Semantics for a Quantum Programming Language by Operator Algebras

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Abstract This paper presents a novel semantics for a quantum programming language by *operator algebras*, which are known to give a formulation for quantum theory that is alternative to the one by Hilbert spaces. We show that the opposite of the category of W^*-algebras and normal completely positive subunital maps is an elementary quantum flow chart category in the sense of Selinger. As a consequence, it gives a denotational semantics for Selinger's first-order functional quantum programming language. The use of operator algebras allows us to accommodate infinite structures and to handle classical and quantum computations in a unified way.

Keywords Quantum Computation, Quantum Programming Languages, Operator Algebras, Denotational Semantics, Complete Partial Orders.

§1 Introduction

Aiming at high-level and structured description of quantum computation/information, many *quantum programming languages* have been proposed and their semantics studied.[17,55] Selinger, in his seminal work,[50] proposed a first-order functional quantum programming language QFC (and QPL), and gave its denotational semantics rigorously in terms of categories. Selinger and Valiron successively studied a higher-order quantum programming language, or the *quantum lambda calculus*.[51–53] It turned out to be challenging to give a denotational semantics for the quantum lambda calculus (with full features, such as the ! modality and recursion). The satisfactory denotational semantics was first given via Geometry of Interaction;[22] but different approaches have been proposed.[31,39] As Pagani et al. stated,[39, §1] the difficulty lies in that (particularly, higher-order) programming languages contain *infinitary* concepts such as infinite types and recursion, while quantum computation is traditionally modelled via *finite* dimensional Hilbert spaces.

The present paper proposes a novel denotational semantics for a quantum programming language by *operator algebras*. Operator algebras, specifically C^*-algebras and W^*-algebras (the latter are also known as von Neumann algebras), give an alternative formulation of quantum theory (sometimes called the *algebraic* formulation[30]). It is worth mentioning that von Neumann himself, who formulated quantum theory by Hilbert spaces,[58] developed the theory of operator algebras[34–36,56,57] (partly in collaboration with Murray), and later preferred the algebraic approach for quantum theory.[43] Operator algebras have been successfully used in areas such as quantum statistical mechanics[5] and quantum field theory.[2,19,20] They have also been of growing importance in the area of quantum information[27]; for example, D'Ariano et al.[12] reexamined the impossibility of quantum bit commitment in the algebraic formalism.

1.1 Contributions and related work

In this paper it is shown that the category $\mathbf{Wstar}_{\mathrm{CPSU}}$ of W^*-algebras and normal completely positive subunital maps is a $\mathbf{Dcppo}_{\perp}$-enriched symmetric monoidal category with $\mathbf{Dcppo}_{\perp}$-enriched products. It follows that the opposite $(\mathbf{Wstar}_{\mathrm{CPSU}})^{\mathrm{op}}$ is an $\boldsymbol{\omega}\mathbf{Cppo}$-enriched *elementary quantum flow chart category*. As a consequence, it gives a denotational semantics for a first-order functional quantum programming language QFC designed by Selinger.[50]

Selinger himself gave a denotational semantics for QFC by the category $\mathbf{Q}$.[50] In comparison to his original model, our model by operator algebras is more flexible in the following two points. First, our semantics accommodates infinite structures, since we discuss general W^*-algebras, not restricting them to finite dimensional ones. Hence our model can interpret infinite types such as the type of natural numbers. In fact, we will see that Selinger's category $\mathbf{Q}$ is (dually) equivalent to the category of finite dimensional W^*-algebras, in §7.2. Second, classical computation naturally arises in *commutative* operator algebras. There is a categorical 'Gelfand' duality between commutative C^*-algebras and compact Hausdorff spaces,[16,37] and similarly commutative W^*-algebras have a relationship to certain measure/measurable spaces. In §8, we will see that several 'classical' categories can be embedded into the categories of W^*-algebras. It will allow us to handle classical and quantum computations in a unified way.

Traditionally, quantum computation is modelled based on finite dimensional Hilbert spaces $\mathbb{C}^n$ (or matrix algebras $\mathcal{M}_n \cong \mathcal{B}(\mathbb{C}^n)$),[22, 31, 39, 50] rather than using operator algebras explicitly. Recently there are works using C^*-algebras,[16, 25] which led the author to the present work. The use of W^*-algebras in this context appeared independently and coincidentally in Rennela's thesis[44] and the present work (the author's thesis[8]). Rennela also showed that the category $\mathbf{Wstar}_{\mathrm{PSU}}$ of W^*-algebras and normal positive subunital maps are **Dcppo**-enriched,[44, Theorem 3.8] which is a similar result to Theorem 4.3 in the present paper. In his latest paper[45], he further showed that $\mathbf{Wstar}_{\mathrm{PSU}}$ is algebraically compact for a certain class of functors. This result enables us to have inductively defined types.

Similar results appeared in the paper by Chiribella et al.,[7] which studied spaces of "quantum operations" between W^*-algebras, and "quantum supermaps" between them. For instance, they showed that (in our terminology) each homset $\mathbf{Wstar}_{\mathrm{CP}}(M, N)$ is bounded directed complete.[7, Proposition 7]

1.2 Organisation of the paper

In §2 we give preliminaries on operator algebras, which contain standard definitions and results. Some less standard results are shown in §3. In §4 we study the order/domain-theoretic aspect of W^*-algebras; in particular we show that $\mathbf{Wstar}_{\mathrm{CPSU}}$ is a $\mathbf{Dcppo}_{\perp}$-enriched symmetric monoidal category with $\mathbf{Dcppo}_{\perp}$-enriched products. We review the notion of quantum operations in §5, and Selinger's work on QFC in §6. In §7 we discuss a semantics for QFC by operator algebras. In §8 we investigate classical computation in commutative operator algebras. We give a conclusion in §9.

This paper is based on the author's master thesis.[8] An earlier version of this paper was presented at the 11th workshop on Quantum Physics and Logic (QPL 2014).[9] Compared to the workshop version, major differences are as follows. New results on the full embeddings of categories are added in §8; in the earlier version we only had the embedding of **Set** (without fullness). The current version also has more detailed preliminaries on operator algebras in §2. A number of proofs deferred to appendices are now included in the main text, except results on (cartesian) traces.

§2 Preliminaries on operator algebras

Here we give preliminaries on operator algebras, i.e. C^*-algebras and W^*-algebras. In particular, we would like to collect basic results on categories of them, which rarely appear in textbooks on operator theory. Almost all results in this section, however, can be found in the papers by Guichardet,[18] Meyer[33] and Kornell.[28]

Let us first introduce basic notations. We denote by $\mathbb{N}$ the set of natural numbers, by $\mathbb{R}$ the set of real numbers, and by $\mathbb{C}$ the set of complex numbers. We also write $\mathbb{R}^+ = [0,\infty)$ for the set of non-negative real numbers.

2.1 C^*-algebras and their constructions

Definition 2.1 (C^*-algebra)

1. A $*$-*algebra* is a complex vector space A with a bilinear and associative 'multiplication' $\cdot\colon A \times A \to A$ and an 'involution' $(-)^*\colon A \to A$ that satisfies: for $x, y \in A$ and $\lambda \in \mathbb{C}$,
$$(x+y)^* = x^* + y^* \qquad (\lambda x)^* = \overline{\lambda} x^* \qquad (x^*)^* = x \qquad (xy)^* = y^* x^* \ .$$
2. A norm $\|-\|$ on a $*$-algebra A is called a C^*-*norm* if it satisfies (besides the usual axioms) $\|xy\| \le \|x\|\|y\|$ and $\|x^*x\| = \|x\|^2$ for all $x, y \in A$.
3. A C^*-*algebra* is a $*$-algebra A with a C^*-norm with respect to which A is complete.

In this paper, we additionally require that C^*-algebras be *unital*, i.e. they have multiplicative units 1. In other words, we refer to unital C^*-algebras as C^*-algebras. A C^*-algebra is *commutative* if the multiplication is commutative; and *finite dimensional* if it is finite dimensional as a vector space.

Definition 2.2

A linear map $f\colon A \to B$ between C^*-algebras is called a $*$-*homomorphism* if it is both *multiplicative*, i.e. $f(xy) = f(x)f(y)$, and *involutive*, i.e. $f(x^*) = f(x)^*$. It is said to be *unital* if $f(1) = 1$.

Although the definition of $*$-homomorphism is purely algebraic, metric properties automatically follow.

Proposition 2.1 ([41, Theorem 1.5.7])

Every $*$-homomorphism $f\colon A \to B$ between C^*-algebras is *short* (a.k.a. *contractive*), i.e. $\|f(x)\| \le \|x\|$ for all $x \in A$ (equivalently $\|f\| \le 1$). Moreover, f is isometric if (and only if) it is injective.

It follows that for any $*$-algebra, there is at most one norm under which it is a C^*-algebra.

Here are a few examples of C^*-algebras.

Example 2.1

1. For a Hilbert space $\mathcal{H}$, the set $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ is a C^*-algebra.
2. As a special case of the previous one, the set $\mathcal{M}_n \cong \mathcal{B}(\mathbb{C}^n)$ of complex $n \times n$ matrices is a finite dimensional C^*-algebra.
3. For a compact Hausdorff space X, the set $C(X)$ of complex valued continuous functions on X is a commutative C^*-algebra. In fact, any commutative C^*-algebra is of this form (up to $*$-isomorphism).

It is important that every C^*-algebra can be represented on a Hilbert space.

Definition 2.3

A *representation* of C^*-algebra A is a pair $(\mathcal{H}, \pi)$ of a Hilbert space $\mathcal{H}$ and a unital[*1] $*$-homomorphism $\pi\colon A \to \mathcal{B}(\mathcal{H})$. It is said to be *faithful* if π is injective.

Theorem 2.1 ([54, Theorem I.9.18])

Every C^*-algebra admits a faithful representation. ■

It follows that C^*-algebras characterise norm-closed $*$-algebras of bounded operators on a Hilbert space.

Next, we describe a couple of important constructions of C^*-algebras. Products of C^*-algebras are simple.

Definition 2.4 (Product of C^*-algebras)

Let $(A_i)_{i\in I}$ be a family of C^*-algebras. The *product* of $(A_i)_i$, denoted by $\prod_i A_i$, has the underlying set

$$\prod\nolimits_{i\in I} A_i \coloneqq \left\{ (x_i)_{i\in I} \;\middle|\; x_i \in A_i \text{ and } \sup_i \|x_i\| < \infty \right\}$$

with coordinatewise operations and a norm $\|(x_i)_i\| = \sup_i \|x_i\|$. Note that $\prod_i A_i$ is a subset of the set-theoretic product of $(A_i)_i$. The empty product is the trivial C^*-algebra $\{0\}$, which is denoted by 1.

If the index set I is finite, say $I = \{1, \ldots, n\}$, then the product is denoted by $A_1 \times \cdots \times A_n$. In this case, the underlying set is simply the set-theoretic product.

[*1] Note that the unitality may not be assumed in the literature.

In the subsequent sections, we mostly use finite products.

Remark 2.1
Products of C^*-algebras are known under various names, such as *direct sum*,[48, Definition 1.1.5] ℓ^∞*-direct sum*[6, §1.3] and *direct product*.[4, §II.8.1] We nevertheless call it simply 'product', since it is a product in the categorical sense; see Proposition 2.6.

Tensor products of C^*-algebras are much more involved than products. For C^*-algebras A and B, we denote by $A \odot B$ the algebraic tensor product of A and B. It is not hard to see that $A \odot B$ is a $*$-algebra in an obvious manner. The $*$-algebra $\mathcal{M}_n(A)$ of matrices with entries from A is a special case by virtue of the $*$-homomorphism $\mathcal{M}_n(A) \cong \mathcal{M}_n \odot A$ (where $\mathcal{M}_n = \mathcal{M}_n(\mathbb{C})$). In this case the situation is simple.

Proposition 2.2 ([54, §IV.3])
For any C^*-algebra A, there is a (unique) C^*-norm on $\mathcal{M}_n(A)$ under which it is complete, that is, $\mathcal{M}_n(A)$ is a C^*-algebra. The norm satisfies the following inequalities. For $[x_{ij}]_{ij} \in \mathcal{M}_n(A)$,

$$\max_{i,j} \|x_{ij}\| \le \big\|[x_{ij}]_{ij}\big\| \le \sum_{i,j} \|x_{ij}\| \ .$$

■

To obtain a C^*-algebra in the general case, we need to complete the $*$-algebra $A \odot B$ under some C^*-norm. The following fact is highly nontrivial; for a proof we refer to the textbook by Takesaki[54, §IV.4] or by Brown and Ozawa.[6, Chapter 3]

Theorem 2.2
Let $A \odot B$ be the algebraic tensor product of C^*-algebras A and B.

1. There is a least and a greatest C^*-norm on $A \odot B$.
2. Every C^*-norm α on $A \odot B$ is a *cross norm* in the following sense:
   - $\alpha(x \otimes y) = \|x\|\|y\|$ for $x \in A$ and $y \in B$;
   - $\alpha^*(\varphi \otimes \psi) = \|\varphi\|\|\psi\|$ for $\varphi \in A^*$ and $\psi \in B^*$, where $\varphi \otimes \psi$ is identified with a functional on $A \odot B$, and α^* is the dual norm of α on $(A \odot B)^*$. ■

A least and a greatest C^*-norm are different in general, and hence there are different kinds of tensor products of C^*-algebras. We use the following one.

Definition 2.5 (Tensor product of C^*-algebras)
Let A and B be C^*-algebras. The least C^*-norm on $A \odot B$ is called the *spatial*

C^*-norm. The *spatial C^*-tensor product* of A and B, denoted by $A \otimes B$, is the completion of $A \odot B$ under the spatial C^*-norm.

The spatial C^*-norm (resp. C^*-tensor product) is also referred to as the *minimal* (or *injective*) C^*-norm (resp. C^*-tensor product). The term 'spatial' is explained in the following result.

Theorem 2.3 ([54, Theorem IV.4.9])
Let A and B be C^*-algebras, and $(\mathcal{H}_A, \pi_A)$ and $(\mathcal{H}_B, \pi_B)$ be faithful representations of A and B respectively. The map $\pi_A \odot \pi_B \colon A \odot B \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, given by $a \otimes b \mapsto \pi_A(a) \otimes \pi_B(b)$, extends to a faithful representation $\pi_A \otimes \pi_B \colon A \otimes B \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ of $A \otimes B$ on $\mathcal{H}_A \otimes \mathcal{H}_B$.

Since faithful representations are isometric, spatial C^*-norms can be obtained via faithful representations.

2.2 Various maps and categories of C^*-algebras

Definition 2.6
Let A be a C^*-algebra. An element $x \in A$ is *self-adjoint* if $x^* = x$; *positive* if there exists $y \in A$ such that $x = y^* y$; an *effect* if both x and $1 - x$ are positive; a *projection* if $x^* = x = x^2$. We write A_{sa} for the set of self-adjoint elements, A^+ for the set of positive elements, $[0,1]_A$ for the set of effects, and $\mathcal{P}r(A)$ for the set of projections. It is easy to see inclusions $\mathcal{P}r(A) \subseteq [0,1]_A \subseteq A^+ \subseteq A_{\mathrm{sa}}$.

There is a standard partial order in a C^*-algebra.

Definition 2.7
Let A be a C^*-algebra. We define a relation $\leq$ on A_{sa} by $x \leq y \iff$ '$y - x$ is positive'. Then, $\leq$ is a partial order.[54, Theorem I.6.1)]

We may write $x \geq 0$ for 'x is positive'. Note also that x is an effect if and only if $0 \leq x \leq 1$, which justifies the notation $[0,1]_A$.

The following lemma relating the order and the norm is useful.

Lemma 2.1
Let A be a C^*-algebra and $x \in A$ a self-adjoint element. For any $M \in \mathbb{R}^+$, $\|x\| \leq M$ if and only if $-M1 \leq x \leq M1$. In particular, for any self-adjoint element $x \in A$ one has $-\|x\|1 \leq x \leq \|x\|1$.

Proof For any $x \in A$, let

$$\mathrm{Sp}(x) := \{\lambda \in \mathbb{C} \mid x - \lambda 1 \text{ is not invertible}\}$$

denote the spectrum. Note that $\mathrm{Sp}(\alpha x + \beta 1) = \alpha \cdot \mathrm{Sp}(x) + \beta$ for any $\alpha, \beta \in \mathbb{C}$ ($\alpha \neq 0$). We will use the following basic facts: for $x \in A_{\mathrm{sa}}$,

- $\|x\| = \sup_{\lambda \in \mathrm{Sp}(x)} |\lambda|$;[54, Proposition I.4.2]
- $\mathrm{Sp}(x) \subseteq \mathbb{R}$;[54, Proposition I.4.3]
- $x \geq 0 \iff \mathrm{Sp}(x) \subseteq \mathbb{R}^+$.[54, Theorem I.6.1]

We then reason as follows.

$$\begin{aligned}
\|x\| \leq M &\iff \sup_{\lambda \in \mathrm{Sp}(x)} |\lambda| \leq M \\
&\iff \forall \lambda \in \mathrm{Sp}(x).\, -M \leq \lambda \leq M \\
&\iff \forall \lambda \in \mathrm{Sp}(x).\, M - \lambda \geq 0 \text{ and } \lambda + M \geq 0 \\
&\iff \mathrm{Sp}(M1 - x) \subseteq \mathbb{R}^+ \text{ and } \mathrm{Sp}(x + M1) \subseteq \mathbb{R}^+ \\
&\iff M1 - x \geq 0 \text{ and } x + M1 \geq 0 \\
&\iff -M1 \leq x \leq M1
\end{aligned}$$

■

Definition 2.8

A linear map $f \colon A \to B$ between C^*-algebras is *positive* if $x \geq 0$ implies $f(x) \geq 0$; and *completely positive* if $\mathcal{M}_n(f)$ is positive for all $n \in \mathbb{N}$, where $\mathcal{M}_n(f) \colon \mathcal{M}_n(A) \to \mathcal{M}_n(B)$ is a map defined by $\mathcal{M}_n(f)([x_{kl}]_{kl}) = [f(x_{kl})]_{kl}$. A (completely) positive map $f \colon A \to B$ is *subunital* if $f(1) \leq 1$.

