitely many equations (in suitable categories of algebras).

Definition B.1 (Finitely presentable object). An object $X \in \mathbb{C}$ is *finitely presentable (FP)* if the functor $\mathbb{C}(X, \_) \colon \mathbb{C} \to \mathbf{Sets}$ preserves filtered colimits.

Definition B.2 (Locally finitely presentable category). A category $\mathbb{C}$ is *locally finitely presentable (LFP)* if it is cocomplete and it has a (small) set $\mathbb{F}$ of FP objects such that every object is a filtered colimit of objects in $\mathbb{F}$.

Remark B.3. (Adámek and Rosický, 1994, Theorem 1.5) A filtered colimit can be rewritten as a directed colimit. Hence every object in an LFP category is a directed colimit of objects in $\mathbb{F}$. Some papers prefer to use directed colimits instead of filtered colimits in the definition of LFP categories, possibly because of simplicity in notations.

Lemma B.4. Let $\mathbb{C}$ be LFP, with a set $\mathbb{F}$ of FP objects as in Definition 3.1; and $X \in \mathbb{C}$. The *canonical diagram* for X with respect to $\mathbb{F}$

$$\mathbb{F}/X \xrightarrow{\pi} \mathbb{F} \hookrightarrow \mathbb{C} \tag{30}$$

is filtered, and X is its colimit. Here π is the projection from the comma category $\mathbb{F}/X$ of $\mathbb{F} \hookrightarrow \mathbb{C}$ and $1 \xrightarrow{X} \mathbb{C}$.

Proof. In case $\mathbb{F}$ contains all the FP objects up to isomorphisms, our claim would be (Adámek and Rosický, 1994, Proposition 1.22). In our current general case, almost the same proof yields our claim, except that we also have to show that the diagram $\mathbb{F}/X$ is filtered.

We shall show that any finite diagram $(Y_I \xrightarrow{f_I} X)_{I \in \mathbb{I}}$ in $\mathbb{F}/X$ has its cocone (in $\mathbb{F}/X$). Firstly we construct a cocone in $\mathbb{C}/X$. Let $(Y_I \xrightarrow{\kappa_I} Y)_I$ be a colimiting cocone in $\mathbb{C}$. The arrows $(f_I)_I$ induce $f \colon Y \to X$, which forms a colimiting cocone

$$\left((Y_I \xrightarrow{f_I} X) \xrightarrow{\kappa_I} (Y \xrightarrow{f} X)\right)_{I \in \mathbb{I}} \quad \text{in } \mathbb{C}/X$$

by Lemma B.5 below.

The finite colimit $Y = \mathrm{Colim}_I\, Y_I$ of FP objects is FP. Therefore Y is a split quotient of some object Y' in $\mathbb{F}$ (Adámek and Rosický, 1994, Remark 1.9). Then we obtain a cocone

$$\left((Y_I \xrightarrow{f_I} X) \xrightarrow{i \circ \kappa_I} (Y' \twoheadrightarrow Y \xrightarrow{f} X)\right)_{I \in \mathbb{I}} \quad \text{in } \mathbb{F}/X$$

where $i \colon Y \rightarrowtail Y'$ is a section of $Y' \twoheadrightarrow Y$. $\qquad \square$

Lemma B.5. Let $\mathbb{C}$ be a cocomplete category and $(X_I)_{I \in \mathbb{I}}$ be a diagram in $\mathbb{C}$. There

exists a canonical isomorphism

$$\left(\left(\operatorname*{Colim}_{I} X_I\right) \xrightarrow{f} Y\right) \;\cong\; \operatorname*{Colim}_{I}\left(X_I \xrightarrow{f_I} Y\right) \quad \text{in } \mathbb{C}/Y \tag{31}$$

for a cocone $(X_I \xrightarrow{f_I} Y)_I$ and the arrow $f\colon \operatorname{Colim}_I X_I \to Y$ that is induced by the universality of colimits. In other words, the colimiting cocone over $(X_I)_I$ in $\mathbb{C}$ induces a colimiting cocone over $(f_I)_I$ in $\mathbb{C}/Y$.

Proof. We have a cocone $(f_I \xrightarrow{\kappa_I} f)_I$ in $\mathbb{C}/Y$ induced by the colimiting cocone $(X_I \xrightarrow{\kappa_I} \operatorname{Colim}_I X_I)_I$ in $\mathbb{C}$, since the diagram below commutes and the arrows $f_I \xrightarrow{\kappa_I} f$ are natural in I.

$$
\begin{array}{ccc}
X_I & \xrightarrow{\quad \kappa_I \quad} & \operatorname{Colim}_I X_I \\
{\scriptstyle f_I}\downarrow & & \downarrow{\scriptstyle f} \\
Y & =\!\!=\!\!=\!\!=\!\!=\!\!= & Y
\end{array}
$$

To prove the isomorphism (31), we shall show that the induced cocone, say c, is colimiting.

Let c' be an arbitrary cocone $(f_I \xrightarrow{g_I} f')_I$ in $\mathbb{C}/Y$. An arrow $g\colon \operatorname{Colim}_I X_I \to X'$ in $\mathbb{C}$ forms an arrow $g\colon c \to c'$ of cocones if and only if for any $I \in \mathbb{I}$ the diagram

$$
\begin{array}{ccccc}
& & \overset{g_I}{\overbrace{\hspace{6cm}}} & & \\
X_I & \xrightarrow{\ \kappa_I\ } & \operatorname{Colim}_I X_I & \dashrightarrow[\;g\;]{} & X' \\
{\scriptstyle f_I}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\
Y & =\!\!=\!\!=\!\!= & Y & =\!\!=\!\!=\!\!= & Y
\end{array}
\tag{32}
$$

commutes. The universality of colimits in $\mathbb{C}$ shows that an arrow g satisfying $g \circ \kappa_I = g_I$ for any $I \in \mathbb{I}$ uniquely exists. Moreover, the arrow g with this condition satisfies $f' \circ g = f$ since $f' \circ g \circ \kappa_I = f' \circ g_I = f_I$. Hence there uniquely exists an arrow $g\colon c \to c'$ of cocones. $\qquad\square$

Lemma B.6. (Adámek and Rosický, 1994, Corollary 1.28 & Proposition 1.61) Let $\mathbb{C}$ be LFP.