For the sake of convenience, we introduce shorthand for kinds of maps as follows: *M* for multiplicative; *I* for involutive; *P* for positive; *CP* for completely positive; *U* for unital; and *SU* for subunital. For example, a CPSU-map refers to a completely positive subunital map, and a MIU-map—a multiplicative involutive unital map—is a synonym for a unital $*$-homomorphism.

Proposition 2.3

For maps between C^*-algebras, there are the following implications.

$$\begin{array}{ccccccc}
\mathrm{MIU} & \Longrightarrow & \mathrm{MI} & & & & \\
\Downarrow & & \Downarrow & & & & \\
\mathrm{CPU} & \Longrightarrow & \mathrm{CPSU} & \Longrightarrow & \mathrm{CP} & & \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
\mathrm{PU} & \Longrightarrow & \mathrm{PSU} & \Longrightarrow & \mathrm{P} & \Longrightarrow & \mathrm{I}
\end{array}$$

Proof MI $\implies$ CPSU: Assume that f is a MI-map. It is easy to see that $\mathcal{M}_n(f)$ is MI too. Because MI implies positive, f is CP. It is also subunital since $f(1)$ is a projection, hence below 1.

A positive map is involutive since any element of a C^*-algebra can be written as a linear combination of positive elements.[54, §I.4)] The other implications are easy. ∎

Proposition 2.4
A positive map $f\colon A \to B$ between C^*-algebras is bounded and $\|f\| = \|f(1)\|$. Moreover, f is subunital if and only if it is short.

Proof For the first statement see e.g. Paulsen's book.[40, Corollary 2.9)] Then, the latter follows from $\|f(1)\| \le 1 \iff f(1) \le 1$ by Lemma 2.1. ∎

The following result is useful.

Proposition 2.5 ([54, Corollary IV.3.5 and Proposition IV.3.9)])
A positive map $f\colon A \to B$ between C^*-algebras is completely positive if at least one of A and B is commutative. ∎

Now we introduce categories of C^*-algebras.

Definition 2.9
Let X be a kind of maps (we use one of MIU, MI, CPU, CPSU, PU, PSU, CP and P). We denote by $\mathbf{Cstar}_{\mathrm{X}}$ the category of C^*-algebras and X-maps; by $\mathbf{CCstar}_{\mathrm{X}}$ the full subcategory of $\mathbf{Cstar}_{\mathrm{X}}$ containing commutative C^*-algebras; and by $\mathbf{FdCstar}_{\mathrm{X}}$ the full subcategory of $\mathbf{Cstar}_{\mathrm{X}}$ containing finite dimensional C^*-algebras.

There are inclusions of categories of C^*-algebras corresponding to Proposition 2.3, e.g. $\mathbf{Cstar}_{\mathrm{MI}} \subseteq \mathbf{Cstar}_{\mathrm{CPSU}}$.

Proposition 2.6
For $\mathrm{X} \in \{\mathrm{MIU}, \mathrm{MI}, \mathrm{CPU}, \mathrm{CPSU}, \mathrm{PU}, \mathrm{PSU}\}$, products of C^*-algebras with obvious projections are categorical products in $\mathbf{Cstar}_{\mathrm{X}}$. In particular, the trivial C^*-algebra 1 is a final object.

Proof Let $(f_i\colon A \to B_i)_{i\in I}$ be a family of X-maps between C^*-algebras ($\mathrm{X} \in \{\mathrm{MIU}, \mathrm{MI}, \mathrm{CPU}, \mathrm{CPSU}, \mathrm{PU}, \mathrm{PSU}\}$). There is a map $\langle f_i \rangle_i \colon A \to \prod_i B_i$ given by $\langle f_i \rangle_i(x) = (f_i(x))_i$, which is well-defined thanks to the shortness of X-maps. For the projections $\pi_i\colon \prod_i B_i \to B_i$, we have $\pi_i \circ \langle f_i \rangle_i = f_i$ for all

$i \in I$, and such a map is unique. It is not hard to see that if f_i is a X-map, so is $\langle f_i \rangle_i$; for complete positivity, use the $*$-isomorphism $\mathcal{M}_n(\prod_i B_i) \cong \prod_i \mathcal{M}_n(B_i)$, which is obtained using the inequalities of Proposition 2.2. ■

Remark 2.2
Finite products of C^*-algebras are biproducts in $\mathbf{Cstar}_{\mathrm{X}}$ ($\mathrm{X} \in \{\mathrm{CP}, \mathrm{P}\}$). The trivial C^*-algebra 1 is initial (hence a zero object) in $\mathbf{Cstar}_{\mathrm{X}}$ ($\mathrm{X} \in \{\mathrm{MI}, \mathrm{CPSU}, \mathrm{PSU}, \mathrm{CP}, \mathrm{P}\}$).

We wish to make the category $\mathbf{Cstar}_{\mathrm{X}}$ symmetric monoidal via the spatial C^*-tensor product $\otimes$. For this we need to consider the tensor product of maps $f \otimes g$. Let $f \colon A \to A'$ and $g \colon B \to B'$ be maps between C^*-algebra. It is easy to form the algebraic tensor product $f \odot g \colon A \odot B \to A' \odot B'$. If $f \odot g$ is bounded under the spatial C^*-norms, then $f \odot g$ extends uniquely to $f \otimes g \colon A \otimes B \to A' \otimes B'$. If at least one of f and g is merely positive, however, $f \odot g$ may be unbounded. Even in the finite dimensional case, the tensor product of positive maps may not be positive. This is why we need complete positivity.

Proposition 2.7
For $\mathrm{X} \in \{\mathrm{MIU}, \mathrm{MI}, \mathrm{CPU}, \mathrm{CPSU}, \mathrm{CP}\}$, the category $\mathbf{Cstar}_{\mathrm{X}}$ is symmetric monoidal with the spatial C^*-tensor product $\otimes$ and the C^*-algebra $\mathbb{C}$ of complex numbers as a unit object.

Proof If $f \colon A \to A'$ and $g \colon B \to B'$ are MI-maps (resp. CP-maps) between C^*-algebras, then $f \odot g$ extends to a MI-map (resp. CP-map) $f \otimes g \colon A \otimes B \to A' \otimes B'$ between the spatial C^*-tensor products.[54, Propositions IV.4.22 and IV.4.23] It is easy to see that f and g are (sub)unital, then $f \otimes g$ is (sub)unital, and therefore the spatial C^*-tensor product $\otimes$ forms a bifunctor on $\mathbf{Cstar}_X$.

It is easy to see that $\mathbb{C}$ is the unit object, and the tensor product is symmetric (up to MIU-isomorphism). To see the associativity, one may take faithful representations and use Theorem 2.3, with the associativity of the Hilbert space tensor product: $(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{H}_3 \cong \mathcal{H}_1 \otimes (\mathcal{H}_2 \otimes \mathcal{H}_3)$. ■

2.3 W^*-algebras

We define W^*-algebras via Sakai's characterisation,[47] as a special kind of C^*-algebras.

Definition 2.10 (W^*-algebra)
A *W^*-algebra* is a C^*-algebra M that admits a predual, i.e. a Banach space X

with an isometric isomorphism $X^* \cong M$. It turns out that a predual of a W^*-algebra M is unique (up to isometric isomorphism).[48, Corollary 1.13.3] The weak* topology on M induced by the predual is referred to as the *ultraweak* topology. A linear map between W^*-algebras is said to be *normal* if it is ultraweakly continuous. We denote the set of normal functionals on M by M_* ($\subseteq M^*$); it is standard that M_* is a predual of M.

Remark 2.3
In this paper, W^*-algebras are unital by definition, since we require that C^*-algebras be unital. In fact, W^*-algebras are necessarily unital. In other words, if a not necessarily unital C^*-algebra admits a predual, then it has a unit.[48, §1.7]

Since the ultraweak topology is by definition the weak* topology, we may apply results for the weak* topology to the ultraweak topology. For example, the addition and the scalar multiplication of a W^*-algebra are ultraweakly continuous. The following basic fact also comes from a general result for the weak* continuity.

Proposition 2.8 ([42, Proposition 2.4.12])
A linear map $f\colon M \to N$ between W^*-algebras is normal if and only if there is a bounded map $g\colon N_* \to M_*$ that makes the following diagram commute.

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ (M_*)^* & \xrightarrow{\quad g^* \quad} & (N_*)^* \end{array}$$ ■

Because such g is unique, it establishes a bijective correspondence between normal maps $f\colon M \to N$ and bounded maps $g\colon N_* \to M_*$. We call such g the *predual map* of f and write $f_* = g$. Moreover, this correspondence is isometric: i.e. $\|f_*\| = \|f\|$.[42, Proposition 2.3.10]

The following nontrivial fact is important.

Proposition 2.9 ([48, §1.7])
Let M be a W^*-algebra. Both M_{sa} and M_+ are ultraweakly closed. The involution $(-)^*\colon M \to M$ is ultraweakly continuous (i.e. normal), and the multiplication $\cdot\colon M \times M \to M$ is separately ultraweakly continuous. ■

Example 2.2

1. Recall that for a Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ is a C^*-algebra. It is well-known that $\mathcal{B}(\mathcal{H})$ is dual of the space $\mathcal{T}(\mathcal{H})$ of trace class operators.

Hence $\mathcal{B}(\mathcal{H})$ is a W^*-algebra.

2. An ultraweakly closed $*$-subalgebra of a W^*-algebra is a W^*-algebra.[48, Definition 1.1.4] In particular, an ultraweakly closed unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ is a W^*-algebra, which is usually called a *von Neumann algebra.*
3. $\mathcal{M}_n \cong \mathcal{B}(\mathbb{C}^n)$ is a W^*-algebra. The predual is itself $\mathcal{M}_n \cong \mathcal{T}(\mathbb{C}^n)$ with the trace norm.
4. For a localisable measure space (X, Σ, μ), the (complex-valued) L^∞ space $L^\infty(X, \Sigma, \mu)$ is a commutative W^*-algebra with a predual $L^1(X, \Sigma, \mu)$. In fact, any commutative W^*-algebra is of this form.[48, §1.18]
5. Less generally, for a set X, the ℓ^∞ space $\ell^\infty(X)$ is a W^*-algebra with a predual $\ell^1(X)$. We will investigate this kind of W^*-algebras in §8.

Remark 2.4

Any finite dimensional C^*-algebra A is a W^*-algebra with the predual $A_* = A^*$, since $A^{**} \cong A$. In the light of Proposition 2.8, every linear map from finite dimensional W^*-algebra to any W^*-algebra is normal.

A representation $(\mathcal{H}, \pi)$ of a W^*-algebra M behaves well when it is *normal*, i.e. the map $\pi \colon M \to \mathcal{B}(\mathcal{H})$ is normal. In that case the image $\pi(M)$ is ultraweakly closed in $\mathcal{B}(\mathcal{H})$,[48, Proposition 1.16.2] that is, $\pi(M)$ is a von Neumann algebra. Moreover one has the following theorem.

Theorem 2.4 ([48, Theorem 1.16.7])

Every W^*-algebra admits a faithful normal representation. ∎

Therefore, W^*-algebras characterise von Neumann algebras. In fact, W^*-algebras are more often studied as concrete von Neumann algebras than as abstract W^*-algebras. Via representations, nonetheless, we may apply results for von Neumann algebras to abstract W^*-algebras.

Products of W^*-algebras are simply products as C^*-algebras. It is well-defined because of the following fact.[48, Definition 1.1.5]

Lemma 2.2

Let $(M_i)_{i \in I}$ be a family of W^*-algebras. Let $\prod_i M_i$ be the product of $(M_i)_i$ as C^*-algebras. Then the ℓ^1-direct sum $\bigoplus_i^1 M_{i*}$ of the preduals, where the norm is given by $\|(\varphi_i)_i\| = \sum_i \|\varphi_i\|$, is a predual of $\prod_i M_i$, i.e. $(\bigoplus_i^1 M_{i*})^* \cong \prod_i M_i$. ∎

On the other hand, the tensor product of W^*-algebras differs from that of C^*-algebras. We shall sketch the construction, which follows the textbooks by

Sakai[48, §1.22] and by Takesaki.[54, §IV.5] Let M and N be W^*-algebras, and $M_* \odot N_*$ the algebraic tensor product of the preduals M_* and N_*. We equip $M_* \odot N_*$ with the dual spatial C^*-norm via the following embedding:

$$M_* \odot N_* \hookrightarrow M^* \odot N^* \hookrightarrow (M \otimes N)^* \ .$$

Let $M_* \otimes N_*$ denote the completion of $M_* \odot N_*$ under this norm, and let

$$I = (M_* \otimes N_*)^\perp = \{\varphi \in (M \otimes N)^{**} \mid \forall x \in M_* \otimes N_*. \, \varphi(x) = 0\}$$

be the annihilator of $M_* \otimes N_*$. It is standard that $(M \otimes N)^{**}/I \cong (M_* \otimes N_*)^*$.[10, Theorem V.2.3] Moreover the composite map

$$M \otimes N \hookrightarrow (M \otimes N)^{**} \twoheadrightarrow (M \otimes N)^{**}/I \cong (M_* \otimes N_*)^*$$

has weak* dense image, and is injective since $(M \otimes N) \cap I = \{0\}$. Now we apply the following two results: for a C^*-algebra A, 1) if I is a closed ideal of A, the quotient A/I is a C^*-algebra; 2) the double dual A^{**} is a C^*- and hence W^*-algebra. We can show that I is a closed ideal of $(M \otimes N)^{**}$, so that $(M_* \otimes N_*)^*$ is a W^*-algebra. Moreover the spatial C^*-tensor product $M \otimes N$ is ultraweakly densely embedded into $(M_* \otimes N_*)^*$.

Definition 2.11 (Tensor product of W^*-algebras)
For W^*-algebras M and N, the W^*-algebra $(M_* \otimes N_*)^*$ is called the *spatial W^*-tensor product* of M and N, and denoted by $M \mathbin{\overline{\otimes}} N$.

As the term 'spatial' suggests, one has the following result (cf. Theorem 2.3).

Theorem 2.5 ([54, Definition IV.5.2])
Let M and N be W^*-algebras, and $(\mathcal{H}_N, \pi_M)$ and $(\mathcal{H}_N, \pi_N)$ be faithful normal representations of M and N respectively. The representation $\pi_M \otimes \pi_N \colon M \otimes N \to \mathcal{B}(\mathcal{H}_M \otimes \mathcal{H}_N)$ of $M \otimes N$ (see Theorem 2.3) extends to a faithful normal representation $\pi_M \mathbin{\overline{\otimes}} \pi_N \colon M \mathbin{\overline{\otimes}} N \to \mathcal{B}(\mathcal{H}_M \otimes \mathcal{H}_N)$ of $M \mathbin{\overline{\otimes}} N$.

Now we define categories of W^*-algebras.

Definition 2.12
Let X be a kind of maps. We denote by $\mathbf{Wstar}_{\mathrm{X}}$ the category of W^*-algebras and *normal* X-maps; by $\mathbf{CWstar}_{\mathrm{X}}$ the full subcategory of $\mathbf{Wstar}_{\mathrm{X}}$ containing commutative W^*-algebras; and by $\mathbf{FdWstar}_{\mathrm{X}}$ the full subcategory of $\mathbf{Wstar}_{\mathrm{X}}$ containing finite dimensional W^*-algebras. Note that $\mathbf{Wstar}_{\mathrm{X}}$ (resp. $\mathbf{CWstar}_{\mathrm{X}}$) is a *non-full* subcategory of $\mathbf{Cstar}_{\mathrm{X}}$ (resp. $\mathbf{CCstar}_{\mathrm{X}}$), since we require that

maps be normal. In the light of Remark 2.4, however, one has $\mathbf{FdWstar}_X = \mathbf{FdCstar}_X$.

Proposition 2.10
For $\mathrm{X} \in \{\mathrm{MIU}, \mathrm{MI}, \mathrm{CPU}, \mathrm{CPSU}, \mathrm{PU}, \mathrm{PSU}\}$, products of W^*-algebras with obvious projections are categorical products in $\mathbf{Wstar}_{\mathrm{X}}$.

Proof It is almost done in Proposition 2.6. The normality of the maps involved can be checked using Proposition 2.8. ∎

Let us think about the functoriality of the spatial W^*-tensor product. Let $f\colon M \to M'$ and $g\colon N \to N'$ be normal maps between W^*-algebra, and $f_*\colon M'_* \to M_*$ and $g_*\colon N'_* \to N_*$ be the predual maps. Then, the algebraic tensor product $f_* \odot g_*\colon M'_* \odot N'_* \to M_* \odot N_*$ is bounded under the dual spatial C^*-norm if $f \odot g\colon M \odot N \to M' \odot N'$ is bounded under the spatial C^*-norm, since the following diagram commute.

$$\begin{array}{ccc} M'_* \odot N'_* & \xrightarrow{f_* \odot g_*} & M_* \odot N_* \\ \big\downarrow & & \big\downarrow \\ (M' \otimes N')^* & \xrightarrow{(f \otimes g)^*} & (M \otimes N)^* \end{array}$$

In that case we obtain a normal map $(f_* \otimes g_*)^*\colon (M_* \otimes N_*)^* \to (M'_* \otimes N'_*)^*$, which we denote by $f \mathbin{\overline{\otimes}} g\colon M \mathbin{\overline{\otimes}} N \to M' \mathbin{\overline{\otimes}} N'$. The normal map $f \mathbin{\overline{\otimes}} g$ extends $f \otimes g$ in the sense that the following diagram commutes.

$$\begin{array}{ccc} M \otimes N & \xrightarrow{f \otimes g} & M' \otimes N' \\ \big\downarrow & & \big\downarrow \\ M \mathbin{\overline{\otimes}} N & \xrightarrow{f \mathbin{\overline{\otimes}} g} & M' \mathbin{\overline{\otimes}} N' \end{array}$$

Proposition 2.11
For $\mathrm{X} \in \{\mathrm{MIU}, \mathrm{MI}, \mathrm{CPU}, \mathrm{CPSU}, \mathrm{CP}\}$, the category $\mathbf{Wstar}_{\mathrm{X}}$ is symmetric monoidal with the spatial W^*-tensor product $\overline{\otimes}$ and the W^*-algebra $\mathbb{C}$ of complex numbers as a unit object.