1 $\mathbb{C}$ is complete.
2 $\mathbb{C}$ has $(\mathrm{StrongEpi}, \mathrm{Mono})$- and $(\mathrm{Epi}, \mathrm{StrongMono})$-factorization structures. $\qquad\square$

For each $X \in \mathbb{C}$, the $(\mathrm{StrongEpi}, \mathrm{Mono})$-factorization structure induces the *image functor* $\mathrm{Im}\colon \mathbb{C}/X \to \mathrm{Sub}(X)$, which is left adjoint to the forgetful functor $\mathrm{Sub}(X) \to \mathbb{C}/X$. An image of a colimit can be calculated as a sup of images.

Lemma B.7. Let $\mathbb{C}$ be LFP and $(X_I \xrightarrow{\kappa_I} X)$ be a colimiting cocone in $\mathbb{C}$. For an arbitrary cocone $(X_I \xrightarrow{f_I} Y)$, we have

$$\mathrm{Im}\, f = \bigvee_I \mathrm{Im}\, f_I \quad \text{in } \mathrm{Sub}(Y)$$

where $f\colon X \to Y$ is induced by the universality of colimits.

Proof. We have

$$\mathrm{Im}\left(\left(\operatorname*{Colim}_{I\in\mathbb{I}} X_I\right) \xrightarrow{f} Y\right) \;=\; \mathrm{Im}\left(\operatorname*{Colim}_{I\in\mathbb{I}}\left(X_I \xrightarrow{f_I} Y\right)\right) \;=\; \bigvee_{I\in\mathbb{I}} \mathrm{Im}\left(X_I \xrightarrow{f_I} Y\right).$$

The former equality is by Lemma B.5; the latter is because $\mathrm{Im}\colon \mathbb{C}/Y \to \mathrm{Sub}(Y)$ is a left adjoint functor. $\qquad\square$

Lemma B.8. Let $\mathbb{C}$ be an LFP category.

1. (Adámek and Rosický, 1994, Proposition 1.59) Filtered colimits commute with finite limits in $\mathbb{C}$. Precisely, the canonical arrow

$$\mathop{\mathrm{Colim}}_{I\in\mathbb{I}} \mathop{\mathrm{Lim}}_{J\in\mathbb{J}} X_{I,J} \to \mathop{\mathrm{Lim}}_{J\in\mathbb{J}} \mathop{\mathrm{Colim}}_{I\in\mathbb{I}} X_{I,J}$$

   is an isomorphism for a diagram $(X_{I,J})_{(I,J)\in\mathbb{I}\times\mathbb{J}}$ in $\mathbb{C}$ such that $\mathbb{I}$ is a filtered category and $\mathbb{J}$ is a finite category.
2. Filtered colimits in $\mathbb{C}$ are stable under pullbacks.

Proof. We prove the item 2. Let $X = \mathrm{Colim}_{I\in\mathbb{I}} X_I$ be a filtered colimit and $f\colon Y \to X$ be an arrow. Apply the item 1 to the diagram

$$\left(\begin{array}{c} Y \\ \downarrow f \\ X_I \xrightarrow[\kappa_I]{} X \end{array}\right)_{I\in\mathbb{I}}$$

where $\mathbb{J} = \left(\begin{array}{c} \cdot \\ \downarrow \\ \cdot \to \cdot \end{array}\right)$. This yields a pullback square

$$\begin{array}{ccc} \mathrm{Colim}_I\, \kappa_I^* Y & \longrightarrow & \mathrm{Colim}_I\, Y \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Colim}_I\, f} \\ \mathrm{Colim}_I\, X_I & \xrightarrow[\mathrm{Colim}_I\, \kappa_I]{} & \mathrm{Colim}_I\, X \end{array} \quad,\quad \text{that is} \quad \begin{array}{ccc} \mathrm{Colim}_I\, \kappa_I^* Y & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ X & =\!=\!= & X \end{array} \quad,$$

because we have $X = \mathrm{Colim}_{I\in\mathbb{I}} X$ and $Y = \mathrm{Colim}_{I\in\mathbb{I}} Y$ for a filtered category $\mathbb{I}$. Since a pullback of $f\colon Y \to X$ along $\mathrm{id}\colon X \to X$ is given by f itself, we obtain $\mathrm{Colim}_I\, \kappa_I^* Y = Y$, as required. $\qquad\square$

The following notion (which is already in Definition B.1) is about the "size" of functors. An intuition (when $\mathbb{C} = \mathbf{Sets}$) is: a functor F is finitary if F's action FX on an arbitrary set X is determined by its action FX' on all the finite subsets $X' \subseteq X$.

Definition B.9 (Finitary functor). A functor $F\colon \mathbb{C} \to \mathbb{D}$ is *finitary* if it preserves filtered colimits.

For an endofunctor $F\colon \mathbb{C} \to \mathbb{C}$, this notion of finitariness is commonly used to bound the "branching degree" of systems as F-coalgebras. For example, the finite powerset functor $\mathcal{P}_\omega$ is finitary; the (full) powerset functor $\mathcal{P}$ is not.

There are many LFP categories, among which are $\mathbf{Sets}$, the category $\mathbf{Posets}$ of posets and monotone functions, and categories of algebras with finitary operations. See (Adámek and Rosický, 1994) for more examples.

Example B.10 (Presheaf categories). Let $\mathbb{A}$ be a small category. The presheaf category $\mathbf{Sets}^{\mathbb{A}}$ is LFP: the set

$$\mathbb{F} := \{\text{finite colimits of representable presheaves } \mathbf{y}A\},$$

where $\mathbf{y}A = \mathbb{A}(A, \_)$, satisfies the conditions of Definition B.1.