Proof If f and g are normal MI-maps, then $f \mathbin{\overline{\otimes}} g$ constructed above is MI too, since $f \mathbin{\overline{\otimes}} g$ extends $f \otimes g$, $M \otimes N$ is ultraweakly dense in $M \mathbin{\overline{\otimes}} N$, and the operations of W^*-algebras are ultraweakly continuous.[48, §1.7] It works similarly for CP-maps.[54, Proposition IV.5.13] It is easy to see that $\overline{\otimes}$ preserves (sub)unitality. To check that $(\overline{\otimes}, \mathbb{C})$ is a symmetric monoidal structure is straightforward except the associativity, for which we may rely on faithful normal representations and Theorem 2.5. ■

§3 Miscellaneous results on C^*- and W^*-algebras

Here we give several (less standard) results on C^*- and W^*-algebras, which will be needed later.

3.1 Distributivity of tensor products

We will show the distributivity of the spatial C^*- and W^*-tensor products over finite products. We do not use the distributivity for C^*-algebras in this paper, but include it for completeness. Such distributivity is very common: for example, the tensor product of vector spaces distributes over direct sums (categorically, biproducts), and the tensor product of Hilbert spaces also distributes over direct sums. These give examples of *rig categories* (or *bimonoidal categories*), categories with two monoidal structures satisfying distributivity.

For later use, we explicitly state the distributivity for vector spaces.

Lemma 3.1
Let U, V, W be a vector space. The canonical maps $\langle \mathrm{id} \odot \pi_1, \mathrm{id} \odot \pi_2 \rangle \colon U \odot (V \times W) \to (U \odot V) \times (U \odot W)$ and $[\mathrm{id} \odot \kappa_1, \mathrm{id} \odot \kappa_2] \colon (U \odot V) \times (U \odot W) \to U \odot (V \times W)$ are inverses of each other. Here $\times$ and $\odot$ denote the direct sum (biproduct) and the tensor product of vector spaces respectively. Finite direct sums are biproducts in the category of vector spaces, so that we have obvious projections π_i, coprojections κ_i, and tupling $\langle -, - \rangle$ and cotupling $[-, -]$ operations. ■

Although products of C^*-algebras are not necessarily coproducts in a category of C^*-algebras, we still use coprojections κ_i and the cotuple notation $[f, g](x, y) = f(x) + g(y)$ for finite products, which coincide with direct sums of vector spaces.

Theorem 3.1
Let A, B, C be C^*-algebras. Then the canonical maps

$$! \colon A \otimes 1 \longrightarrow 1$$

$$\langle \mathrm{id} \otimes \pi_1, \mathrm{id} \otimes \pi_2 \rangle \colon A \otimes (B \times C) \longrightarrow (A \otimes B) \times (A \otimes C)$$

are (unital) $*$-isomorphisms. Therefore, for each C^*-algebra A, a functor $A \otimes (-)$

on $\mathbf{Cstar}_{\mathrm{MIU}}$ preserves finite products.

Proof Note that $A \odot 1 \cong 1$ and the only possible norm is the trivial one. Therefore $A \otimes 1 \cong A \odot 1 \cong 1$ and the first one is proved.

We will show the latter one. Let

$$\iota_1 \colon A \odot (B \times C) \hookrightarrow A \otimes (B \times C) \ ,$$

$$\iota_2 \colon A \odot B \hookrightarrow A \otimes B \ , \qquad \iota_3 \colon A \odot C \hookrightarrow A \otimes C$$

be the canonical dense embeddings. It is easy to check that the following diagram commutes.

$$\begin{tikzcd} A \odot (B \times C) \arrow[r, "\langle \mathrm{id} \odot \pi_1, \mathrm{id} \odot \pi_2 \rangle"] \arrow[d, "\iota_1"'] & (A \odot B) \times (A \odot C) \arrow[d, "\iota_2 \times \iota_3"] \\ A \otimes (B \times C) \arrow[r, "\langle \mathrm{id} \otimes \pi_1, \mathrm{id} \otimes \pi_2 \rangle"] & (A \otimes B) \times (A \otimes C) \end{tikzcd}$$

Note that the injection maps $\kappa_1 \colon B \to B \times C$ and $\kappa_2 \colon C \to B \times C$ are CP, so that we have the spatial C^*-tensor products of maps $\mathrm{id} \otimes \kappa_1 \colon A \otimes B \to A \otimes (B \times C)$ and $\mathrm{id} \otimes \kappa_2 \colon A \otimes C \to A \otimes (B \times C)$. Then the following diagram commutes.

$$\begin{tikzcd} (A \odot B) \times (A \odot C) \arrow[r, "{[\mathrm{id} \odot \kappa_1, \mathrm{id} \odot \kappa_2]}"] \arrow[d, "\iota_2 \times \iota_3"'] & A \odot (B \times C) \arrow[d, "\iota_1"] \\ (A \otimes B) \times (A \otimes C) \arrow[r, "{[\mathrm{id} \otimes \kappa_1, \mathrm{id} \otimes \kappa_2]}"] & A \otimes (B \times C) \end{tikzcd}$$

Now, it is easy to see that $\iota_2 \times \iota_3 \colon (A \odot B) \times (A \odot C) \to (A \otimes B) \times (A \otimes C)$ has dense image. By Lemma 3.1, $\langle \mathrm{id} \odot \pi_1, \mathrm{id} \odot \pi_2 \rangle$ and $[\mathrm{id} \odot \kappa_1, \mathrm{id} \odot \kappa_2]$ are inverses of each other, and therefore the commutativity of the above two diagrams shows that $\langle \mathrm{id} \otimes \pi_1, \mathrm{id} \otimes \pi_2 \rangle$ and $[\mathrm{id} \otimes \kappa_1, \mathrm{id} \otimes \kappa_2]$ are inverses on dense subsets. Since both maps are bounded (i.e. norm-continuous), they are inverses of each other. Therefore $\langle \mathrm{id} \otimes \pi_1, \mathrm{id} \otimes \pi_2 \rangle$ is a bijective $*$-homomorphism, and hence a $*$-isomorphism. ■

The result for W^*-algebras is shown similarly, but we need the following lemma.

Lemma 3.2
Let M and N be W^*-algebras. Suppose that $M' \subseteq M$ and $N' \subseteq N$ are ultraweakly dense subsets. Then $M' \times N'$ is ultraweakly dense in the product $M \times N$.

Proof Use the fact that the predual of $M \times N$ is the ℓ^1-direct sum $M_* \oplus^1 N_*$ of the preduals. ∎

Theorem 3.2
Let M, N, L be W^*-algebras. Then the canonical maps

$$!\colon M \mathbin{\overline{\otimes}} 1 \longrightarrow 1$$

$$\langle \mathrm{id} \mathbin{\overline{\otimes}} \pi_1, \mathrm{id} \mathbin{\overline{\otimes}} \pi_2 \rangle\colon M \mathbin{\overline{\otimes}} (N \times L) \longrightarrow (M \mathbin{\overline{\otimes}} N) \times (M \mathbin{\overline{\otimes}} L)$$

are (normal unital) $*$-isomorphisms. Therefore, for each W^*-algebra M, a functor $M \mathbin{\overline{\otimes}} (-)$ on $\mathbf{Wstar}_{\mathrm{MIU}}$ preserves finite products.

Proof Note that $1_* \cong 1$. Hence $M_* \otimes 1_* \cong M_* \odot 1_* \cong 1$ and $M \mathbin{\overline{\otimes}} 1 = (M_* \otimes 1_*)^* \cong 1$, which proves the first one.

The latter is shown in the same way as the latter of Theorem 3.1, using the ultraweak density and continuity instead of the norm. Note that the canonical embedding $M \odot N \to M \mathbin{\overline{\otimes}} N$ is ultraweakly dense, and use Lemma 3.2. ∎

3.2 Results on ultraweak limits

We will show some results on the spatial W^*-tensor products and the ultraweak limits. For a net $(x_i)_i$ in a W^*-algebra, we denote the ultraweak limit of $(x_i)_i$, if exists, by $\mathrm{uwlim}_i\, x_i$.

Lemma 3.3
Let A be a finite dimensional W^*-algebra (note that $A_* = A^*$), and let M be a W^*-algebra. Then, the algebraic tensor product $A^* \odot M_*$ is already complete under the dual spatial C^*-norm. Moreover, the canonical embedding

$$A \odot M \longhookrightarrow (A^* \odot M_*)^*$$

is surjective, so that $A \odot M \cong (A^* \odot M_*)^*$. Therefore, $A \odot M$ is a W^*-algebra with the predual $A^* \odot M_*$.

Proof Fix a normalised basis $\{a_1, \dots, a_n\}$ of A. We denote its dual basis by $\{\hat{a}_1, \dots, \hat{a}_n\}$, which is a basis of A^*. Note that every element $\chi \in A^* \odot M_*$ is uniquely written as $\chi = \sum_{i=1}^n \hat{a}_i \otimes \varphi_i$. For arbitrary $x \in M$ with $\|x\| \leq 1$, we have $\|a_i \otimes x\| = \|a_i\|\|x\| = \|x\| \leq 1$ because a C^*-norm is a cross-norm. Then

$$\begin{aligned} |\varphi_i(x)| &= \Big|\Big(\sum_{i=1}^{n} \hat{a}_i \otimes \varphi_i\Big)(a_i \otimes x)\Big| \\ &= |\chi(a_i \otimes x)| \\ &\leq \sup\{|\chi(z)| \mid z \in A \otimes M, \|z\| \leq 1\} =: \|\chi\| \ , \end{aligned}$$

so that $\|\varphi_i\| \coloneqq \sup\{|\varphi_i(x)| \mid x \in M, \|x\| \leq 1\} \leq \|\chi\|$ for each i. Now, assume that $(\chi_j)_j = (\sum_{i=1}^{n} \hat{a}_i \otimes \varphi_{ij})$ is a Cauchy sequence in $A^* \odot M_*$. Because

$$\|\varphi_{ik} - \varphi_{ij}\| \leq \Big\|\sum_{i=1}^{n} \hat{a}_i \otimes (\varphi_{ik} - \varphi_{ij})\Big\| = \|\chi_k - \chi_j\| \ ,$$

$(\varphi_{ij})_j$ is a Cauchy sequence for each i. Let $\varphi_i = \lim_{j\to\infty} \varphi_{ij}$ and $\chi = \sum_{i=1}^{n} \hat{a}_i \otimes \varphi_i$. Then

$$\|\chi - \chi_j\| = \Big\|\sum_{i=1}^{n} \hat{a}_i \otimes (\varphi_i - \varphi_{ij})\Big\| \leq \sum_{i=1}^{n} \|\hat{a}_i\|\|\varphi_i - \varphi_{ij}\| \to 0 \quad \text{when } j \to \infty \ .$$

Hence $A^* \odot M_*$ is complete.

Let $\theta\colon A \odot M \to (A^* \odot M_*)^*$ be the canonical embedding. Take arbitrary $\Phi \in (A^* \odot M_*)^*$. For each i, define $\Phi_i\colon M_* \to \mathbb{C}$ by $\Phi_i(\varphi) = \Phi(\hat{a}_i \otimes \varphi)$. Clearly Φ_i is linear, and bounded because

$$\|\Phi_i(\varphi)\| = \|\Phi(\hat{a}_i \otimes \varphi)\| \leq \|\Phi\|\|\hat{a}_i \otimes \varphi\| = \|\Phi\|\|\hat{a}_i\|\|\varphi\| \ .$$

Hence $\Phi_i \in (M_*)^*$. Then we have $\theta(\sum_{i=1}^{n} a_i \otimes \iota^{-1}(\Phi_i)) = \Phi$, where $\iota\colon M \to (M_*)^*$ is the canonical isomorphism, because

$$\begin{aligned} \theta\Big(\sum_{i=1}^{n} a_i \otimes \iota^{-1}(\Phi_i)\Big)(\hat{a}_j \otimes \varphi) &= \sum_{i=1}^{n} \hat{a}_j(a_i)\varphi(\iota^{-1}(\Phi_i)) \\ &= \varphi(\iota^{-1}(\Phi_j)) \\ &= \Phi_j(\varphi) = \Phi(\hat{a}_j \otimes \varphi) \ . \end{aligned}$$ ■

Lemma 3.4

In the setting of Lemma 3.3, fix a basis $\{a_1, \dots, a_n\}$ of A. Let $(z_j)_j = (\sum_{i=1}^{n} a_i \otimes x_{ij})_j$ be a net in $A \odot M$, and let $z = \sum_{i=1}^{n} a_i \otimes x_i \in A \odot M$. Then, $z_j \to z$ ultraweakly in $A \odot M$ if and only if $x_{ij} \to x_i$ ultraweakly in M for all $i \in \{1, \dots, n\}$.

Proof Assume that $z_j \to z$ ultraweakly in $A \odot M$. It means for all $\chi \in A^* \odot M_*$ one has $\chi(z_j) \to \chi(z)$. Then for all i and for all $\varphi \in M_*$,

$$\varphi(x_{ij}) = (\hat{a}_i \otimes \varphi)\Big(\sum_{i=1}^{n} a_i \otimes x_{ij}\Big) \to (\hat{a}_i \otimes \varphi)\Big(\sum_{i=1}^{n} a_i \otimes x_i\Big) = \varphi(x_i) \ ,$$

that is, $x_{ij} \to x_i$ ultraweakly.

Conversely, assume $x_{ij} \to x_i$ ultraweakly in M for all $i \in \{1, \ldots, n\}$. Then, for $\sum_{i=1}^{n} \hat{a}_i \otimes \varphi_i \in A^* \odot M_*$,

$$\begin{aligned}
\Big|\Big(\sum_{i=1}^{n} \hat{a}_i \otimes \varphi_i\Big)(z - z_j)\Big| &= \Big|\Big(\sum_{i=1}^{n} \hat{a}_i \otimes \varphi_i\Big)\Big(\sum_{i=1}^{n} a_i \otimes (x_i - x_{ij})\Big)\Big| \\
&= \Big|\sum_{i=1}^{n} \varphi_i(x_i - x_{ij})\Big| \\
&\leq \sum_{i=1}^{n} |\varphi_i(x_i) - \varphi_i(x_{ij})| \to 0
\end{aligned}$$

because $\varphi_i(x_{ij}) \to \varphi_i(x_i)$ for all i. Hence $z_j \to z$ ultraweakly in $A \odot M$. ■

In particular, taking $A = \mathcal{M}_n$, we obtain the following result.

Corollary 3.1
Let M be a W^*-algebra. Then $\mathcal{M}_n(M)$ is also a W^*-algebra. Let $(x_j)_j = ([x_{klj}]_{kl})_j$ be a net in $\mathcal{M}_n(M)$, and let $x = [x_{kl}]_{kl} \in \mathcal{M}_n(M)$. Then, $x_j \to x$ ultraweakly in $\mathcal{M}_n(M)$ if and only if $x_{klj} \to x_{kl}$ ultraweakly in M for all $k, l \in \{1, \ldots, n\}$. In other words, one has $\mathrm{uwlim}_j [x_{klj}]_{kl} = [\mathrm{uwlim}_j\, x_{klj}]_{kl}$. ■

The following result shows the compatibility of the ultraweak limit and the W^*-tensor product.

Lemma 3.5
Let M, N be W^*-algebras. Let $x \in M$, and assume that a norm-bounded net $(y_i)_i$ converges ultraweakly to y in N. Then a net $(x \otimes y_i)_i$ converges ultraweakly to $x \otimes y$ in $M \mathbin{\overline{\otimes}} N$.

Proof Recall that

$$y_i \to y \ \text{ ultraweakly in } N \iff \forall \varphi \in N_*.\, \varphi(y_i) \to \varphi(y)$$

$$x \otimes y_i \to x \otimes y \ \text{ ultraweakly in } M \mathbin{\overline{\otimes}} N \iff \forall \xi \in (M \mathbin{\overline{\otimes}} N)_*.\, \xi(x \otimes y_i) \to \xi(x \otimes y) \ .$$

For any $\varphi \in M_*$ and $\psi \in N_*$,

$$\begin{aligned}(\varphi \otimes \psi)(x \otimes y_i) &= \varphi(x) \cdot \psi(y_i) \\ &\to \varphi(x) \cdot \psi(y) = (\varphi \otimes \psi)(x \otimes y) \ ,\end{aligned}$$

because $\psi(y_i) \to \psi(y)$. Hence we have $\chi(x \otimes y_i) \to \chi(x \otimes y)$ for all $\chi \in M_* \odot N_*$. Now, take arbitrary $\xi \in M_* \otimes N_* \cong (M \overline{\otimes} N)_*$. There exists a sequence $(\chi_j)_j$ in $M_* \odot N_*$ convergent to ξ under the dual spatial C^*-norm (therefore, we have inequality like $|\xi(z)| \le \|\xi\|\|z\|$). Then

$$\begin{aligned}&|\xi(x \otimes y) - \xi(x \otimes y_i)| \\ &\le |\xi(x \otimes y) - \chi_j(x \otimes y)| + |\chi_j(x \otimes y) - \chi_j(x \otimes y_i)| + |\chi_j(x \otimes y_i) - \xi(x \otimes y_i)| \\ &\le \|\xi - \chi_j\|\|x \otimes y\| + |\chi_j(x \otimes y) - \chi_j(x \otimes y_i)| + \|\xi - \chi_j\|\|x \otimes y_i\| \\ &= \|\xi - \chi_j\|\|x\|\|y\| + |\chi_j(x \otimes y) - \chi_j(x \otimes y_i)| + \|\xi - \chi_j\|\|x\|\|y_i\| \\ &= |\chi_j(x \otimes y) - \chi_j(x \otimes y_i)| + \|\xi - \chi_j\|\|x\|(\|y\| + \|y_i\|) \ .\end{aligned}$$

Take arbitrary $\varepsilon > 0$. Because $(y_i)_i$ is norm-bounded and $\chi_j \to \xi$, we have, for large enough j,

$$|\chi_j(x \otimes y) - \chi_j(x \otimes y_i)| + \|\xi - \chi_j\|\|x\|(\|y\| + \|y_i\|) < |\chi_j(x \otimes y) - \chi_j(x \otimes y_i)| + \varepsilon \ .$$

Since $\chi_j(x \otimes y_i) \to \chi_j(x \otimes y)$, for sufficiently large i we have

$$|\xi(x \otimes y) - \xi(x \otimes y_i)| < |\chi_j(x \otimes y) - \chi_j(x \otimes y_i)| + \varepsilon < 2\varepsilon \ .$$

This proves $\xi(x \otimes y_i) \to \xi(x \otimes y)$. Hence $x \otimes y_i$ converges ultraweakly to $x \otimes y$ in $M \overline{\otimes} N$. ■

§4 Order and domain theory in operator algebras

4.1 Recap of complete partial orders

We will briefly review the notion of complete partial orders, which plays a central role in domain theory,[1)] and is fundamental for denotational semantics of programming languages.