Lemma B.11. Let $\mathbb{C}$ be LFP, with $\mathbb{F} \subseteq \mathbb{C}$ as in Definition 3.1; and $X \in \mathbb{C}$.

1 (Adámek and Rosický, 1994, Proposition 1.57) The slice category $\mathbb{C}/X$ is LFP, which is guaranteed by the set $\mathbb{F}_{\mathbb{C}/X} = \mathbb{F}/X$ of FP objects.
2 The poset $\mathrm{Sub}(X)$ of subobjects is LFP (i.e. it is an *algebraic lattice*, meaning a complete lattice in which each element is a join of compact elements), which is guaranteed by the set

$$\mathbb{F}_{\mathrm{Sub}(X)} = \{\mathrm{Im}\, f \mid f \in \mathbb{F}/X\}$$
$$= \{(P \rightarrowtail X) \mid \text{there exist an object } Z \in \mathbb{F} \text{ and a strong epi } Z \twoheadrightarrow P\}\ ,$$

of FP objects (i.e. compact elements) where $\mathrm{Im}\colon \mathbb{C}/X \to \mathrm{Sub}(X)$ denotes the image functor defined by the (StrongEpi, Mono)-factorization.

Proof. We shall prove the item 2. A proof that $\mathrm{Sub}(X)$ is LFP without explicit description of $\mathbb{F}_{\mathrm{Sub}(X)}$ is found e.g. in (Porst, 2011, Theorem 5).

The lattice $\mathrm{Sub}(X)$ is a reflective subcategory of $\mathbb{C}/X$ by the reflection $\mathrm{Im}\colon \mathbb{C}/X \to \mathrm{Sub}(X)$. Thus, $\mathrm{Sub}(X) \subseteq \mathbb{C}/X$ is closed under filtered colimits by (Adámek and Rosický, 1994, Corollary 1.60). Hence by (Adámek and Rosický, 1994, Theorem 1.39), $\mathrm{Sub}(X)$ is LFP, with FP objects $\{\mathrm{Im}\, f \mid f \in \mathbb{F}/X\}$. $\square$

Appendix C. Fibrations

We follow (Jacobs, 1999), although we focus on the simpler notion of poset fibration.

C.1. *Introduction (via Indexed Posets)*

This paper's interest is in coinductive predicates, hence in predicate logic. The most straightforward formalization of predicate is as a subset $P \subseteq X$ of a set (a "universe") X: an element $x \in X$ *satisfies* P if $x \in P$. Accompanying is the natural notion of entailment: P *entails* Q if $P \subseteq Q$. This way we obtain the poset $(2^X, \subseteq)$ of predicates over X.

However it is not on a single universe X that we consider predicates. For example, in a situation where there are two Kripke models $c = (X, \to, V_X)$, $d = (Y, \to, V_Y)$ and a "homomorphism" $f\colon X \to Y$, a natural question is if the interpretation of a formula $\nu u.\alpha$ is preserved by f. (It is; see Proposition 3.13). Here we are comparing the predicate $[\![\nu u.\alpha]\!]_c \subseteq X$ with the predicate $[\![\nu u.\alpha]\!]_d \subseteq Y$ *reindexed* via $f\colon X \to Y$. The latter is concretely described as the *inverse image*

$$f^{-1}\big([\![\nu u.\alpha]\!]_d\big)\ =\ \big\{x \in X \mid f(x) \in [\![\nu u.\alpha]\!]_d\big\}.$$

Therefore a reindexing structure is also relevant to predicate logic: a function $f\colon X \to Y$ induces reindexing $f^{-1}\colon 2^Y \to 2^X$. Additionally, the map f^{-1} is monotone.
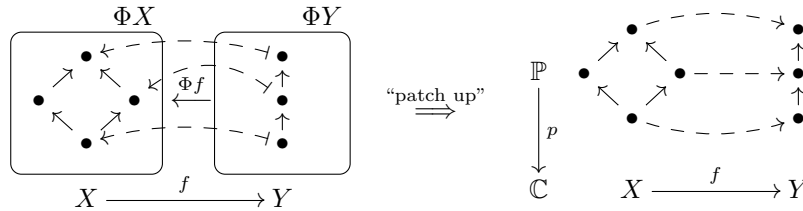
To summarize: 1) predicates on a universe X form a poset; 2) a function $f\colon X \to Y$ between universes induces a monotone reindexing function from the collection of predicates over X to that over Y. Such a situation is nicely described as a (contravariant)

functor

$$\Phi \,:\, \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{Posets} \ , \tag{33}$$

where **Posets** is the category of posets and monotone functions. The functor Φ assigns, to each "universe" $X \in \mathbb{C}$, the poset ΦX of predicates over X. Moreover, $f\colon X \to Y$ in $\mathbb{C}$ induces a reindexing map $\Phi f\colon \Phi Y \to \Phi X$. This functor Φ is a special case of an *indexed category* (Jacobs, 1999, §1.10).

In the current paper, however, we favor an equivalent presentation of such a structure by a *fibration*, since we find the latter to be more amenable to generalization of structures in ordinary category theory (such as limits). The equivalence between index categories and fibrations is well-known; here we sketch the *Grothendieck construction* from the former to the latter. Its idea is to "patch up" the posets $(\Phi X)_{X \in \mathbb{C}}$ and form a big category $\mathbb{P}$, as in the following figure.

On the right we add some arrows (denoted by $-\dashrightarrow$) so that we have an arrow $(\Phi f)(Q) \to Q$ in $\mathbb{P}$ for each $Q \in \Phi Y$. (On the left the correspondence $\vdash\!-\!\dashrightarrow$ depicts the action of the map Φf.) The above diagram in $\mathbb{P}$ should be understood as a Hasse diagram: those arrows which arise from composition are not depicted.