Definition 4.1

A poset is *directed complete* if every directed subset has a supremum; *bounded directed complete* if every directed subset that is bounded from above has a supremum; ω-*complete* if every ω-chain ($(x_n)_{n\in\mathbb{N}}$ with $x_n \leq x_{n+1}$) has a supremum; and *pointed* if it has a least element (denoted by $\perp$).

A (bounded) directed complete poset is abbreviated as a *(b)dcpo*, and an ω-complete poset as an ω*cpo*.

A *monotone net*, a net $(x_i)_i$ on a poset satisfying $i \leq j \implies x_i \leq x_j$, gives a convenient description of a directed subset.[1, §2.1.4)] Each directed subset is a monotone net indexed by itself. We use directed subsets and monotone nets interchangeably.

Definition 4.2
A map between posets is *Scott-continuous* if it preserves suprema of directed subsets; and ω-*(Scott-)continuous* if it preserves suprema of ω-chains. A map between pointed posets is *strict* if it preserves the least element.

Note that every dcpo is an ωcpo, and every Scott-continuous map is ω-continuous. The next theorem is very fundamental.

Theorem 4.1
Every ω-continuous endomap f on a pointed ωcpo has a least (pre-)fixed point, which is given by $\bigvee_n f^n(\perp)$. ■

We fix the notations of categories we use in this paper.

Definition 4.3
We denote by $\mathbf{Dcppo}_\perp$ the category of pointed dcpos and strict Scott-continuous maps, and by $\boldsymbol{\omega}\mathbf{Cppo}$ the category of pointed ωcpos and ω-continuous maps.

The *product* of posets is given by the product of the underlying sets with the coordinatewise order. They are categorical products in both $\mathbf{Dcppo}_\perp$ and $\boldsymbol{\omega}\mathbf{Cppo}$. The following fact is useful.

Lemma 4.1 ([1, Lemma 3.2.6)])
Let P, Q, R be posets. Then a map $f\colon P \times Q \to R$ is Scott-continuous (resp. ω-continuous) if and only if it is separately Scott-continuous (resp. separately ω-continuous). ■

The category $\boldsymbol{\omega}\mathbf{Cppo}$ is a cartesian closed category, and $\mathbf{Dcppo}_\perp$ is a symmetric monoidal closed category via the smash product.[1, §3.2–3)] It allows us to speak of $\boldsymbol{\omega}\mathbf{Cppo}$- and $\mathbf{Dcppo}_\perp$-enrichment of categories.[26)] In this paper we use the following explicit definition.

Definition 4.4
A category $\mathbf{C}$ is $\boldsymbol{\omega}$**Cppo**-*enriched* (resp. $\mathbf{Dcppo}_\perp$-*enriched*) if each homset $\mathbf{C}(X,Y)$ is a pointed ωcpo (resp. a pointed dcpo) and the composition $\circ\colon \mathbf{C}(Y,Z)\times \mathbf{C}(X,Y)\to\mathbf{C}(X,Z)$ is ω-continuous (resp. bi-strict Scott-continuous). Here 'bi-strict' means strictness in each variable, i.e. $f\circ\perp=\perp$ and $\perp\circ g=\perp$.

Furthermore, a monoidal structure $(\otimes, I)$ on $\mathbf{C}$ is $\boldsymbol{\omega}$**Cppo**-*enriched* (resp. $\mathbf{Dcppo}_\perp$-*enriched*) if the monoidal product $\otimes\colon \mathbf{C}(X,Y)\times\mathbf{C}(Z,W)\to\mathbf{C}(X\otimes Z, Y\otimes W)$ is ω-continuous (resp. bi-strict Scott-continuous).

We also use the following term, which comes from a general notion of enriched (conical) limits.[26, §3.8)]

Definition 4.5
Let $\mathbf{C}$ be an $\boldsymbol{\omega}$**Cppo**-enriched category (resp. a $\mathbf{Dcppo}_\perp$-enriched category). A product $\prod_i X_i$ in $\mathbf{C}$ is $\boldsymbol{\omega}$**Cppo**-*enriched* (resp. $\mathbf{Dcppo}_\perp$-*enriched*) if the canonical bijections $\mathbf{C}(Y,\prod_i X_i)\cong\prod_i\mathbf{C}(Y,X_i)$ are isomorphisms in $\boldsymbol{\omega}$**Cppo** (resp. in $\mathbf{Dcppo}_\perp$), where the right-hand side is the product of posets. In both cases, it just means that $\mathbf{C}(Y,\prod_i X_i)\cong\prod_i\mathbf{C}(Y,X_i)$ are order-isomorphisms.

4.2 Orders in operator algebras

Recall from Definition 2.7 that C^*-algebras are equipped with partial orders $\leq$ on self-adjoint elements defined by: $a\leq b \iff$ '$b-a$ is positive'. Many notions in operator algebras can be characterised by the order $\leq$. For example, it is easy to see that a linear map $f\colon A\to B$ between C^*-algebras is positive if and only if it is restricted to a monotone map $f\colon A_{\mathrm{sa}}\to B_{\mathrm{sa}}$. It turns out that the order in a W^*-algebra has a significant property, called *monotone completeness*, which distinguishes W^*-algebras from C^*-algebras.

Definition 4.6
A C^*-algebra A is *monotone complete* (or *monotone closed*) if every norm-bounded directed subset of A_{sa} has a supremum in A_{sa}.

Proposition 4.1 ([48, Lemma 1.7.4)])
Every W^*-algebra is monotone complete. Moreover, the supremum of a norm-bounded directed set of self-adjoint elements is obtained as the ultraweak limit. ■

Furthermore, the normality of positive maps between W^*-algebras is characterised as follows.

Proposition 4.2 ([11, Corollary 46.5])
Let $f\colon M \to N$ be a positive map between W^*-algebra. Then f is normal (i.e. ultraweakly continuous) if and only if it preserves the supremum of every norm-bounded directed subset of M_{sa}. ■

It is worth noting that W^*-algebras can be characterised by the monotone completeness with an additional condition.

Theorem 4.2 ([54, Theorem III.3.16])
A C^*-algebra is a W^*-algebra if and only if it is monotone complete and admits sufficiently many normal positive functionals (i.e. they separate the points). Here the normality is defined by the latter condition in Proposition 4.2. ■

We can rephrase monotone completeness in terms of domain theory.

Proposition 4.3
Let A be a C^*-algebra. The following are equivalent.

1. A is monotone complete.
2. A_{sa} is bounded directed complete.
3. A^+ is bounded directed complete.
4. $[0,1]_A$ is directed complete.

Proof Without loss of generality, we may assume directed subsets are bounded from below. Then, in the light of Lemma 2.1, norm-boundedness and order-boundedness coincide. It follows that $1 \iff 2$.

$2 \implies 3 \implies 4$ is trivial. For the converse, note that A_{sa} is an ordered $\mathbb{R}$-vector space with a (strong) order unit 1 (by Lemma 2.1). Then, a bounded directed subset of A_{sa} can be shifted into a bounded directed subset of A^+, which can be scaled into a directed subset of $[0,1]_A$. Because shifting and scaling (by a positive number) preserve suprema, the converse follows. ■

For any W^*-algebra M, therefore, M_{sa} is a bdcpo; M^+ is a pointed bdcpo; and $[0,1]_M$ is a pointed dcpo. We have a corresponding result for normal maps, which is proved in a similar 'shifting and scaling' argument using Proposition 4.2.

Proposition 4.4
Let $f\colon M \to N$ be a positive map between W^*-algebras. The following are equivalent.

1. f is normal.

2. The restriction $f|_{M_{\mathrm{sa}}}\colon M_{\mathrm{sa}} \to N_{\mathrm{sa}}$ is Scott-continuous.
3. The restriction $f|_{M^+}\colon M^+ \to N^+$ is Scott-continuous.

Moreover, a positive subunital map $f\colon M \to N$ between W^*-algebras is normal if and only if the restriction $f|_{[0,1]_M}\colon [0,1]_M \to [0,1]_N$ is Scott-continuous. ■

4.3 $\mathbf{Dcppo}_\perp$-enrichment of the category of W^*-algebras

The goal of this subsection is to show that the category $\mathbf{Wstar}_{\mathrm{CPSU}}$ is $\mathbf{Dcppo}_\perp$-enriched. We also see that products and the monoidal structure $(\overline{\otimes}, \mathbb{C})$ on $\mathbf{Wstar}_{\mathrm{CPSU}}$ are $\mathbf{Dcppo}_\perp$-enriched.

Definition 4.7
Let M, N be W^*-algebras. We define a partial order $\sqsubseteq$ on $\mathbf{Wstar}_{\mathrm{CPSU}}(M,N)$ by

$$f \sqsubseteq g \overset{\text{def}}{\iff} g - f \text{ is completely positive} \ .$$

We use the following lemma.

Lemma 4.2
Let A and B be C^*-algebras. Any PSU-map $f\colon A \to B$ can be restricted to a map $f\colon [0,1]_A \to [0,1]_B$ between their effects such that $f(0) = 0$; $f(x+y) = f(x) + f(y)$ for all $x, y \in [0,1]_A$ with $x + y \le 1$; and $f(rx) = rf(x)$ for all $x \in [0,1]_A$ and $r \in [0,1]$. Conversely, any such map $f\colon [0,1]_A \to [0,1]_B$ can be extended to a PSU-map $f\colon A \to B$ uniquely.

Proof The first statement is straightforward. Let $f\colon [0,1]_A \to [0,1]_B$ be a map satisfying the conditions. By Lemma 2.1, one has $x \le \|x\|1$ and hence $x/\|x\| \le 1$, Then define $f\colon A^+ \to B^+$ by $f(x) = \|x\| f(x/\|x\|)$, which satisfies $f(0) = 0$, $f(x+y) = f(x) + f(y)$, and $f(rx) = rf(x)$ for all $x, y \in A^+$ and $r \in \mathbb{R}^+$. It is standard that the map $f\colon A^+ \to B^+$ extends to a positive linear map $f\colon A \to B$, using the fact that any element of a C^*-algebra can be written as a linear combination of positive elements.[54, §I.4] The map is subunital since $f(1) \in [0,1]_B$. It is easy to see that this extension is unique. ■

Note in particular that a PSU-map $f\colon A \to B$ is determined by the values on $[0,1]_A$.

Proposition 4.5
For any W^*-algebras M and N, $\mathbf{Wstar}_{\mathrm{CPSU}}(M,N)$ with the order $\sqsubseteq$ is a pointed

dcpo.

Proof First of all, it is easy to see that $\mathbf{Wstar}_{\mathrm{CPSU}}(M,N)$ is a pointed poset; the zero map is a least element. We will show the direct completeness.

Let $(f_i)_i$ be a monotone net in $\mathbf{Wstar}_{\mathrm{CPSU}}(M,N)$. For each $x \in [0,1]_M$, $(f_i(x))_i$ is a monotone net in a dcpo $[0,1]_N$. Hence we define $f(x) \coloneqq \sup_i f_i(x)$ for $x \in [0,1]_M$. It is easy to see that $f\colon [0,1]_M \to [0,1]_N$ satisfies the conditions of Lemma 4.2, so that we obtain a PSU-map $f\colon M \to N$.

Each map $f_i\colon M \to N$ is normal, therefore by Proposition 4.4, the restriction $f_i\colon [0,1]_M \to [0,1]_N$ is Scott-continuous. Then, for a monotone net $(x_j)_j$ in $[0,1]_M$,

$$f\Big(\sup_j x_j\Big) = \sup_i f_i\Big(\sup_j x_j\Big) = \sup_i\Big(\sup_j f_i(x_j)\Big) = \sup_j\Big(\sup_i f_i(x_j)\Big) = \sup_j f(x_j) \enspace .$$

Hence $f\colon [0,1]_M \to [0,1]_N$ is Scott-continuous, i.e. $f\colon M \to N$ is normal.

Note that $f(x) \coloneqq \sup_i f_i(x) = \operatorname{uwlim}_i f_i(x)$ for all $x \in [0,1]_M$ by Proposition 4.1. Recall from the proof of Lemma 4.2 that any $x \in M$ can be decomposed to a (finite) linear combination $x = \sum_j \lambda_j x_j$ with $x_j \in [0,1]_M$. Then

$$\begin{aligned} f(x) &= \sum\nolimits_j \lambda_j f(x_j) \\ &= \sum\nolimits_j \lambda_j \operatorname{uwlim}_i f_i(x_j) \\ &\overset{\star}{=} \operatorname{uwlim}_i \sum\nolimits_j \lambda_j f_i(x_j) \\ &= \operatorname{uwlim}_i f_i(x) \enspace , \end{aligned}$$

where the equality $\overset{\star}{=}$ holds by the ultraweak continuity of the addition and the scalar multiplication. Therefore we have $f(x) = \operatorname{uwlim}_i f_i(x)$ for all $x \in M$.

Finally we show that f is CP, and that f is indeed a supremum of $(f_i)_i$. For $[x_{kl}]_{kl} \in \mathcal{M}_n(M)^+$,

$$\begin{aligned} \mathcal{M}_n(f)([x_{kl}]_{kl}) &= [f(x_{kl})]_{kl} \\ &= [\operatorname{uwlim}_i f_i(x_{kl})]_{kl} \\ &= \operatorname{uwlim}_i [f_i(x_{kl})]_{kl} && \text{by Corollary 3.1} \\ &= \operatorname{uwlim}_i \mathcal{M}_n(f_i)([x_{kl}]_{kl}) \enspace . \end{aligned}$$

Note that f_i is CP, so that $\mathcal{M}_n(f_i)([x_{kl}]_{kl})$ is positive for all i. Moreover, $f_i \sqsubseteq f_j$ implies $\mathcal{M}_n(f_i)([x_{kl}]_{kl}) \le \mathcal{M}_n(f_j)([x_{kl}]_{kl})$. Hence $(\mathcal{M}_n(f_i)([x_{kl}]_{kl}))_i$ is a positive monotone net in $\mathcal{M}_n(N)$, which is bounded because each f_i is subunital and so is $\mathcal{M}_n(f_i)$. By Proposition 4.1 we obtain

$$\mathcal{M}_n(f)([x_{kl}]_{kl}) = \mathrm{uwlim}_i\, \mathcal{M}_n(f_i)([x_{kl}]_{kl}) = \sup_i \mathcal{M}_n(f_i)([x_{kl}]_{kl}) \ .$$

Thus $\mathcal{M}_n(f)([x_{kl}]_{kl}) \geq \mathcal{M}_n(f_i)([x_{kl}]_{kl}) \geq 0$, so that f is CP and $f_i \sqsubseteq f$ for all i. Let $f' \in \mathbf{Wstar}_{\mathrm{CPSU}}(M,N)$ with $f_i \sqsubseteq f'$ for all i. Then, for $[x_{kl}]_{kl} \in \mathcal{M}_n(M)^+$, we have $\mathcal{M}_n(f_i)([x_{kl}]_{kl}) \leq \mathcal{M}_n(f')([x_{kl}]_{kl})$ for all i. Hence $\mathcal{M}_n(f)([x_{kl}]_{kl}) = \sup_i \mathcal{M}_n(f_i)([x_{kl}]_{kl}) \leq \mathcal{M}_n(f')([x_{kl}]_{kl})$. It follows that $f \sqsubseteq f'$. ■

We denote the supremum of $(f_i)_i$ by $\bigsqcup_i f_i$. As shown in the proof, one has $(\bigsqcup_i f_i)(x) = \mathrm{uwlim}_i\, f_i(x)$ ($= \sup_i f_i(x)$ for $x \in M^+$). Next, we show that the composition in $\mathbf{Wstar}_{\mathrm{CPSU}}$ has a desired property.

Proposition 4.6
Let M, N, L be W^*-algebras. The composition

$$\circ\colon \mathbf{Wstar}_{\mathrm{CPSU}}(N,L) \times \mathbf{Wstar}_{\mathrm{CPSU}}(M,N) \longrightarrow \mathbf{Wstar}_{\mathrm{CPSU}}(M,L)$$

is bi-strict Scott-continuous.

Proof The bi-strictness is obvious because bottom maps $\perp$ are zero maps. We show the Scott-continuity separately in each variable (Lemma 4.1). Let $(g_i)_i$ be a monotone net in $\mathbf{Wstar}_{\mathrm{CPSU}}(N,L)$ and let $f \in \mathbf{Wstar}_{\mathrm{CPSU}}(M,N)$. It is easy to see $(-) \circ f$ is monotone, and hence $(g_i \circ f)_i$ is a monotone net in $\mathbf{Wstar}_{\mathrm{CPSU}}(M,L)$. Then for each $x \in [0,1]_M$,

$$\Big(\big(\bigsqcup_i g_i\big)\circ f\Big)(x) = \big(\bigsqcup_i g_i\big)(f(x)) = \sup_i g_i(f(x)) = \sup_i (g_i \circ f)(x) = \big(\bigsqcup_i (g_i \circ f)\big)(x)$$

so that $(\bigsqcup_i g_i) \circ f = \bigsqcup_i (g_i \circ f)$.