Formally:

Definition C.1 (The Grothendieck construction). Given $\Phi\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Posets}$, we define the category $\mathbb{P}_\Phi$ by

— its object is a pair (X, P) of an object $X \in \mathbb{C}$ and an element P of the poset ΦX; and

— its arrow $f\colon (X, P) \to (Y, Q)$ is an arrow $f\colon X \to Y$ in $\mathbb{C}$ such that

$$P \le (\Phi f)(Q).$$

Here $\le$ refers to the order of ΦX.

Thus arises a category $\mathbb{P} = \mathbb{P}_\Phi$ that incorporates: the order structure of each of the posets $(\Phi X)_{X \in \mathbb{C}}$; and the reindexing structure by $(\Phi f)_{f\colon \mathbb{C}\text{-arrow}}$. For fixed $X \in \mathbb{C}$, the objects of the form (X, P) and the arrows id_X between them form a subcategory of $\mathbb{P}$. This is denoted by $\mathbb{P}_X$ and called the *fiber* over X. It is obvious that $\mathbb{P}_X$ is a poset that is isomorphic to ΦX.

Moreover, there is a canonical projection functor $p\colon \mathbb{P} \to \mathbb{C}$ that carries (X, P) to X.

C.2. *Formal Definition of (Poset) Fibration*

We axiomatize those structures which arise in the way described above.

Definition C.2 ((Poset) fibration). A *(poset) fibration* $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ consists of two categories $\mathbb{P}, \mathbb{C}$ and a functor $p \colon \mathbb{P} \to \mathbb{C}$, that satisfy the following properties.

—— Each fiber $\mathbb{P}_X$ is a poset. Here the *fiber* $\mathbb{P}_X$ for $X \in \mathbb{C}$ is the subcategory of $\mathbb{P}$ consisting of objects $P \in \mathbb{P}$ such that $pP = X$ and arrows $f \colon P \to Q$ such that $pf = \mathrm{id}_X$ (such arrows are said to be *vertical*).

—— Given $f \colon X \to Y$ in $\mathbb{C}$ and $Q \in \mathbb{P}_Y$, there is an object $f^*Q \in \mathbb{P}_X$ and a $\mathbb{P}$-arrow $\overline{f}Q \colon f^*Q \to Q$ with the following universal property. For any $P \in \mathbb{P}_X$ and $g \colon P \to Q$ in $\mathbb{P}$, if $pg = f$ then g factors through $\overline{f}(Q)$ uniquely via a vertical arrow. That is, there exists a unique g' such that $g = \overline{f}(Q) \circ g'$ and $pg' = \mathrm{id}_X$.

$$
\begin{array}{ccc}
\mathbb{P} & Q & f^*Q \xrightarrow{\overline{f}(Q)} Q \\
\Big\downarrow p & \Longrightarrow & {\scriptstyle g'}\Big\uparrow \nearrow {\scriptstyle g} \\
\mathbb{C} \quad X \xrightarrow{\ f\ } Y & & X \xrightarrow{\ f\ } Y
\end{array}
$$

—— The correspondences $(\_)^*$ and $\overline{(\_)}$ are functorial:

$$
\begin{aligned}
\mathrm{id}_Y^* Q &= Q \ , & (g \circ f)^*(Q) &= f^*(g^*Q), \\
\overline{\mathrm{id}_Y}(Q) &= \mathrm{id}_Q \ , & \overline{g \circ f}(Q) &= \overline{g}Q \circ \overline{f}(g^*Q).
\end{aligned}
$$

The last equality can be depicted as follows.

$$
\begin{array}{cc}
\mathbb{P} & f^*(g^*Q) \xrightarrow{\overline{f}(g^*Q)} g^*Q \xrightarrow{\overline{g}Q} Q \\
\Big\downarrow p & \ \| \\
 & (g \circ f)^*Q \xrightarrow[\overline{g \circ f}(Q)]{} \\
\mathbb{C} & X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z
\end{array}
$$

The category $\mathbb{P}$ is called the *total category* of the fibration; $\mathbb{C}$ is the *base category*. The arrow $\overline{f}Q \colon f^*Q \to Q$ is called the *Cartesian lifting* of f and Q. An arrow in $\mathbb{P}$ is *Cartesian* (or *reindexing*) if it coincides with $\overline{f}Q$ for some f and Q.

In the case where $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ is induced by an indexed category $\Phi \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Posets}$ via Definition C.1, a Cartesian lifting is obviously given by $f^*(Q) = (\Phi f)(Q)$.

In the current paper we focus on poset fibrations (which we shall simply call *fibrations*). In a (general) fibration a fiber $\mathbb{P}_X$ is not just a poset but a category, and this elicits a lot of technical subtleties. Nevertheless, it should not be hard to generalize the current paper's results to general, not necessarily poset, fibrations (especially to the split ones). We shall often denote a vertical arrow in $\mathbb{P}$ (i.e. an arrow inside a fiber) by $\leq$.

The dual notion of a fibration is an opfibration.

Definition C.3. An *opfibration* $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ consists of two categories $\mathbb{P}, \mathbb{C}$ and a functor $p \colon \mathbb{P} \to \mathbb{C}$ such that $\begin{smallmatrix}\mathbb{P}^{\mathrm{op}}\\\downarrow p^{\mathrm{op}}\\\mathbb{C}^{\mathrm{op}}\end{smallmatrix}$ is a fibration. Concretely, in an opfibration $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$, for an arrow $f \colon X \to Y$ in $\mathbb{C}$ and $P \in \mathbb{P}_X$, there is an object $\coprod_f P \in \mathbb{P}_Y$ and a $\mathbb{P}$-arrow $P \to \coprod_f P$ satisfying an

appropriate universal property. This arrow $P \to \coprod_f P$ in $\mathbb{P}$ is said to be *opcartesian* (or *opreindexing*).

A *bifibration* $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$ is a fibration as well as an opfibration.

Note that we do not assume the Beck-Chevalley condition for a bifibration. A fibration with *coproducts* $\coprod_f$ *between fibers*—introduced later in Definition C.10—carries a canonical opfibration structure, too.

Lemma C.4. (Jacobs, 1999, Lemma 9.1.2). A fibration $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$ is a bifibration if and only if for any arrow $f \colon X \to Y$ in $\mathbb{C}$ the reindexing functor $f^* \colon \mathbb{P}_Y \to \mathbb{P}_X$ has a left adjoint $\coprod_f \dashv f^*$. $\square$

C.3. *Examples*

Example C.5 (Subobject fibration). Let $\mathbb{C}$ be a (well-powered) category with finite limits. The category $\mathrm{Sub}(\mathbb{C})$ is defined by: its object is a pair (P, X) of $X \in \mathbb{C}$ and its subobject $P \rightarrowtail X$ (we write $(P \rightarrowtail X) \in \mathrm{Sub}(\mathbb{C})$); and its arrow $(P \rightarrowtail X) \xrightarrow{f} (V \rightarrowtail Y)$ is a $\mathbb{C}$-arrow $f \colon X \to Y$ that restricts to $P \to Q$. That is, given an arrow $f \colon X \to Y$ in $\mathbb{C}$,

$$
\begin{array}{c} f \text{ is an arrow in } \mathrm{Sub}(\mathbb{C}) \\ (P \overset{m}{\rightarrowtail} X) \xrightarrow{f} (Q \overset{n}{\rightarrowtail} Y) \end{array}
\iff
\exists f' \text{ such that }
\begin{array}{c}
P \overset{f'}{\dashrightarrow} Q \\
m \downarrow \quad \quad \downarrow n \\
X \xrightarrow{f} Y
\end{array} . \tag{34}
$$

The projection $(P \rightarrowtail X) \mapsto X$ defines a functor; thus arises the *subobject fibration* $\begin{smallmatrix} \mathrm{Sub}(\mathbb{C}) \\ \downarrow \\ \mathbb{C} \end{smallmatrix}$ of $\mathbb{C}$. In particular, given $X \xrightarrow{f} Y$ in $\mathbb{C}$ and $(Q \rightarrowtail Y) \in \mathrm{Sub}(Y)$, the Cartesian lifting $f^* Q$ is defined by a pullback.

$$
\begin{array}{c}
f^* Q \overset{\overline{f}Q}{\dashrightarrow} Q \\
m \downarrow \quad \lrcorner \quad \downarrow n \\
X \xrightarrow{f} Y
\end{array}
$$

A special case is the following most straightforward modeling of predicate logic. It arises from the contravariant powerset functor $2^{(-)} \colon \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Posets}$ via Definition C.1.

Example C.6 ($\begin{smallmatrix} \mathbf{Pred} \\ \downarrow \\ \mathbf{Sets} \end{smallmatrix}$). The subobject fibration $\begin{smallmatrix} \mathrm{Sub}(\mathbf{Sets}) \\ \downarrow \\ \mathbf{Sets} \end{smallmatrix}$ of $\mathbf{Sets}$ is denoted by $\begin{smallmatrix} \mathbf{Pred} \\ \downarrow \\ \mathbf{Sets} \end{smallmatrix}$. An object of its total category is often denoted by $(U \subseteq X)$. Reindexing is given by inverse images.

More concretely, in the category $\mathbf{Pred}$, an object is a pair (P, X) of a set X and its subset $P \subseteq X$; an arrow $(P \subseteq X) \xrightarrow{f} (Q \subseteq Y)$ is a function $X \xrightarrow{f} Y$ that restricts to $P \to Q$ (i.e. $P \subseteq f^{-1} Q$).

Example C.7 (Rel). The fibration $\begin{smallmatrix} \mathbf{Rel} \\ \downarrow \\ \mathbf{Sets} \end{smallmatrix}$ can be introduced from $\begin{smallmatrix} \mathbf{Pred} \\ \downarrow \\ \mathbf{Sets} \end{smallmatrix}$ via the following change-of-base.

$$
\begin{array}{ccc}
\mathbf{Rel} & \longrightarrow & \mathbf{Pred} \\
\downarrow & \lrcorner & \downarrow \\
\mathbf{Sets} & \xrightarrow{X \mapsto X \times X} & \mathbf{Sets}
\end{array}
$$

Concretely, an object of **Rel** is a pair (X, R) of a set X and a relation $R \subseteq X \times X$; an arrow $f : (X, R) \to (Y, S)$ is a function $f : X \to Y$ such that xRx' implies $f(x)Sf(x')$. See (Jacobs, 1999, p. 14).

Example C.8 (Family fibration). The *family fibration* $\begin{smallmatrix} \mathrm{Fam}(\Omega) \\ \downarrow \\ \mathbf{Sets} \end{smallmatrix}$ over a poset Ω is introduced as follows. An object in the fiber $\mathrm{Fam}(\Omega)_X$ is a function $f : X \to \Omega$; and an arrow $(X \xrightarrow{f} \Omega) \xrightarrow{k} (Y \xrightarrow{g} \Omega)$ in the total category $\mathrm{Fam}(\Omega)$ is a function $k : X \to Y$ such that $f(x) \le g(k(x))$ for each $x \in X$. See e.g. (Jacobs, 1999, Definition 1.2.1) for more details.

C.4. *Structures in a Fibration*

In a fibration $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$, a $\mathbb{C}$-arrow $X \xrightarrow{f} Y$ induces a correspondence $\mathbb{P}_Y \xrightarrow{f^*} \mathbb{P}_X$ via reindexing. This is easily seen to be a monotone map (i.e. a functor between posets as categories).

Definition C.9 (Fiberwise (co)limits). A fibration $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$ is said to have *fiberwise limits* if:

— each fiber $\mathbb{P}_X$ has, as a category, all limits (meaning it has arbitrary inf's $\bigwedge$); and

— for each $\mathbb{C}$-arrow $X \xrightarrow{f} Y$, the reindexing functor $\mathbb{P}_Y \xrightarrow{f^*} \mathbb{P}_X$ preserves these limits.

In this case each fiber $\mathbb{P}_X$ has a final object (denoted by $\top_X$).

Similarly, a fibration has *fiberwise colimits* if each fiber has them and they are preserved by reindexing.