Let $(f_i)_i$ be a monotone net in $\mathbf{Wstar}_{\mathrm{CPSU}}(M,N)$ and let $g \in \mathbf{Wstar}_{\mathrm{CPSU}}(N,L)$. It is easy to see $g \circ (-)$ is monotone, and hence $(g \circ f_i)_i$ is a monotone net in $\mathbf{Wstar}_{\mathrm{CPSU}}(M,L)$. Then for each $x \in [0,1]_M$,

$$\big(g \circ \big(\bigsqcup_i f_i\big)\big)(x) = g(\sup_i f_i(x)) = \sup_i g(f_i(x)) = \sup_i (g \circ f_i)(x) = \big(\bigsqcup_i (g \circ f_i)\big)(x) \ ,$$

where we used the normality of g (and Proposition 4.2) for the second equality. It shows $g \circ (\bigsqcup_i f_i) = \bigsqcup_i (g \circ f_i)$. ■

Therefore, we proved:

Theorem 4.3
The category $\mathbf{Wstar}_{\mathrm{CPSU}}$ is $\mathbf{Dcppo}_\perp$-enriched. ■

We furthermore show that products and the monoidal product $(\overline{\otimes}, \mathbb{C})$ are $\mathbf{Dcppo}_\perp$-enriched.

Theorem 4.4
Products (i.e. products) in $\mathbf{Wstar}_{\mathrm{CPSU}}$ are $\mathbf{Dcppo}_\perp$-enriched.

Proof It is straightforward to see that the canonical bijections

$$\mathbf{Wstar}_{\mathrm{CPSU}}\big(N, \prod_i M_i\big) \cong \prod_i \mathbf{Wstar}_{\mathrm{CPSU}}(N, M_i)$$

are order isomorphisms. ■

Theorem 4.5
The monoidal structure $(\overline{\otimes}, \mathbb{C})$ on $\mathbf{Wstar}_{\mathrm{CPSU}}$ is $\mathbf{Dcppo}_\perp$-enriched. Namely, for W^*-algebras M, M', N, N', the map

$$\overline{\otimes}\colon \mathbf{Wstar}_{\mathrm{CPSU}}(M, M') \times \mathbf{Wstar}_{\mathrm{CPSU}}(N, N') \longrightarrow \mathbf{Wstar}_{\mathrm{CPSU}}(M \overline{\otimes} N, M' \overline{\otimes} N')$$

is bi-strict Scott-continuous.

Proof By Lemma 4.1 and the symmetry, it suffices to show that, for $f \in \mathbf{Wstar}_{\mathrm{CPSU}}(M, M')$,

$$f \overline{\otimes}(-)\colon \mathbf{Wstar}_{\mathrm{CPSU}}(N, N') \to \mathbf{Wstar}_{\mathrm{CPSU}}(M \overline{\otimes} N, M' \overline{\otimes} N')$$

is strict Scott-continuous. Let $\perp \in \mathbf{Wstar}_{\mathrm{CPSU}}(N, N')$ be the least element, i.e. the zero map. Then

$$(f \overline{\otimes} \perp)(x \otimes y) = f(x) \otimes \perp(y) = f(x) \otimes 0 = 0$$

for all $x \in M, y \in N$. Hence $(f \overline{\otimes} \perp)(z) = 0$ for all $z \in M \odot N$. Because $M \odot N$ is ultraweakly dense in $M \overline{\otimes} N$ and $f \overline{\otimes} \perp$ is normal (i.e. ultraweakly continuous), we obtain $(f \overline{\otimes} \perp)(z) = 0$ for all $z \in M \overline{\otimes} N$. Therefore $f \overline{\otimes} \perp = \perp$.

Let $g, g' \in \mathbf{Wstar}_{\mathrm{CPSU}}(N, N')$ with $g \sqsubseteq g'$. By definition $g' - g$ is completely positive, and so is $f \overline{\otimes}(g' - g)$. Notice that $f \overline{\otimes}(g' - g) = f \overline{\otimes} g' - f \overline{\otimes} g$ because they coincide on $M \odot N$. Then $f \overline{\otimes} g' - f \overline{\otimes} g$ is completely positive, and $f \overline{\otimes} g \sqsubseteq f \overline{\otimes} g'$. Hence $f \overline{\otimes}(-)$ is monotone.

Let $(g_i)_i$ be a monotone net in $\mathbf{Wstar}_{\mathrm{CPSU}}(N, N')$. By the monotonicity, $(f \overline{\otimes} g_i)_i$ is a monotone net in $\mathbf{Wstar}_{\mathrm{CPSU}}(M \overline{\otimes} N, M' \overline{\otimes} N')$. By a similar argument to above, to prove $f \overline{\otimes}(\bigsqcup_i g_i) = \bigsqcup_i (f \overline{\otimes} g_i)$, it suffice to show $(f \overline{\otimes}(\bigsqcup_i g_i))(x \otimes y) = (\bigsqcup_i f \overline{\otimes} g_i)(x \otimes y)$ for all $x \in M, y \in N$. This is shown as follows.

$$
\begin{aligned}
\bigl(f \mathbin{\overline{\otimes}} (\textstyle\bigsqcup_i g_i)\bigr)(x \otimes y) &= f(x) \mathbin{\overline{\otimes}} \bigl(\textstyle\bigsqcup_i g_i\bigr)(y) \\
&= f(x) \otimes (\mathrm{uwlim}_i\, g_i(y)) \\
&= \mathrm{uwlim}_i (f(x) \otimes g_i(y)) && \text{by Lemma 3.5} \\
&= \mathrm{uwlim}_i (f \mathbin{\overline{\otimes}} g_i)(x \otimes y) \\
&= \bigl(\textstyle\bigsqcup_i f \mathbin{\overline{\otimes}} g_i\bigr)(x \otimes y) && \blacksquare
\end{aligned}
$$

Remark 4.1
It is worth noting that $\mathbf{Cstar}_{\mathrm{CPSU}}$ is not a $\mathbf{Dcppo}_{\perp}$-enriched category, nor an $\boldsymbol{\omega}\mathbf{Cppo}$-enriched category. We have an order-isomorphism $\mathbf{Cstar}_{\mathrm{CPSU}}(\mathbb{C}, A) \cong [0,1]_A$, whereas there is a C^*-algebra A such that $[0,1]_A$ is not even ω-complete; consider $A = C([0,1])$ for instance.

Remark 4.2
The order $\sqsubseteq$ defined in Definition 4.7 does not agree with the order $f \sqsubseteq' g \overset{\text{def}}{\iff} g - f$ is positive. It happens that the difference of two completely positive maps is positive but not completely positive.[59] The order $\sqsubseteq'$ still works well for $\mathbf{Wstar}_{\mathrm{PSU}}$, which turns out to be $\mathbf{Dcppo}_{\perp}$-enriched too.[45]

Remark 4.3
One can define a *partial sum* $\ovee$ on the homset $\mathbf{Wstar}_{\mathrm{CPSU}}(M, N)$ by '$f \ovee g$ is defined' if $f + g \in \mathbf{Wstar}_{\mathrm{CPSU}}(M, N)$ ($\iff f + g$ is subunital $\iff f(1) + g(1) \leq 1$), and in that case $f \ovee g \coloneqq f + g$. It is straightforward to see that $\mathbf{Wstar}_{\mathrm{CPSU}}(M, N)$ is a *generalised effect algebra* with this partial sum $\ovee$, and the order $\sqsubseteq$ coincides with the canonical order $\leq$ in a generalised effect algebra: $f \leq g \overset{\text{def}}{\iff} \exists h.\, f \ovee h = g$. The fact that the order $\sqsubseteq$ is directed complete allows us to define the infinite partial sum as the supremum of finite sums. It then turns out that $(\mathbf{Wstar}_{\mathrm{CPSU}})^{\mathrm{op}}$ is *partially additive* in the sense of Arbib and Manes.[3, 32]

§5 Quantum operations

In this section we discuss quantum operations, which are now a fundamental notion in quantum theory, and has a close connection with operator algebras.

Recall that for a Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$, i.e. the set of bounded operators on $\mathcal{H}$, is a W^*-algebra with the predual $\mathcal{T}(\mathcal{H})$, i.e. the set of trace class operators on $\mathcal{H}$. For every normal map $f\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$, there exists a corresponding bounded map $f_*\colon \mathcal{T}(\mathcal{K}) \to \mathcal{T}(\mathcal{H})$ between preduals. Since the duality $\mathcal{B}(\mathcal{H}) \cong \mathcal{T}(\mathcal{H})^*$ is given by $S \mapsto \mathrm{tr}(S(-))$, we have the following equation that relates f and f_*

$$\mathrm{tr}(f(S) \cdot T) = \mathrm{tr}(S \cdot f_*(T)) \tag{1}$$

for all $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{T}(\mathcal{K})$. Although $\mathcal{T}(\mathcal{H})$ is not a C^*-algebra in general, we still have the notion of positivity of operators (i.e. positivity in $\mathcal{B}(\mathcal{H})$), so that we can define positivity of maps $f_*\colon \mathcal{T}(\mathcal{K}) \to \mathcal{T}(\mathcal{H})$. We can also define complete positivity via the identification $\mathcal{M}_n(\mathcal{T}(\mathcal{H})) \cong \mathcal{M}_n \odot \mathcal{T}(\mathcal{H}) \cong \mathcal{T}(\mathbb{C}^n \otimes \mathcal{H})$. We write $\mathcal{T}(\mathcal{H})^+ = \mathcal{B}(\mathcal{H})^+ \cap \mathcal{T}(\mathcal{H})$ for the set of positive trace class operators.

Proposition 5.1 below was essentially already shown in Kraus's early work on quantum operations,[29, §2] and also found in Heinosaari and Ziman's book[23, §4.1.2] (although both books assume separability of Hilbert spaces). For completeness and for later reference, we include a proof.

Lemma 5.1
Let $\mathcal{H}$ be a Hilbert space. A bounded operator $S \in \mathcal{B}(\mathcal{H})$ is positive if and only if $\mathrm{tr}(ST) \in \mathbb{R}^+$ for all $T \in \mathcal{T}(\mathcal{H})^+$. ■

Proposition 5.1
In the correspondence between normal maps $f\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ and bounded maps $f_*\colon \mathcal{T}(\mathcal{K}) \to \mathcal{T}(\mathcal{H})$, f is positive (resp. CP) if and only if f_* is positive (resp. CP). In that case, moreover one has:

1. f is subunital if and only if f_* is trace-nonincreasing, i.e. $\mathrm{tr}(f_*(T)) \leq \mathrm{tr}(T)$ for all $T \in \mathcal{T}(\mathcal{K})^+$.
2. f is unital if and only if f_* is trace-preserving, i.e. $\mathrm{tr}(f_*(T)) = \mathrm{tr}(T)$ for all $T \in \mathcal{T}(\mathcal{K})$.

Proof The first part follows easily from the equation (1) and Lemma 5.1. To see 1, note that

$$\mathrm{tr}(T) - \mathrm{tr}(f_*(T)) = \mathrm{tr}(T) - \mathrm{tr}(f(1) \cdot T) = \mathrm{tr}((1 - f(1)) \cdot T) \ .$$

By Lemma 5.1, therefore, $\mathrm{tr}(f_*(T)) \leq \mathrm{tr}(T)$ for all $T \in \mathcal{T}(\mathcal{H})^+$ if and only if $1 - f(1)$ is positive, i.e. f is subunital. We can show 2 similarly. ■

Therefore, there is a bijective correspondence between normal CPSU-maps (resp. normal CPU-maps) $f\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ and CP trace-nonincreasing maps (resp. CP trace-preserving maps) $f_*\colon \mathcal{T}(\mathcal{K}) \to \mathcal{T}(\mathcal{H})$.[*2] Such maps are physically meaningful "operations",[29)] and widely used in quantum theory; see e.g. the textbooks by Nielsen and Chuang[38, §8.2)] and by Heinosaari and Ziman.[23, Chapter 4)]

Definition 5.1
Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. A CP trace-nonincreasing (resp. trace-preserving) map $\mathcal{E}\colon \mathcal{T}(\mathcal{K}) \to \mathcal{T}(\mathcal{H})$ is called a *quantum operation* (resp. *quantum channel*).

The correspondence of $f\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ and $f_*\colon \mathcal{T}(\mathcal{K}) \to \mathcal{T}(\mathcal{H})$ is understood as the well-known duality between the Heisenberg and Schrödinger picture: f transforms observables (i.e. self-adjoint operators), while f_* transforms states (i.e. density operators).

In the context of quantum computation, finite dimensional Hilbert spaces $\mathbb{C}^n$ are often concerned. In that case, the situation is much simpler, since $\mathcal{B}(\mathbb{C}^n) = \mathcal{T}(\mathbb{C}^n) \cong \mathcal{M}_n$ and maps are always continuous. Quantum operations are CP trace-preserving maps $\mathcal{E}\colon \mathcal{M}_m \to \mathcal{M}_n$, which are in bijective correspondence with CPSU-maps $\mathcal{E}^*\colon \mathcal{M}_n \to \mathcal{M}_m$.

§6 Selinger's QFC and its semantics

Quantum flow chart, or QFC, is a first-order functional quantum programming language equipped with loop and recursion, proposed by Selinger.[50)] In this section we only review the semantics for this language; for the other details we refer to the original paper.[50)]

Selinger gave a denotational semantics of QFC by the category $\mathbf{Q}$, which is described in what follows.

Definition 6.1
For $n \in \mathbb{N}$, let $\mathcal{M}_n$ denote the algebra of complex $n \times n$ matrices. The category $\mathbf{CPM}_s$ is defined as follows.

- An object is a natural number.
- An arrow $f\colon m \to n$ is a CP-map $f\colon \mathcal{M}_m \to \mathcal{M}_n$.

The category $\mathbf{CPM}$ is the finite biproduct completion of $\mathbf{CPM}_s$. Specifically:

- An object is a sequence $\vec{n} = (n_1, \dots, n_k)$ of natural numbers.
- An arrow $f\colon \vec{m} \to \vec{n}$, say $\vec{m} = (m_1, \dots, m_l)$ and $\vec{n} = (n_1, \dots, n_k)$, is a $l \times k$ matrix $[f_{ij}]_{ij}$ of arrows $f_{ij}\colon m_j \to n_i$ in $\mathbf{CPM}_s$, i.e. CP-maps

[*2] Positive maps $\mathcal{T}(\mathcal{K}) \to \mathcal{T}(\mathcal{H})$ are always bounded w.r.t. trace norm.[13, Lemma 2.1)]

$$f_{ij}\colon \mathcal{M}_{m_j} \to \mathcal{M}_{n_i} \ (i = 1, \dots, l \text{ and } j = 1, \dots, k).$$

Note that matrices $(f_{ij}\colon \mathcal{M}_{m_j} \to \mathcal{M}_{n_i})_{ij}$ are in bijective correspondence with maps $f\colon \prod_j \mathcal{M}_{m_j} \to \prod_i \mathcal{M}_{n_i}$ between (algebraic) products, which are biproducts in the category of vector spaces. We define the *trace* of a tuple of matrices $(A_j)_i \in \prod_j \mathcal{M}_{n_i}$ to be the sum of traces: $\mathrm{tr}((A_i)_i) = \sum_i \mathrm{tr}(A_i)$. Then, we say a map $f\colon \prod_j \mathcal{M}_{m_j} \to \prod_i \mathcal{M}_{n_i}$ in **CPM** is *trace-nonincreasing* if $\mathrm{tr}(f((A_j)_j)) \leq \mathrm{tr}((A_j)_j)$ for all (coordinatewise) positive $(A_j)_j \in \prod_j \mathcal{M}_{m_j}$. More explicitly, $f = [f_{ij}]_{ij}$ is trace-nonincreasing if $\sum_{ij} \mathrm{tr}(f_{ij}(A_j)) \leq \sum_j \mathrm{tr}(A_j)$.

Definition 6.2
The category **Q** is the subcategory of **CPM** containing all the objects, but only trace-nonincreasing maps.

Arrows $f\colon \vec{m} \to \vec{n}$ in **Q** are precisely quantum operations $f\colon \mathcal{M}_m \to \mathcal{M}_n$ when $\vec{m} = (m)$ and $\vec{n} = (n)$, i.e. their lengths are 1. Therefore arrows in **Q** can be understood as "generalised" quantum operations.

Selinger showed that the category **Q** has enough structures to give a denotational semantics for QFC; that is, each quantum flow chart can be interpreted as an arrow in **Q**.[50, §6.5] Furthermore, he axiomatised a category that gives semantics for QFC.

Definition 6.3 (Selinger[50, §6.6])
An *elementary quantum flow chart category* is a symmetric monoidal category $(\mathbf{C}, \otimes, I)$ with traced finite coproducts $(+, 0, \mathrm{Tr})$ such that:

- for each $A \in \mathbf{C}$, $A \otimes (-)$ is a traced monoidal functor;
- there is a distinguished object $\mathbf{qbit} \in \mathbf{C}$ with arrows $\iota\colon I + I \to \mathbf{qbit}$ and $p\colon \mathbf{qbit} \to I + I$ such that $p \circ \iota = \mathrm{id}$.

Theorem 6.1 (Selinger[50, §6.6])
Let $\mathbf{C}$ be an elementary quantum flow chart category. Let η be an assignment of an arrow $\eta_S\colon \mathbf{qbit}^{\otimes n} \to \mathbf{qbit}^{\otimes n}$ in $\mathbf{C}$ to each built-in n-ary operator symbol S. Then we have an interpretation $[\![-]\!]_\eta$ of quantum flow charts without recursion in $\mathbf{C}$, mapping each quantum flow chart X to an arrow $[\![X]\!]_\eta$ in $\mathbf{C}$. If $\mathbf{C}$ is additionally $\boldsymbol{\omega}\mathbf{Cppo}$-enriched, then quantum flow charts with recursion can be interpreted in $\mathbf{C}$. ∎

The category **Q** is, of course, an example of an elementary quantum flow chart category which is also $\boldsymbol{\omega}\mathbf{Cppo}$-enriched. In fact, Selinger first showed that

$\mathbf{Q}$ is $\boldsymbol{\omega}\mathbf{Cppo}$-enriched, and then constructed a trace Tr for coproducts using the $\boldsymbol{\omega}\mathbf{Cppo}$-enrichment. As Selinger mentioned[50, §6.4)] (though he did not give a proof), the construction of a trace from the $\boldsymbol{\omega}\mathbf{Cppo}$-enrichment works in the general case. Specifically, we have the following theorem.

Theorem 6.2

1. Every $\boldsymbol{\omega}\mathbf{Cppo}$-enriched cocartesian category with right-strict composition (i.e. $f \circ \bot = \bot$) is traced.
2. Let $\mathbf{C}$ and $\mathbf{D}$ be $\boldsymbol{\omega}\mathbf{Cppo}$-enriched cocartesian categories with right-strict composition, which are traced by 1. Every $\boldsymbol{\omega}\mathbf{Cppo}$-enriched cocartesian functor between $\mathbf{C}$ and $\mathbf{D}$ satisfying $F\bot = \bot$ is traced. ∎

Here, a cocartesian category refers to a monoidal category whose monoidal structure is given by finite coproducts. For the sake of completeness, the proofs are included in Appendix. To summarise, the following is sufficient to obtain an $\boldsymbol{\omega}\mathbf{Cppo}$-enriched elementary quantum flow chart category.

Theorem 6.3

A category is an $\boldsymbol{\omega}\mathbf{Cppo}$-enriched elementary quantum flow chart category if it is an $\boldsymbol{\omega}\mathbf{Cppo}$-enriched symmetric monoidal category $(\mathbf{C}, \otimes, I)$ with $\boldsymbol{\omega}\mathbf{Cppo}$-enriched finite coproducts $(+, 0)$ such that:

- the composition is right-strict (i.e. $f \circ \bot = \bot$);
- for each $A \in \mathbf{C}$, a functor $A \otimes (-)$ preserves finite coproducts and bottom arrows;
- $\mathbf{C}$ has a distinguished object $\mathbf{qbit}$ with arrows $\iota\colon I + I \to \mathbf{qbit}$ and $p\colon \mathbf{qbit} \to I + I$ such that $p \circ \iota = \mathrm{id}$. ∎

§7 Semantics for QFC by operator algebras

7.1 $(\mathbf{Wstar}_{\mathrm{CPSU}})^{\mathrm{op}}$ is an elementary quantum flow chart category

We have proved that $\mathbf{Wstar}_{\mathrm{CPSU}}$ is an $\mathbf{Dcppo}_{\bot}$-enriched symmetric monoidal category with $\mathbf{Dcppo}_{\bot}$-enriched products (Theorems 4.3, 4.4 and 4.5). Moreover, the monoidal product distributes over finite products (Theorem 3.2). In the light of Theorem 6.3, we have almost already shown that the opposite category $(\mathbf{Wstar}_{\mathrm{CPSU}})^{\mathrm{op}}$ is an $\boldsymbol{\omega}\mathbf{Cppo}$-enriched elementary quantum flow chart category.[*3] What remains is to give a distinguished object **qbit** with arrows ι, p.

Not surprisingly, we take $\mathbf{qbit} := \mathcal{M}_2$, the algebra of complex 2×2-matrices. We define two maps $\iota\colon \mathbb{C} \times \mathbb{C} \to \mathcal{M}_2$ and $p\colon \mathcal{M}_2 \to \mathbb{C} \times \mathbb{C}$ in $(\mathbf{Wstar}_{\mathrm{CPSU}})^{\mathrm{op}}$, i.e. $\iota\colon \mathcal{M}_2 \to \mathbb{C} \times \mathbb{C}$ and $p\colon \mathbb{C} \times \mathbb{C} \to \mathcal{M}_2$ in $\mathbf{Wstar}_{\mathrm{CPSU}}$ by

$$\iota\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) = (x, w) \ , \qquad\qquad p(x, y) = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \ .$$

It is straightforward to check that the two maps are positive, hence CP by Proposition 2.5. They are clearly unital, and automatically normal because they are finite dimensional. Therefore ι and p are indeed maps in $\mathbf{Wstar}_{\mathrm{CPSU}}$. It is clear that $\iota \circ p = \mathrm{id}$, hence $p \circ \iota = \mathrm{id}$ in $(\mathbf{Wstar}_{\mathrm{CPSU}})^{\mathrm{op}}$. Now, we showed:

Theorem 7.1
The opposite category of $\mathbf{Wstar}_{\mathrm{CPSU}}$ is an $\boldsymbol{\omega}\mathbf{Cppo}$-enriched elementary quantum flow chart category with $\mathbf{qbit} = \mathcal{M}_2$. ■

Moreover, we have all unitary operators in $\mathbf{Wstar}_{\mathrm{CPSU}}$. Let U be an n-ary unitary operator, i.e. a $2^n \times 2^n$ unitary matrix. We assign an arrow $\eta_U\colon (\mathcal{M}_2)^{\overline{\otimes}\, n} \to (\mathcal{M}_2)^{\overline{\otimes}\, n}$ in $\mathbf{Wstar}_{\mathrm{CPSU}}$ by:

$$(\mathcal{M}_2)^{\overline{\otimes}\, n} \cong \mathcal{M}_{2^n} \longrightarrow \mathcal{M}_{2^n} \cong (\mathcal{M}_2)^{\overline{\otimes}\, n} \ , \qquad A \longmapsto U^\dagger A U \ .$$

Therefore, by Theorem 6.1, quantum flow charts with recursion, in which any unitary operators may be used, can be interpreted in $(\mathbf{Wstar}_{\mathrm{CPSU}})^{\mathrm{op}}$.

7.2 Revisit of Selinger's original semantics

We here study Selinger's category $\mathbf{Q}$ from an operator algebraic point of view. Recall from §5 that there is a bijective correspondence between CP-maps $f\colon \mathcal{M}_n \to \mathcal{M}_m$ and $g\colon \mathcal{M}_m \to \mathcal{M}_n$, when we consider finite dimensional Hilbert spaces. Namely $\mathbf{CPM}_s(n, m) \cong \mathbf{CPM}_s(m, n)$, and we have the categorical self-duality $(\mathbf{CPM}_s)^{\mathrm{op}} \cong \mathbf{CPM}_s$. It easily extends to $\mathbf{CPM}^{\mathrm{op}} \cong \mathbf{CPM}$: for $f = [f_{ji}]_{ji}\colon \vec{n} \to \vec{m}$, take $g = [g_{ij}]_{ij}\colon \vec{m} \to \vec{n}$ where $g_{ij}\colon m_j \to n_i$ is the arrow corresponding to $f_{ji}\colon n_i \to m_j$.

Recall again from §5 that the corresponding maps $f_{ji}\colon \mathcal{M}_{n_i} \to \mathcal{M}_{m_j}$ and $g_{ij}\colon \mathcal{M}_{m_j} \to \mathcal{M}_{n_i}$ are related via trace, by the equation (1). It is straightforward

[*3] Note that the $\mathbf{Dcppo}_\perp$-enrichment implies the $\boldsymbol{\omega}\mathbf{Cppo}$-enrichment.

to see that the maps $f = [f_{ji}]_{ji} \colon \prod_i \mathcal{M}_{n_i} \to \prod_j \mathcal{M}_{m_j}$ and $g = [g_{ij}]_{ij} \colon \prod_j \mathcal{M}_{m_j} \to \prod_i \mathcal{M}_{n_i}$ in correspondence are also related via (extended) trace:

$$\mathrm{tr}\big(f((A_i)_i) \cdot (B_j)_j\big) = \mathrm{tr}\big((A_i)_i \cdot g((B_j)_j)\big) \tag{2}$$

for $(A_i)_i \in \prod_i \mathcal{M}_{n_i}$ and $(B_j)_j \in \prod_j \mathcal{M}_{m_j}$. Here the trace is defined as the sum of traces of coordinates, as in §6, and the multiplication is coordinatewise: $(A_i)_i \cdot (A'_i)_i = (A_i \cdot A'_i)_i$.

We can easily generalise Lemma 5.1 to the current situation. Note that the positivity here is the coordinatewise positivity.

Lemma 7.1

A tuple of matrices $(A_i)_i \in \prod_i \mathcal{M}_{n_i}$ is positive if and only if $\mathrm{tr}((A_i)_i \cdot (B_i)_i) \in \mathbb{R}^+$ for all positive $(B_i)_i \in \prod_i \mathcal{M}_{n_i}$. ■

Using this lemma and the equation (2), we obtain the following result by a similar reasoning to Proposition 5.1.

Proposition 7.1

Let $f \colon \prod_i \mathcal{M}_{n_i} \to \prod_j \mathcal{M}_{m_j}$ and $g \colon \prod_j \mathcal{M}_{m_j} \to \prod_i \mathcal{M}_{n_i}$ be maps corresponding via $\mathbf{CPM}^{\mathrm{op}} \cong \mathbf{CPM}$. Then f is subunital, i.e. $f((1)_i) \leq (1)_j$ if and only if g is trace-nonincreasing, i.e. $\mathrm{tr}(g((B_j)_j)) \leq \mathrm{tr}((B_j)_j)$ for all positive $(B_j)_j \in \mathcal{M}_{m_j}$. ■

This proposition identifies the maps that are dual to the ones in $\mathbf{Q}$. Let us define the category $\mathbf{Q}_{\mathrm{H}}$ to be the subcategory of $\mathbf{CPM}$ containing all the objects, but only subunital maps (H stands for 'Heisenberg picture'). Then the previous proposition gives the following isomorphism of categories.

Proposition 7.2

We have an isomorphism of categories $\mathbf{Q}_{\mathrm{H}} \cong \mathbf{Q}^{\mathrm{op}}$. ■

This can be considered as a categorical expression of the duality of the Heisenberg versus Schrödinger picture in the finite dimensional case. Note that $\mathbf{Q}_{\mathrm{H}}(\vec{n}, \vec{m}) \cong \mathbf{Wstar}_{\mathrm{CPSU}}(\prod_i \mathcal{M}_{n_i}, \prod_j \mathcal{M}_{m_j})$ by definition. Therefore, by defining $I(\vec{n}) = \prod_i \mathcal{M}_{n_i}$, we obtain a full embedding $I \colon \mathbf{Q}_{\mathrm{H}} \to \mathbf{Wstar}_{\mathrm{CPSU}}$.[*4] In fact, we have the following better result.

Theorem 7.2

We have an equivalence of categories $\mathbf{Q}_{\mathrm{H}} \simeq \mathbf{FdWstar}_{\mathrm{CPSU}}$ ($= \mathbf{FdCstar}_{\mathrm{CPSU}}$).

[*4] By an *embedding (of categories)* we mean a faithful functor that is injective on objects.

Proof We can clearly restrict the full embedding $I\colon \mathbf{Q}_{\mathrm{H}} \to \mathbf{Wstar}_{\mathrm{CPSU}}$ to $I\colon \mathbf{Q}_{\mathrm{H}} \to \mathbf{FdWstar}_{\mathrm{CPSU}}$. It is known that every finite dimensional C^*-algebra is of the form $\prod_i \mathcal{M}_{n_i}$, up to $*$-isomorphism.[54, Theorem I.11.2] Therefore $I\colon \mathbf{Q}_{\mathrm{H}} \to \mathbf{FdWstar}_{\mathrm{CPSU}}$ is essentially surjective, and hence an equivalence. ■

Thus we also have an equivalence $\mathbf{Q} \simeq (\mathbf{FdWstar}_{\mathrm{CPSU}})^{\mathrm{op}}$. Let us make two observations on the results of this subsection. First, our semantics is dual to Selinger's: namely, our semantics by $\mathbf{Wstar}_{\mathrm{CPSU}}$ is given in the Heisenberg picture, while Selinger's semantics by $\mathbf{Q}$ is in the Schrödinger picture. Our semantics is also related to *quantum weakest preconditions* of D'Hondt and Panangaden.[14] *Healthy predicate transformers* in their work correspond to CPSU-maps $f\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$, because such maps restricts to maps $f\colon [0,1]_{\mathcal{B}(\mathcal{H})} \to [0,1]_{\mathcal{B}(\mathcal{K})}$ between the sets of effects (*predicates* in their work).

Second, in the light of the equivalence $\mathbf{Q}_{\mathrm{H}} \simeq \mathbf{FdWstar}_{\mathrm{CPSU}}$, the category $\mathbf{Wstar}_{\mathrm{CPSU}}$ can be thought of as an infinite dimensional extension of $\mathbf{Q}_{\mathrm{H}} \cong \mathbf{Q}^{\mathrm{op}}$. Working in the category $\mathbf{Wstar}_{\mathrm{CPSU}}$ rather than $\mathbf{Q}$ enables us to handle infinite types. The classical type bit in QFC is interpreted by $[\![\mathsf{bit}]\!] = \mathbb{C} \times \mathbb{C}$. We can obviously interpret the type trit by $[\![\mathsf{trit}]\!] = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$, and more generally the type of n-level classical system by $\prod_{i=1}^{n} \mathbb{C}$. It will be then natural to interpret the type nat of natural numbers by $[\![\mathsf{nat}]\!] = \prod_{i\in\mathbb{N}} \mathbb{C}$, as Selinger also suggested.[50, §7.3] The infinite product $\prod_{i\in\mathbb{N}} \mathbb{C}$ exists in $\mathbf{Wstar}_{\mathrm{CPSU}}$, but not in $\mathbf{Q}$. As a quantum analogue, interpretations such as $[\![\mathsf{qbit}]\!] = \mathcal{M}_2 \cong \mathcal{B}(\mathbb{C}^2)$ and $[\![\mathsf{qtrit}]\!] = \mathcal{M}_3 \cong \mathcal{B}(\mathbb{C}^3)$ can be generalised to an interpretation of the type of countable level quantum system ("quantum natural numbers" in a sense) by $\mathcal{B}(\ell^2)$, where $\ell^2 = \ell^2(\mathbb{N})$ is the Hilbert space of countable dimension. We have $\mathcal{B}(\ell^2)$ in $\mathbf{Wstar}_{\mathrm{CPSU}}$, but not in $\mathbf{Q}$.

Remark 7.1
We also have a full embedding $\mathbf{CPM} \to \mathbf{Wstar}_{\mathrm{CP}}$, and an equivalence $\mathbf{CPM} \simeq \mathbf{FdWstar}_{\mathrm{CP}}$.

§8 Classical computation in commutative operator algebras

As mentioned in the previous section, the category $\mathbf{Wstar}_{\mathrm{CPSU}}$ can accommodate the infinite classical type nat. In this section we generalise this observation, and show that the categories of commutative W^*-algebras can accommodate any classical data types modelled by sets, and classical computation between such data types, including probabilistic computation. Categorically speaking, we will prove that the following four categories can be embedded to the categories of commutative W^*-algebras.

Definition 8.1
We denote by **Set** the category of sets and functions, and by **Pfn** the category of sets and partial functions. We denote by $\mathcal{D}$ the (infinite) distribution monad on **Set**, and by $\mathcal{K}\ell(\mathcal{D})$ the Kleisli category of $\mathcal{D}$. Specifically, objects of $\mathcal{K}\ell(\mathcal{D})$ are sets; and arrows $f\colon X \nrightarrow Y$ of $\mathcal{K}\ell(\mathcal{D})$ are functions $f\colon X \to \mathcal{D}Y$, where $\mathcal{D}Y = \{\varphi\colon Y \to [0,1] \mid \sum_{y\in Y}\varphi(y) = 1\}$ is the set of probability distributions. The identities $\eta_X\colon X \nrightarrow X$ in $\mathcal{K}\ell(\mathcal{D})$ are functions $\eta_X\colon X \to \mathcal{D}X$ defined by $\eta_X(x)(x) = 1$ and $\eta_X(x)(y) = 1$ $(x \neq y)$. For $f\colon X \nrightarrow Y$ and $g\colon Y \nrightarrow Z$ in $\mathcal{K}\ell(\mathcal{D})$, the composition $g \odot f$ in $\mathcal{K}\ell(\mathcal{D})$ is $g^{\#} \circ f$ in **Set**, where $g^{\#}\colon \mathcal{D}Y \to \mathcal{D}Z$ is the Kleisli extension of g, defined by $g^{\#}(\varphi)(z) = \sum_{y\in Y} g(y)(z)\varphi(y)$. Finally, we denote by $\mathcal{D}_{\leq 1}$ the (infinite) subdistribution monad on **Set**, and by $\mathcal{K}\ell(\mathcal{D}_{\leq 1})$ the Kleisli category of $\mathcal{D}_{\leq 1}$. The set $\mathcal{D}_{\leq 1}Y = \{\varphi\colon Y \to [0,1] \mid \sum_{y\in Y}\varphi(y) \leq 1\}$ consists of subdistributions, and the Kleisli category $\mathcal{K}\ell(\mathcal{D}_{\leq 1})$ is defined in a similar manner to $\mathcal{D}$.

The category **Set** models deterministic computation, while $\mathcal{K}\ell(\mathcal{D})$ models probabilistic computation. The categories **Pfn** and $\mathcal{K}\ell(\mathcal{D}_{\leq 1})$ model partial variants (i.e. computation which may not terminate) of the two computations.

Definition 8.2
Let X be a set. We define:

$$
\begin{aligned}
c_{00}(X) &:= \{\varphi\colon X \to \mathbb{C} \mid \varphi \text{ has finite support}\} \\
\ell^{\infty}(X) &:= \left\{\varphi\colon X \to \mathbb{C} \;\middle|\; \sup_{x\in X}|\varphi(x)| < \infty\right\} \\
\ell^{1}(X) &:= \left\{\varphi\colon X \to \mathbb{C} \;\middle|\; \sum_{x\in X}|\varphi(x)| < \infty\right\} .
\end{aligned}
$$

It is standard that $\ell^{\infty}(X)$ and $\ell^{1}(X)$ are Banach spaces with pointwise operations, and norms $\|\varphi\|_{\infty} = \sup_{x\in X}|\varphi(x)|$ and $\|\varphi\|_{1} = \sum_{x\in X}|\varphi(x)|$ respectively. Moreover $c_{00}(X)$ is a dense subspace of $\ell^{1}(X)$. We write $\delta\colon X \to c_{00}(X)$ for Kronecker's delta, which is defined by $\delta(x)(x) = 1$ and $\delta(x)(x') = 0$ $(x \neq x')$. Then $\{\delta(x)\}_{x\in X}$ forms a basis of $c_{00}(X)$.