The following notions must be distinguished from "fiberwise (co)products."

Definition C.10 ((Co)products between fibers). A fibration $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$ is said to have *products (between fibers)* if

— each reindexing functor $f^* : \mathbb{P}_Y \to \mathbb{P}_X$ has a right adjoint $f^* \dashv \prod_f$; and

— the functors $(\prod_f)_f$ satisfy the so-called *Beck-Chevalley condition*. See (Jacobs, 1999, §1.9).

Similarly, a fibration has *coproducts (between fibers)* if each reindexing has a left adjoint $\coprod_f$ and they satisfy the Beck-Chevalley condition.

The prototype example $\begin{smallmatrix} \mathbf{Pred} \\ \downarrow \\ \mathbf{Sets} \end{smallmatrix}$ has fiberwise (co)limits: each fiber is a complete lattice; and $\bigwedge$ and $\bigvee$ are preserved by inverse images. It has products $\prod$ and coproducts $\coprod$ between fibers, too: specifically $\coprod_f$ is given by the *direct image* of the function f. See (Jacobs, 1999, §1.9).

Throughout the paper we rely on the following result. It extends Lemma 3.6. Note that colimits are preserved by opreindexings in a bifibration.

Lemma C.11. Let $\begin{smallmatrix} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{smallmatrix}$ be a fibration. Assume that $\mathbb{C}$ is complete; then the following are equivalent.

1   The fibration p has fiberwise limits.

2   The total category $\mathbb{P}$ is complete and $p\colon \mathbb{P} \to \mathbb{C}$ preserves limits.

If this is the case, a limit of a small diagram $(P_I)_{I\in\mathbb{I}}$ in $\mathbb{P}$ can be given by

$$\bigwedge_{I\in\mathbb{I}}(\pi_I^* P_I) \qquad \text{over } \mathrm{Lim}_{I\in\mathbb{I}} X_I.$$

Here $X_I := pP_I$; $(\mathrm{Lim}_{I\in\mathbb{I}} X_I \overset{\pi_I}{\to} X_I)_{I\in\mathbb{I}}$ is a limiting cone in $\mathbb{C}$; and $\bigwedge_{I\in\mathbb{I}}$ denotes the limit computed in the fiber $\mathbb{P}_{\mathrm{Lim}_I X_I}$.

(Sort of) dually, let $\overset{\mathbb{P}}{\underset{\mathbb{C}}{\downarrow p}}$ be a bifibration (such as a fibration with coproducts $\coprod$ between fibers, see Lemma C.4). Assume that $\mathbb{C}$ is cocomplete; then the following are equivalent.

1   Any fiber $\mathbb{P}_X$ has colimits.

2   The total category $\mathbb{P}$ is cocomplete and $p\colon \mathbb{P} \to \mathbb{C}$ preserves colimits.

In this case a colimit of a small diagram $(P_I)_{I\in\mathbb{I}}$ in $\mathbb{P}$ can be given by

$$\bigvee_{I\in\mathbb{I}}(\coprod_{\kappa_I} P_I) \quad \text{over } \mathrm{Colim}_I X_I,$$

where $X_I := pP_I$ and $(X_I \overset{\kappa_I}{\to} \mathrm{Colim}_I X_I)_{I\in\mathbb{I}}$ is a colimiting cocone in $\mathbb{C}$. $\qquad\square$

In contrast to the above results that are on limits *in* the total category $\mathbb{P}$ of a fibration, Lemma C.12 allows one to compute limits over $\mathbb{P}$ *as a diagram*. It is well-known that an iterated limit $\mathrm{Lim}_{X\in\mathbb{C}} \mathrm{Lim}_{Y\in\mathbb{D}} F(X,Y)$ is isomorphic to the limit $\mathrm{Lim}_{(X,Y)\in\mathbb{C}\times\mathbb{D}} F(X,Y)$. This kind of isomorphism exists even if the category $\mathbb{D}$ "depends" on $X \in \mathbb{C}$ in the following sense. (Note that $\overset{\mathbb{C}\times\mathbb{D}}{\underset{\mathbb{C}}{\downarrow \pi_1}}$ is at the same time a fibration and an opfibration.)

Lemma C.12. Let $\overset{\mathbb{P}}{\underset{\mathbb{C}}{\downarrow p}}$ be a fibration and $F\colon \mathbb{P} \to \mathbb{E}$ be a functor. If $\mathrm{Lim}_{P\in\mathbb{P}_X} FP$ exists for each $X \in \mathbb{C}$, then we have a canonical isomorphism

$$\underset{X\in\mathbb{C}}{\mathrm{Lim}}\,\underset{P\in\mathbb{P}_X}{\mathrm{Lim}}\, FP \;\cong\; \underset{P\in\mathbb{P}}{\mathrm{Lim}}\, FP \;,$$

where one side exists if the other side does.

Dually, let $\overset{\mathbb{P}}{\underset{\mathbb{C}}{\downarrow p}}$ be an opfibration and $F\colon \mathbb{P} \to \mathbb{E}$ be a functor. If $\mathrm{Colim}_{P\in\mathbb{P}_X} FP$ exists for each $X \in \mathbb{C}$, then we have a canonical isomorphism

$$\underset{X\in\mathbb{C}}{\mathrm{Colim}}\,\underset{P\in\mathbb{P}_X}{\mathrm{Colim}}\, FP \;\cong\; \underset{P\in\mathbb{P}}{\mathrm{Colim}}\, FP \;,$$

where one side exists if the other side does.