Proposition 8.1
For a set X, $\ell^\infty(X)$ is a commutative W^*-algebra with a predual $\ell^1(X)$.

Proof One has $\ell^\infty(X) \cong \prod_{x\in X}\mathbb{C}$ and $(\prod_{x\in X}\mathbb{C})_* \cong \bigoplus^1_{x\in X}\mathbb{C}_* \cong \bigoplus^1_{x\in X}\mathbb{C} \cong \ell^1(X)$, using Lemma 2.2. Alternatively, it is easy to directly check that $\ell^\infty(X)$ is a C^*-algebra with pointwise operations, and the duality $\ell^\infty(X) \cong \ell^1(X)^*$ is well-known. ■

Notice that $\ell^\infty(2) \cong \mathbb{C}\times\mathbb{C} = [\![\mathsf{bit}]\!], \ell^\infty(3) \cong \mathbb{C}\times\mathbb{C}\times\mathbb{C} = [\![\mathsf{trit}]\!] \ldots, \ell^\infty(\mathbb{N}) \cong \prod_{i\in\mathbb{N}}\mathbb{C} = [\![\mathsf{nat}]\!]$ are W^*-algebras interpreting familiar classical types. In general, if a type t is interpreted by a set X (i.e. $[\![t]\!] = X \in \mathbf{Set}$), then we may interpret t by a W^*-algebra $\ell^\infty(X)$. We also wish to interpret a program between such classical types as a map between W^*-algebras. For this, we will investigate the structure of $\ell^\infty(X)$ and maps between them. Because $\ell^\infty(X)$ is commutative, we do not need to care about the *complete* positivity of maps, see Proposition 2.5.

The following is an immediate consequence from definition.

Lemma 8.1
Let X be a set, and $\varphi \in \ell^\infty(X)$ an element of a W^*-algebra $\ell^\infty(X)$.

1. φ is self-adjoint if and only if $\varphi(x) \in \mathbb{R}$ for all $x \in X$.
2. φ is positive if and only if $\varphi(x) \in \mathbb{R}^+$ for all $x \in X$.
3. φ is an effect if and only if $\varphi(x) \in [0,1]$ for all $x \in X$.
4. φ is a projection if and only if $\varphi(x) \in \{0,1\}$ for all $x \in X$. ■

Similarly, the order structure is simply pointwise.

Lemma 8.2
Let X be a set. For self-adjoint elements $\varphi, \psi \in \ell^\infty(X)$, $\varphi \le \psi$ if and only if $\varphi(x) \le \psi(x)$ for all $x \in X$. Moreover, if $(\varphi_i)_i$ is a norm-bounded monotone net of self-adjoint elements in $\ell^\infty(X)$, then the supremum $\sup_i \varphi_i$ is calculated pointwisely: one has $(\sup_i \varphi_i)(x) = \sup_i \varphi_i(x)$ for each $x \in X$. ■

For a set X, we denote by $\mathcal{P}_{\mathrm{fin}}(X)$ the finite powerset of X, i.e. the set of finite subsets of X. Note that $\mathcal{P}_{\mathrm{fin}}(X)$ is a directed set via the inclusion order. Then, the following is easily obtained using the previous lemma.

Lemma 8.3
Let X be a set. Then $(\sum_{x\in F}\delta(x))_{F\in\mathcal{P}_{\mathrm{fin}}(X)}$ is a norm-bounded monotone net

of positive elements in $\ell^\infty(X)$, and we have

$$\sup_{F \in \mathcal{P}_{\mathrm{fin}}(X)} \sum_{x \in F} \delta(x) = 1 \ .$$

■

Lemma 8.4

Let X be a set. Then $c_{00}(X)$ is ultraweakly dense in $\ell^\infty(X)$.

Proof By Proposition 4.1 and Lemma 8.3, one has $\mathrm{uwlim}_{F \in \mathcal{P}_{\mathrm{fin}}(X)} \sum_{x \in F} \delta(x) = 1$. Recall that the multiplication in a W^*-algebra is separately ultraweak continuous (Proposition 2.9). For each $\varphi \in \ell^\infty(X)$, therefore,

$$\varphi = \varphi \cdot 1 = \varphi \cdot \operatorname*{uwlim}_{F \in \mathcal{P}_{\mathrm{fin}}(X)} \sum_{x \in F} \delta(x) = \operatorname*{uwlim}_{F \in \mathcal{P}_{\mathrm{fin}}(X)} \varphi \sum_{x \in F} \delta(x) \ ,$$

and it is easy to see that $\varphi \sum_{x \in F} \delta(x) \in c_{00}(X)$ for $F \in \mathcal{P}_{\mathrm{fin}}(X)$. ■

Lemma 8.5

Let X, Y be sets and $f \colon \ell^\infty(Y) \to \ell^\infty(X)$ a normal positive map. For each $x \in X$, one has $\sum_{y \in Y} f(\delta(y))(x) = f(1)(x)$.

Proof Consider a norm-bounded monotone net $(\sum_{y \in F} \delta(y))_{F \in \mathcal{P}_{\mathrm{fin}}(Y)}$ of positive elements in $\ell^\infty(Y)$. Then $(f(\sum_{y \in F} \delta(y)))_{F \in \mathcal{P}_{\mathrm{fin}}(Y)}$ is also a norm-bounded monotone net of positive elements in $\ell^\infty(X)$, and we have

$$\begin{aligned} \sup_{F \in \mathcal{P}_{\mathrm{fin}}(Y)} \sum_{y \in F} f(\delta(y)) &= \sup_{F \in \mathcal{P}_{\mathrm{fin}}(Y)} f\Big(\sum_{y \in F} \delta(y)\Big) \\ &= f\Big(\sup_{F \in \mathcal{P}_{\mathrm{fin}}(Y)} \sum_{y \in F} \delta(y)\Big) && \text{since } f \text{ is normal} \\ &= f(1) && \text{by Lemma 8.3} \ . \end{aligned}$$

Hence, for $x \in X$,

$$\begin{aligned} \sum_{y \in Y} f(\delta(y))(x) &= \sup_{F \in \mathcal{P}_{\mathrm{fin}}(Y)} \sum_{y \in F} f(\delta(y))(x) && \text{(def. of the infinite sum)} \\ &= \sup_{F \in \mathcal{P}_{\mathrm{fin}}(Y)} \Big(\sum_{y \in F} f(\delta(y))\Big)(x) \\ &= \Big(\sup_{F \in \mathcal{P}_{\mathrm{fin}}(Y)} \sum_{y \in F} f(\delta(y))\Big)(x) && \text{by Lemma 8.2} \\ &= f(1)(x) \ . \end{aligned}$$

■

Proposition 8.2
Let X, Y be sets and $f\colon \ell^\infty(Y) \to \ell^\infty(X)$ a normal positive map.

1. f is subunital if and only if $\sum_{y\in Y} f(\delta(y))(x) \leq 1$ for all $x \in X$.
2. f is unital if and only if $\sum_{y\in Y} f(\delta(y))(x) = 1$ for all $x \in X$.
3. f is MI if and only if f is subunital and $f(\delta(y))(x) \in \{0,1\}$ for all $x \in X$ and $y \in Y$.

Proof 1 and 2 follow immediately from Lemmas 8.1 and 8.5. To show 3, assume that f is MI. Note that MI-maps preserve projections, and always subunital. For each $y \in Y$, $\delta(y)$ is a projection, so that $f(\delta(y))$ is a projection too. Hence $f(\delta(y))(x) \in \{0,1\}$ for all $x \in X$. Conversely, assume that f is subunital and $f(\delta(y))(x) \in \{0,1\}$ for all $x \in X$ and $y \in Y$. By 1 one has $\sum_{y\in Y} f(\delta(y))(x) \leq 1$ for all $x \in X$. If $y \neq y'$, therefore, at least one of $f(\delta(y))(x)$ and $f(\delta(y'))(x)$ must be zero, i.e. $f(\delta(y))(x)f(\delta(y'))(x) = 0$. Hence $y \neq y'$ implies $f(\delta(y))f(\delta(y')) = 0$. Then $f(\delta(y)\delta(y')) = f(\delta(y))f(\delta(y'))$ for all $y, y' \in Y$, since: if $y = y'$ then $f(\delta(y)\delta(y)) = f(\delta(y)) = f(\delta(y))f(\delta(y))$; and if $y \neq y'$ then $f(\delta(y)\delta(y')) = f(0) = 0 = f(\delta(y))f(\delta(y'))$. It follows that f is multiplicative on $c_{00}(Y)$, and by the ultraweak density, f is multiplicative. Since positive maps are involutive, f is MI. ∎

Now, we obtain the embedding result for $\mathcal{K}\ell(\mathcal{D}_{\leq 1})$.

Theorem 8.1
The mapping $X \mapsto \ell^\infty(X)$ gives rise to a full embedding

$$\ell^\infty\colon \mathcal{K}\ell(\mathcal{D}_{\leq 1}) \to (\mathbf{Wstar}_{\mathrm{PSU}})^{\mathrm{op}} \ .$$

For a function $f\colon X \to \mathcal{D}_{\leq 1}Y$, a map $\ell^\infty(f)\colon \ell^\infty(Y) \to \ell^\infty(X)$ is given by $\ell^\infty(f)(\varphi)(x) = \sum_{y\in Y} \varphi(y)f(x)(y)$.

Proof It is easy to see $\ell^\infty(f)$ is linear and positive. It is subunital because

$$\ell^\infty(f)(1)(x) = \sum_{y\in Y} 1 \cdot f(x)(y) = \sum_{y\in Y} f(x)(y) \leq 1 \ .$$

It is also normal since we can give a predual map $\ell^1(f)\colon \ell^1(X) \to \ell^1(Y)$ by $\ell^1(f)(\varphi)(y) = \sum_{x\in X} f(x)(y)\varphi(x)$, which is bounded and makes the following diagram commute.

$$\begin{array}{ccc} \ell^\infty(Y) & \xrightarrow{\ell^\infty(f)} & \ell^\infty(X) \\ \cong\downarrow & & \downarrow\cong \\ \ell^1(Y)^* & \xrightarrow{\ell^1(f)^*} & \ell^1(X)^* \end{array}$$

The mapping is functorial. Recall that an identity $\eta_X\colon X \to \mathcal{D}_{\le 1}X$ in $\mathcal{K}\ell(\mathcal{D}_{\le 1})$ is given by $\eta_X(x)(x) = 1$ and $\eta_X(x)(x') = 0$ $(x \ne x')$. Then for $\varphi \in \ell^\infty(X)$ and $x \in X$,

$$\ell^\infty(\eta_X)(\varphi)(x) = \sum_{x' \in X} \varphi(x')\eta_X(x)(x') = \varphi(x) \ .$$

Hence $\ell^\infty(\eta_X)(\varphi) = \varphi$, so that $\ell^\infty(\eta_X) = \mathrm{id}$. For $f\colon X \to \mathcal{D}_{\le 1}Y$, $g\colon Y \to \mathcal{D}_{\le 1}Z$, $\varphi \in \ell^\infty(Z)$, and $x \in X$,

$$\begin{aligned} \ell^\infty(g \odot f)(\varphi)(x) &= \sum_{z \in Z} \varphi(z)(g \odot f)(x)(z) \\ &= \sum_{z \in Z} \varphi(z) g^{\#}(f(x))(z) \\ &= \sum_{z \in Z} \varphi(z)\Big(\sum_{y \in Y} g(y)(z) f(x)(y)\Big) \\ &= \sum_{z \in Z} \sum_{y \in Y} \varphi(z) g(y)(z) f(x)(y) \\ &= \sum_{y \in Y} \sum_{z \in Z} \varphi(z) g(y)(z) f(x)(y) \\ &= \sum_{y \in Y} \Big(\sum_{z \in Z} \varphi(z) g(y)(z)\Big) f(x)(y) \\ &= \sum_{y \in Y} \ell^\infty(g)(\varphi)(y) f(x)(y) \\ &= \ell^\infty(f)(\ell^\infty(g)(\varphi))(x) \\ &= (\ell^\infty(f) \circ \ell^\infty(g))(\varphi)(x) \ . \end{aligned}$$

Hence $\ell^\infty(g \odot f)(\varphi) = (\ell^\infty(f) \circ \ell^\infty(g))(\varphi)$, so that $\ell^\infty(g \odot f) = \ell^\infty(f) \circ \ell^\infty(g)$. Therefore ℓ^∞ is a functor, which is obviously injective on objects.

To show that the functor is full and faithful, we will define an inverse to the map

$$\ell^\infty\colon \mathcal{K}\ell(\mathcal{D}_{\le 1})(X, Y) \to \mathbf{Wstar}_{\mathrm{PSU}}(\ell^\infty(Y), \ell^\infty(X)) \ .$$

Note the following equation

$$\ell^\infty(f)(\delta(y))(x) = \sum_{y' \in Y} \delta(y)(y') f(x)(y') = f(x)(y) \ . \tag{3}$$

for $f\colon X \to \mathcal{D}_{\leq 1} Y$. Keeping it in mind, we define a map

$$\Phi\colon \mathbf{Wstar}_{\mathrm{PSU}}(\ell^\infty(Y), \ell^\infty(X)) \to \mathcal{K}\ell(\mathcal{D}_{\leq 1})(X, Y)$$

by $\Phi(g)(x)(y) = g(\delta(y))(x)$. This is well-defined because $\sum_{y \in Y} \Phi(g)(x)(y) = \sum_{y \in Y} g(\delta(y))(x) \leq 1$ by Proposition 8.2. We show that the map Φ is indeed an inverse to ℓ^∞. For $f\colon X \to \mathcal{D}_{\leq 1} Y$, $x \in X$ and $y \in Y$,

$$\Phi(\ell^\infty(f))(x)(y) = \ell^\infty(f)(\delta(y))(x) \overset{(3)}{=} f(x)(y) \ .$$

Hence $\Phi(\ell^\infty(f)) = f$. For $g\colon \ell^\infty(Y) \to \ell^\infty(X)$, $y \in Y$ and $x \in X$,

$$\ell^\infty(\Phi(g))(\delta(y))(x) \overset{(3)}{=} \Phi(g)(x)(y) = g(\delta(y))(x) \ .$$

Hence $\ell^\infty(\Phi(g))(\delta(y)) = g(\delta(y))$ for each $y \in Y$. Because g and $\ell^\infty(\Phi(g))$ are normal, and $c_{00}(Y)$ is ultraweakly dense in $\ell^\infty(Y)$, it follows that $\ell^\infty(\Phi(g)) = g$. Therefore ℓ^∞ is full and faithful. ■

This embedding of $\mathcal{K}\ell(\mathcal{D}_{\leq 1})$ is in a sense the most general case. It is not hard to restrict this embedding to the other cases.

Corollary 8.1
The mapping $X \mapsto \ell^\infty(X)$ gives rise to the following full embeddings.

$$\ell^\infty\colon \mathbf{Set} \longrightarrow (\mathbf{CWstar}_{\mathrm{MIU}})^{\mathrm{op}}$$

$$\ell^\infty\colon \mathbf{Pfn} \longrightarrow (\mathbf{CWstar}_{\mathrm{MI}})^{\mathrm{op}}$$

$$\ell^\infty\colon \mathcal{K}\ell(\mathcal{D}) \longrightarrow (\mathbf{CWstar}_{\mathrm{PU}})^{\mathrm{op}} \ .$$

Proof Note the following (non-full) embeddings and inclusions of categories, all of which are identity on objects.

$$\begin{array}{ccc} \mathbf{Set} & \longrightarrow & \mathbf{Pfn} \\ \downarrow & & \downarrow \\ \mathcal{K}\ell(\mathcal{D}) & \longrightarrow & \mathcal{K}\ell(\mathcal{D}_{\leq 1}) \end{array} \qquad \begin{array}{ccc} \mathbf{Wstar}_{\mathrm{MIU}} & \longrightarrow & \mathbf{Wstar}_{\mathrm{MI}} \\ \downarrow & & \downarrow \\ \mathbf{Wstar}_{\mathrm{PU}} & \longrightarrow & \mathbf{Wstar}_{\mathrm{PSU}} \end{array}$$

Here $\mathbf{Set} \to \mathcal{K}\ell(\mathcal{D})$ is the canonical (left adjoint) functor for the monad $\mathcal{D}$, and

$F\colon \mathbf{Pfn} \to \mathcal{K}\ell(\mathcal{D}_{\leq 1})$ is defined by, for a partial function $f\colon X \rightharpoonup Y$,

$$Ff(x)(y) = \begin{cases} 1 & \text{if } f(x) = y \\ \\ 0 & \text{if } f(x) \neq y \text{ or } f(x) \text{ is undefined} \end{cases} .$$

The embeddings $\mathbf{Set} \to \mathbf{Pfn}$ and $\mathcal{K}\ell(\mathcal{D}) \to \mathcal{K}\ell(\mathcal{D}_{\leq 1})$ are obvious ones.

Let $f\colon X \nrightarrow Y$ be an arrow in $\mathcal{K}\ell(\mathcal{D}_{\leq 1})$, i.e. a function $f\colon X \to \mathcal{D}_{\leq 1}Y$. Note that:

- f comes from $\mathcal{K}\ell(\mathcal{D})$ if and only if $\sum_{y\in Y} f(x)(y) = 1$ for all $x \in X$.
- f comes from $\mathbf{Pfn}$ if and only if $f(x)(y) \in \{0, 1\}$ for all $x \in X$ and $y \in Y$.
- f comes from $\mathbf{Set}$ if and only if it satisfies both of the previous two conditions.