Proof. Let $\overset{\mathbb{P}}{\underset{\mathbb{C}}{\downarrow p}}$ be a fibration. For $f \in \mathbb{C}(X,Y)$, a canonical arrow

$$\underset{P\in\mathbb{P}_X}{\mathrm{Lim}}\, FP \to \underset{Q\in\mathbb{P}_Y}{\mathrm{Lim}}\, FQ \tag{35}$$

is obtained via the universality of limits as below:

$$\begin{array}{ccc}
\mathrm{Lim}_{P\in\mathbb{P}_X} FP & \dashrightarrow & \mathrm{Lim}_{Q\in\mathbb{P}_Y} FQ \\
{\scriptstyle \pi_{f^*Q}}\downarrow & & \downarrow{\scriptstyle \pi_Q} \\
F(f^*Q) & \xrightarrow[F(\overline{f}Q)]{} & FQ \;.
\end{array} \tag{36}$$

Indeed, we have

$$
\begin{array}{ccc}
F(f^*Q) & \xrightarrow{\ F(\overline{f}Q)\ } & FQ \\
{\scriptstyle F(f^*g)}\downarrow & & \downarrow{\scriptstyle Fg} \\
F(f^*Q') & \xrightarrow[\ F(\overline{f}Q')\ ]{} & FQ'
\end{array}
$$

for $g \in \mathbb{P}_Y(Q, Q')$ because of the naturality of Cartesian liftings.

For each $E \in \mathbb{E}$ we have

$$\mathbb{E}(E, \operatorname*{Lim}_{X \in \mathbb{C}} \operatorname*{Lim}_{P \in \mathbb{P}_X} FP)$$

$$\cong \operatorname*{Lim}_{X \in \mathbb{C}} \operatorname*{Lim}_{P \in \mathbb{P}_X} \mathbb{E}(E, FP)$$

$$\cong \operatorname*{Lim}_{X \in \mathbb{C}} \big\{ \, \big(h_P \in \mathbb{E}(E, FP)\big)_{P \in \mathbb{P}_X} \mid Fg \circ h_P = h_{P'} \text{ for any } g \in \mathbb{P}_X(P, P') \, \big\}$$

$$\cong \big\{ \, \big(\big(h_P \in \mathbb{E}(E, FP)\big)_{P \in \mathbb{P}_X}\big)_{X \in \mathbb{C}}$$
$$\mid Fg \circ h_P = h_{P'} \text{ for any } g \in \mathbb{P}_X(P, P');$$
$$F(\overline{f}Q) \circ h_{f^*Q} = h_Q \text{ for any } f \in \mathbb{C}(X, Y) \text{ and } Q \in \mathbb{P}_Y \big\}$$

the postcomposition of the arrow (35) maps $(h_P)_P$ to $(F(\overline{f}Q) \circ h_{f^*Q})_Q$ by (36)

$$= \{ (h_P \in \mathbb{E}(E, FP))_{P \in \mathbb{P}} \mid Ff \circ h_P = h_Q \text{ for any } f \in \mathbb{P}(P, Q)\}$$

by the factorization $P \xrightarrow{g} f^*Q \xrightarrow{\overline{f}} Q$ of $f \colon P \to Q$ into vertical g and Cartesian $\overline{f}$

$$\cong \operatorname*{Lim}_{P \in \mathbb{P}} \mathbb{E}(E, FP)$$

$$= \mathbb{E}(E, \operatorname*{Lim}_{P \in \mathbb{P}} FP).$$

Applying the Yoneda Lemma yields the claim. $\qquad\square$

C.5. *Fiberwise Opposite*

Let op: **Posets** $\to$ **Posets** be a functor that maps $(\mathbf{P}, \leq)$ to $(\mathbf{P}, \leq)^{\mathrm{op}} = (\mathbf{P}, \geq)$. Assuming a fibration $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ is induced—by the Grothendieck construction—by an indexed category $\Phi \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Posets}$, the composite $\mathbb{C}^{\mathrm{op}} \xrightarrow{\Phi} \mathbf{Posets} \xrightarrow{\mathrm{op}} \mathbf{Posets}$ induces a fibration in which each fiber is opposed. This is what is denoted by $\begin{smallmatrix}\mathbb{P}^{(\mathrm{op})}\\\downarrow p^{(\mathrm{op})}\\\mathbb{C}\end{smallmatrix}$ in the following lemma.

Lemma C.13 (Fiberwise opposite, (Bénabou, 1975)). Let $\begin{smallmatrix}\mathbb{P}\\\downarrow p\\\mathbb{C}\end{smallmatrix}$ be a fibration. There exists a canonical fibration $\begin{smallmatrix}\mathbb{P}^{(\mathrm{op})}\\\downarrow p^{(\mathrm{op})}\\\mathbb{C}\end{smallmatrix}$ such that: $(\mathbb{P}^{(\mathrm{op})})_X = (\mathbb{P}_X)^{\mathrm{op}}$; and reindexing functors coincide, as in the commutative diagram

$$
\begin{array}{ccc}
(\mathbb{P}^{(\mathrm{op})})_Y & \xrightarrow{\ f^* \text{ in } p^{(\mathrm{op})}\ } & (\mathbb{P}^{(\mathrm{op})})_X \\
\| & & \| \\
(\mathbb{P}_Y)^{\mathrm{op}} & \xrightarrow[\ (f^* \text{ in } p)^{\mathrm{op}}\ ]{} & (\mathbb{P}_X)^{\mathrm{op}}
\end{array}
\qquad \text{for } f \colon X \to Y.
$$

This fibration $p^{(\mathrm{op})}$ is called the *fiberwise opposite* of p.

Proof. We describe the construction of $\mathbb{P}^{(\mathrm{op})}$; it is simple in the current setting where we focus on *poset* fibrations. The objects are the same as those of $\mathbb{P}$, and the arrows are defined by

$$\mathbb{P}^{(\mathrm{op})}(P,Q) = \{f \colon pP \to pQ \mid P \geq f^*Q\}.$$

It is easy to show that id_{pP} induces an arrow $P \to P$ in $\mathbb{P}^{(\mathrm{op})}$; this gives the identity arrow. The composite $g \circ f \in \mathbb{P}^{(\mathrm{op})}(P,R)$ of $f \in \mathbb{P}^{(\mathrm{op})}(P,Q)$ and $g \in \mathbb{P}^{(\mathrm{op})}(Q,R)$ is given by composition in $\mathbb{C}$, too. $\qquad\square$

Appendix D. Omitted Proofs

D.1. *Proof of Proposition 6.16*

We shall prove the item 1. In the topos $\mathbf{Sets}^{\mathbb{A}}$, there exists an (Epi, Mono)-factorization, which induces the image functor $\mathrm{Im}\colon \mathbf{Sets}^{\mathbb{A}}/P \to \mathrm{Sub}(P)$ that is surjective on objects. In particular, a subpresheaf of P can be thought of as an image of some arrow with codomain P.

Let $(Q \xrightarrow{\theta} P) \in \mathbf{Sets}^{\mathbb{A}}/P$. By Lemma 6.14, we may assume $Q = \mathrm{Colim}_{I \in \mathbb{I}}(\mathbf{y}B_I)$ for some diagram $(B_I)_{I \in \mathbb{I}}$. By Example 6.10 and Lemma B.7, we have $\mathrm{Im}\,\theta = \bigvee_{I \in \mathbb{I}} \mathrm{Im}\,\theta_I$ where the arrow θ_I is the composite $\left(\mathbf{y}B_I \to \mathrm{Colim}_I(\mathbf{y}B_I) \xrightarrow{\theta} P\right)$.

Letting $P = \mathbf{y}A$, we obtain $\mathrm{Im}(Q \xrightarrow{\theta} \mathbf{y}A) = \bigvee_I \mathrm{Im}(\mathbf{y}B_I \xrightarrow{\mathbf{y}f_I} \mathbf{y}A)$ for a family $(f_I)_I$ such that $\theta_I = \mathbf{y}f_I$; such a family $(f_I)_I$ exists since the functor $\mathbf{y}$ is full and faithful. This proves the item 1.

We shall now prove the item 2. We observe that an epi $A \twoheadrightarrow C$ in $\mathbb{A}$ induces a mono $\mathbf{y}C \rightarrowtail \mathbf{y}A$ in $\mathbf{Sets}^{\mathbb{A}}$: this is because the functor $\mathbf{y}\colon \mathbb{A}^{\mathrm{op}} \to \mathbf{Sets}^{\mathbb{A}}$ preserves all existing limits, including the pullback

$$
\begin{array}{ccc}
C & \xrightarrow{\mathrm{id}_C} & C \\
{\scriptstyle \mathrm{id}_C}\downarrow & \llcorner & \downarrow{\scriptstyle m} \\
C & \xrightarrow{m} & A
\end{array}
$$

in $\mathbb{A}^{\mathrm{op}}$. (The diagram is a pullback if and only if m is a mono in $\mathbb{A}^{\mathrm{op}}$, i.e. an epi in $\mathbb{A}$.) Thus there is a monotone function $\mathrm{Quot}(A) \to \mathrm{Sub}(\mathbf{y}A)$.

Regarding monos in $\mathbb{A}$, we can show the following sublemma (its only-if direction will not be used later).

Sublemma D.1. Let $m\colon C \to B$ be an arrow in $\mathbb{A}$. The arrow $\mathbf{y}m\colon \mathbf{y}B \to \mathbf{y}C$ is an epi in $\mathbf{Sets}^{\mathbb{A}}$ if and only if the arrow m is a split mono.

Proof. The following are equivalent (folklore): for an arrow e in $\mathbb{B}$,

1 The arrow e is an absolute epi, i.e. $F(e)$ is an epi for any functor F with the domain $\mathbb{B}$,

2 the arrow $\mathbf{y}e$ in $\mathbf{Sets}^{\mathbb{B}^{\mathrm{op}}}$ is an epi, and

3 the arrow e is a split epi.

The sublemma is part of this fact for $\mathbb{B} = \mathbb{A}^{\mathrm{op}}$.

To be concrete, we take a retraction $r \colon B \to C$ of a split mono m in $\mathbb{A}$. By $r \circ m = \mathrm{id}_C$, we have $\mathbf{y}m \circ \mathbf{y}r = \mathrm{id}_{\mathbf{y}C}$, which shows that $\mathbf{y}m$ is a (split) epi in $\mathbf{Sets}^{\mathbb{A}}$.

Conversely, let $m \colon C \to B$ be an arrow in $\mathbb{A}$ such that $\mathbf{y}m \colon \mathbf{y}B \to \mathbf{y}C$ is an epi in $\mathbf{Sets}^{\mathbb{A}}$. Because colimits are computed component-wise in the functor category $\mathbf{Sets}^{\mathbb{A}}$, the function $(\mathbf{y}m)_C \colon \mathbb{A}(B,C) \to \mathbb{A}(C,C)$ is surjective. Hence, there exists $r \in \mathbb{A}(B,C)$ such that $(\mathbf{y}m)_C(r) = \mathrm{id}_C \in \mathbb{A}(C,C)$, that is, $r \circ m = \mathrm{id}_C$. Therefore the arrow m has a retraction r. $\qquad\square$

Therefore an $(\mathrm{Epi}, \mathrm{SplitMono})$-factorization $A \xrightarrow{e} C \xrightarrow{m} B$ in $\mathbb{A}$ induces an $(\mathrm{Epi}, \mathrm{Mono})$-factorization $\mathbf{y}B \xrightarrow{\mathbf{y}m} \mathbf{y}C \xrightarrow{\mathbf{y}e} \mathbf{y}A$. This yields

$$\{\mathrm{Im}(\mathbf{y}B \xrightarrow{\mathbf{y}f} \mathbf{y}A) \mid B \in \mathbb{A},\ f \colon A \to B\} = \{(\mathbf{y}C \xrightarrow{\mathbf{y}e} \mathbf{y}A) \in \mathrm{Sub}(\mathbf{y}A) \mid C \in \mathbb{A},\ e \colon A \twoheadrightarrow C\}$$

$$\cong \{(A \xrightarrow{e} C) \in \mathrm{Quot}(A) \mid C \in \mathbb{A}\} = \mathrm{Quot}(A),$$

where the last isomorphism holds because the functor $\mathbf{y}$ is full and faithful. Hence the item 2 reduces to the item 1.