Recall, from the equation (3), that $\ell^\infty(f)(\delta(y))(x) = f(x)(y)$. In the light of Proposition 8.2, we can restrict the embedding $\ell^\infty\colon \mathcal{K}\ell(\mathcal{D}_{\leq 1}) \to (\mathbf{Wstar}_{\mathrm{PSU}})^{\mathrm{op}}$ of Theorem 8.1 to the desired three embeddings. ∎

As a consequence, $\mathbf{Set}$, $\mathbf{Pfn}$, $\mathcal{K}\ell(\mathcal{D})$ and $\mathcal{K}\ell(\mathcal{D}_{\leq 1})$ can be simply thought of as full subcategories of commutative W^*-algebras with corresponding kind of maps. If programs are interpreted in $\mathbf{Set}$, $\mathbf{Pfn}$, $\mathcal{K}\ell(\mathcal{D})$ or $\mathcal{K}\ell(\mathcal{D}_{\leq 1})$, they are also interpreted in the categories of commutative W^*-algebras. Furthermore, the cartesian product of sets corresponds to the spatial W^*-tensor product.

Theorem 8.2
Let X and Y be sets. There is a (normal unital) $*$-isomorphism: $\ell^\infty(X) \mathbin{\overline{\otimes}} \ell^\infty(Y) \cong \ell^\infty(X \times Y)$.

Proof It is known that, when at least one of W^*-algebras M, N is commutative, the dual spatial C^*-norm on $M_* \odot N_*$ coincides with the projective (i.e. greatest) cross norm.[48, Proposition 1.22.12)] The projective tensor product $\ell^1(X) \otimes \ell^1(Y)$ of $\ell^1(X)$ and $\ell^1(Y)$ can be concretely given by:[46, Example 2.6)]

$$\ell^1(X; \ell^1(Y)) \coloneqq \left\{ f\colon X \to \ell^1(Y) \;\middle|\; \sum_{x\in X} \|f(x)\|_1 < \infty \right\} .$$

It is then easy to see that $\ell^1(X; \ell^1(Y)) \cong \ell^1(X \times Y)$, so that we have $\ell^1(X) \otimes \ell^1(Y) \cong \ell^1(X \times Y)$ (isometrically). By dualising it, we obtain a normal isomorphism

$$\ell^\infty(X) \mathbin{\overline{\otimes}} \ell^\infty(Y) \cong (\ell^1(X) \otimes \ell^1(Y))^* \cong \ell^1(X \times Y)^* \cong \ell^\infty(X \times Y) .$$

We need to show that it is a $*$-isomorphism. Let $\Theta\colon \ell^\infty(X) \odot \ell^\infty(Y) \to \ell^\infty(X \times Y)$ be a map given by $\Theta(\varphi \otimes \psi)(x, y) = \varphi(x)\varphi(y)$ on the elementary tensors. Then it is straightforward to check that Θ is a $*$-homomorphism, and that the following diagram commutes.

$$\begin{tikzcd} \ell^\infty(X) \odot \ell^\infty(Y) \arrow[d, tail] \arrow[dr, "\Theta"] & \\ \ell^\infty(X) \mathbin{\overline{\otimes}} \ell^\infty(Y) \arrow[r, "\cong"'] & \ell^\infty(X \times Y) \end{tikzcd}$$

It follows that the isomorphism $\ell^\infty(X) \mathbin{\overline{\otimes}} \ell^\infty(Y) \cong \ell^\infty(X \times Y)$ is a $*$-isomorphism, because the canonical embedding $\ell^\infty(X) \odot \ell^\infty(Y) \to \ell^\infty(X) \mathbin{\overline{\otimes}} \ell^\infty(Y)$ is ultraweakly dense. ■

Thus, the classical product type corresponds precisely to the spatial W^*-tensor product. For example, assume that a program $f\colon \mathsf{nat}, \mathsf{nat} \to \mathsf{nat}$ with multiple inputs is interpreted by a function $[\![f]\!]\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ between sets. Then it can also be interpreted by a map

$$\ell^\infty(\mathbb{N}) \xrightarrow{\ell^\infty([\![f]\!])} \ell^\infty(\mathbb{N} \times \mathbb{N}) \cong \ell^\infty(\mathbb{N}) \mathbin{\overline{\otimes}} \ell^\infty(\mathbb{N})$$

between W^*-algebras.

Remark 8.1

W^*-algebras are often referred to as noncommutative measure/measurable spaces. Indeed, the following results have been known for a long time.

- For a measure space (X, Σ, μ), $L^\infty(X, \Sigma, \mu)$ is a commutative W^*-algebra if (and only if) (X, Σ, μ) is localisable.[49)]
- Any commutative W^*-algebra is $*$-isomorphic to $L^\infty(X, \Sigma, \mu)$ for some localisable measure space (X, Σ, μ).[48, Proposition 1.18.1)]

For C^*-algebras, which are called noncommutative topological spaces, a categorical 'Gelfand' duality between commutative C^*-algebras and compact Hausdorff spaces has been known, since early times.[37)] In contrast, it seems that an analogous categorical result relating W^*-algebras to measure/measureable spaces was not fully elaborated until Robert Furber did so very recently.[15)] In his thesis, he showed that the category of commutative W^*-algebras and normal unital $*$-homomorphisms is dually equivalent to the category of strictly localisable compact complete measure spaces and certain equivalence classes of "normal" measureable maps, and also that strict localisability and compactness are necessary

$$
\begin{tikzcd}
\mathbf{Q} \arrow[r, "\simeq"] & (\mathbf{FdWstar}_{\mathrm{CPSU}})^{\mathrm{op}} \arrow[ddr, hook, bend left, "\mathrm{full}"] & \\
\mathbf{Set} \arrow[r, hook, "\mathrm{full}"] & (\mathbf{CWstar}_{\mathrm{MIU}})^{\mathrm{op}} \arrow[dr, hook] & \\
\mathbf{Pfn} \arrow[r, hook, "\mathrm{full}"] & (\mathbf{CWstar}_{\mathrm{MI}})^{\mathrm{op}} \arrow[r, hook] & (\mathbf{Wstar}_{\mathrm{CPSU}})^{\mathrm{op}} \\
\mathcal{K}\ell(\mathcal{D}) \arrow[r, hook, "\mathrm{full}"] & (\mathbf{CWstar}_{\mathrm{PU}})^{\mathrm{op}} \arrow[ur, hook] & \\
\mathcal{K}\ell(\mathcal{D}_{\le 1}) \arrow[r, hook, "\mathrm{full}"] & (\mathbf{CWstar}_{\mathrm{PSU}})^{\mathrm{op}} \arrow[uur, hook, bend right, "\mathrm{full}"'] &
\end{tikzcd}
$$

Fig. 1 An equivalence and full embeddings to the categories of W^*-algebras

for the duality. Given the general duality result, at least the first embedding of Corollary 8.1 can be obtained as a special case, by considering sets as measure spaces via counting measures. Because the general case is so involved, it would be nice to have the simple special case separately.

§9 Conclusion

We studied operator algebras from a domain-theoretic and categorical point of view, and showed that the category $\mathbf{Wstar}_{\mathrm{CPSU}}$ of W^*-algebras and normal CPSU-maps is a $\mathbf{Dcppo}_{\perp}$-enriched symmetric monoidal category with $\mathbf{Dcppo}_{\perp}$-enriched products. In particular, the opposite $(\mathbf{Wstar}_{\mathrm{CPSU}})^{\mathrm{op}}$ is an elementary quantum flow chart category, which gives a denotational semantics for QFC. We furthermore obtained an equivalence and full embeddings of various familiar categories to the categories of W^*-algebras, see Fig. 1.

In parallel with the present work, Rennela[45)] recently showed that $\mathbf{Wstar}_{\mathrm{PSU}}$ is algebraically compact for a certain class of ("von Neumann") functors. His and our results demonstrate that operator algebras, especially W^*-algebras, provide a flexible and promising model for quantum computation. It is still an open problem to give a denotational semantics by operator algebras for a *higher-order* quantum programming language, or the quantum lambda calculus. Kornell's (unpublished) paper[28)] showed that the symmetric monoidal category $((\mathbf{Wstar}_{\mathrm{MIU}})^{\mathrm{op}}, \overline{\otimes}, \mathbb{C})$ is *closed*. This result may be helpful.

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§A Traces on $\boldsymbol{\omega}$Cppo-enriched cartesian categories

In this appendix, we will give a proof of Theorem 6.2. We in fact show the dual statement as Theorems A.1 and A.3 below, because cartesian categories are more standard than cocartesian categories.

First, let us clarify the terminology. A *cartesian category* is a (symmetric) monoidal category whose monoidal structure is given by finite products. In other words, it is just a category with (a choice of) finite products. A functor between cartesian categories is *cartesian* if it preserves finite products. For an $\boldsymbol{\omega}$**Cppo**-enriched cartesian category, we require that the cartesian product functor $\times$ be $\boldsymbol{\omega}$**Cppo**-enriched, or equivalently, the tupling $\langle -, - \rangle$ be ω-continuous.

The following is the first theorem we wish to prove.

Theorem A.1

Every $\boldsymbol{\omega}$**Cppo**-enriched cartesian category with left-strict composition (i.e. $\perp \circ f = \perp$) is traced. For $f\colon A \times X \to B \times X$, the trace $\mathrm{Tr}(f)\colon A \to B$ is given by

$$\mathrm{Tr}(f) \coloneqq \pi_1 \circ \bigvee_{n \in \mathbb{N}} \mathrm{Tr}^{(n)}(f) \ , \tag{4}$$

where $\mathrm{Tr}^{(n)}(f)\colon A \to B \times X$ is defined by

$$\mathrm{Tr}^{(0)}(f) = \perp$$

$$\mathrm{Tr}^{(n+1)}(f) = f \circ \big\langle \mathrm{id}_A, \pi_2 \circ \mathrm{Tr}^{(n)}(f) \big\rangle \ .$$

We use the well-known theorem of Hasegawa and Hyland. The complete proof is found in Hasegawa's thesis.[21, Theorem 7.1.1]

Theorem A.2 (Hasegawa/Hyland)

A cartesian category is traced if and only if it has a Conway operator. A trace operator Tr and a Conway operator Fix are related bijectively as follows:

$$\mathrm{Tr}(f) = \pi_1 \circ f \circ \big\langle \mathrm{id}_A, \mathrm{Fix}(\pi_2 \circ f) \big\rangle \tag{5}$$

for $f\colon A \times X \to B \times X$, and

$$\mathrm{Fix}(g) = \mathrm{Tr}(\Delta_X \circ g)$$

for $g\colon A \times X \to X$, where $\Delta_X = \langle \mathrm{id}_X, \mathrm{id}_X \rangle$ is the diagonal map. ■

Thanks to this theorem, the problem is reduced to a little easier problem on a Conway operator. The result we need is already shown by Hoshino et al.[24, Lemma A.1]

Lemma A.1
Every **$\boldsymbol{\omega}$Cppo**-enriched cartesian category with left-strict composition (i.e. $\bot \circ f = \bot$) has a Conway operator Fix. For $g\colon A \times X \to X$, $\mathrm{Fix}(g)\colon A \to X$ is given by

$$\mathrm{Fix}(g) \coloneqq \bigvee_{n\in\mathbb{N}} \mathrm{Fix}^{(n)}(g) \ , \tag{6}$$

where $\mathrm{Fix}^{(n)}(g)\colon A \to X$ is defined by

$$\begin{aligned} \mathrm{Fix}^{(0)}(g) &= \bot \\ \mathrm{Fix}^{(n+1)}(g) &= g \circ \left\langle \mathrm{id}_A, \mathrm{Fix}^{(n)}(g) \right\rangle \ . \end{aligned}$$

■

Remark A.1
In the light of Theorem 4.1, the arrow $\mathrm{Fix}(g)$ is the least (pre-)fixed point of a ω-continuous map $g \circ \langle \mathrm{id}_A, - \rangle \colon \mathbf{C}(A, X) \to \mathbf{C}(A, X)$.

Now we prove Theorem A.1 as follows.

Proof of Theorem A.1 Let $\mathbf{C}$ be an **$\boldsymbol{\omega}$Cppo**-enriched cartesian category with left-strict composition. Then, by Theorem A.2 and Lemma A.1 $\mathbf{C}$ is traced. We still need to check the equation (4). For $f\colon A \times X \to B \times X$,

$$\begin{aligned} \mathrm{Tr}(f) &= \pi_1 \circ f \circ \left\langle \mathrm{id}_A, \mathrm{Fix}(\pi_2 \circ f) \right\rangle && \text{by (5)} \\ &= \pi_1 \circ f \circ \left\langle \mathrm{id}_A, \bigvee_{n\in\mathbb{N}} \mathrm{Fix}^{(n)}(\pi_2 \circ f) \right\rangle && \text{by (6)} \\ &= \pi_1 \circ \bigvee_{n\in\mathbb{N}} \left(f \circ \left\langle \mathrm{id}_A, \mathrm{Fix}^{(n)}(\pi_2 \circ f) \right\rangle \right) \ . \end{aligned}$$

It is straightforward to see, by induction on n, that

$$\mathrm{Tr}^{(n)}(f) \leq f \circ \left\langle \mathrm{id}_A, \mathrm{Fix}^{(n)}(\pi_2 \circ f) \right\rangle \leq \mathrm{Tr}^{(n+1)}(f)$$

for all $n \in \mathbb{N}$, which shows

$$\bigvee_{n\in\mathbb{N}} \left(f \circ \left\langle \mathrm{id}_A, \mathrm{Fix}^{(n)}(\pi_2 \circ f) \right\rangle \right) = \bigvee_{n\in\mathbb{N}} \mathrm{Tr}^{(n)}(f) \ .$$

Hence we have

$$\mathrm{Tr}(f) = \pi_1 \circ \bigvee_{n\in\mathbb{N}} \mathrm{Tr}^{(n)}(f) \enspace .$$

■

Next, we will show the theorem on cartesian functors.

Lemma A.2
Let $\mathbf{C}$ and $\mathbf{D}$ be traced cartesian categories, which also have the corresponding Conway operators by Theorem A.2. For a cartesian functor $F\colon \mathbf{C} \to \mathbf{D}$, the following are equivalent.

1. F is traced, that is, for each arrow $f\colon A \times X \to B \times X$ in $\mathbf{C}$, one has $F\,\mathrm{Tr}(f) = \mathrm{Tr}(\phi_{B,X} \circ Ff \circ \phi_{A,X}^{-1})$.
2. For each arrow $g\colon A \times X \to X$ in $\mathbf{C}$, one has $F\,\mathrm{Fix}(g) = \mathrm{Fix}(Fg \circ \phi_{A,X}^{-1})$.

Here $\phi_{A,X} \coloneqq \langle F\pi_1, F\pi_2\rangle\colon F(A \times X) \to FA \times FX$ denotes the canonical isomorphism.

Proof We will use the relationship between Tr and Fix in Theorem A.2, and the equations $\pi_i \circ \phi_{A,X} = F\pi_i$ and $\phi_{A,X} \circ F\langle h, k\rangle = \langle Fh, Fk\rangle$.

Assume the condition 1. For $g\colon A \times X \to X$ in $\mathbf{C}$,

$$\begin{aligned} F\,\mathrm{Fix}(g) &= F\,\mathrm{Tr}(\Delta_X \circ g) \\ &= \mathrm{Tr}(\phi_{X,X} \circ F(\Delta_X \circ g) \circ \phi_{A,X}^{-1}) \\ &= \mathrm{Tr}(\Delta_{FX} \circ Fg \circ \phi_{A,X}^{-1}) \\ &= \mathrm{Fix}(Fg \circ \phi_{A,X}^{-1}) \enspace . \end{aligned}$$

Conversely, assume the condition 2. For $f\colon A \times X \to B \times X$ in $\mathbf{C}$,

$$\begin{aligned} F\,\mathrm{Tr}(f) &= F(\pi_1 \circ f \circ \langle \mathrm{id}_A, \mathrm{Fix}(\pi_2 \circ f)\rangle) \\ &= \pi_1 \circ \phi_{B,X} \circ Ff \circ \phi_{A,X}^{-1} \circ \langle \mathrm{id}_{FA}, F\,\mathrm{Fix}(\pi_2 \circ f)\rangle \\ &= \pi_1 \circ \phi_{B,X} \circ Ff \circ \phi_{A,X}^{-1} \circ \langle \mathrm{id}_{FA}, \mathrm{Fix}(F(\pi_2 \circ f) \circ \phi_{A,X}^{-1})\rangle \\ &= \pi_1 \circ \phi_{B,X} \circ Ff \circ \phi_{A,X}^{-1} \circ \langle \mathrm{id}_{FA}, \mathrm{Fix}(\pi_2 \circ \phi_{B,X} \circ Ff \circ \phi_{A,X}^{-1})\rangle \\ &= \mathrm{Tr}(\phi_{B,X} \circ Ff \circ \phi_{A,X}^{-1}) \enspace . \end{aligned}$$

■

Theorem A.3
Let $\mathbf{C}$ and $\mathbf{D}$ be $\boldsymbol{\omega}\mathbf{Cppo}$-enriched cartesian categories, which are traced by

Theorem A.1. Then, every $\boldsymbol{\omega}$**Cppo**-enriched cartesian functor between **C** and **D** satisfying $F\bot = \bot$ is traced.

Proof By Lemma A.2, it suffices to show $F\,\mathrm{Fix}(g) = \mathrm{Fix}(Fg \circ \phi^{-1}_{A,X})$ for $g\colon A \times X \to X$ in **C**. By definition of a Conway operator, one has $g \circ \langle \mathrm{id}_A, \mathrm{Fix}(g)\rangle = \mathrm{Fix}(g)$, so that

$$F\,\mathrm{Fix}(g) = F(g \circ \langle \mathrm{id}_A, \mathrm{Fix}(g)\rangle) = Fg \circ \phi^{-1}_{A,X} \circ \langle \mathrm{id}_{FA}, F\,\mathrm{Fix}(g)\rangle \ .$$

By Remark A.1 we obtain $\mathrm{Fix}(Fg \circ \phi^{-1}_{A,X}) \leq F\,\mathrm{Fix}(g)$. On the other hand, we can show $F\,\mathrm{Fix}^{(n)}(g) \leq \mathrm{Fix}^{(n+1)}(Fg \circ \phi^{-1}_{A,X})$ by induction on n (here we use $F\bot = \bot$), so that $F\,\mathrm{Fix}(g) \leq \mathrm{Fix}(Fg \circ \phi^{-1}_{A,X})$ by the ω-continuity of F. ■