Coalgebraic Fixed Point Logics in a Fibration

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1 Introduction

Coalgebraic fixed point logics are coalgebraic modal logics plus fixed point operators μ and ν, which for example allow us to describe safety and liveness properties. There are several studies on coalgebraic fixed point logics [2,3,10,11], but all of them focus on the category **Set**. Our aim is to understand coalgebraic fixed point logics in more abstract and general terms, much like understanding coalgebraic modal logics by the dual adjunction (Stone-like duality) framework [1], which applies the categories other than **Set** [9].

For that purpose, we shall use *fibrations* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$, which are commonly used as models of predicate logics in the field of categorical logic [8]. Our work is much inspired by Hermida and Jacobs' work [7], which studies induction and coinduction in a fibrational setting. Recently, Ghani and his colleagues are actively investigating induction and coinduction in a fibration in the context of type theory [4,5].

2 Predicate lifting

In usual coalgebraic modal logics (on **Set**), modal operators can be interpreted by predicate liftings $2^{-} \Rightarrow 2^{F-}$. This approach is generalized into a fibrational setting [7].

Definition 1. A *predicate lifting* of $F : \mathbb{C} \to \mathbb{C}$ along a fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ is a functor $\varphi : \mathbb{P} \to \mathbb{P}$ such that (F, φ) is a map of fibrations.

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\varphi} & \mathbb{P} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{C} & \xrightarrow[F]{} & \mathbb{C} \end{array}$$

Ordinary monotone predicate liftings $2^{-} \Rightarrow 2^{F-}$ coincide with predicate liftings along $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Set}\end{smallmatrix}$.

3 Coinductive predicate

The author's recent work [6] studies coinductive predicates in a fibrational setting. It can be thought of as a fragment of coalgebraic fixed point logics where we can use only one greatest fixed point operator ν at the outermost position. Now a predicate lifting φ of F along $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ plays a role of a (co)recursive definition. That is:

Definition 2. The φ*-coinductive predicate* in an F-coalgebra $X \xrightarrow{c} FX$ is the final coalgebra of a functor $\mathbb{P}_X \xrightarrow{\varphi} \mathbb{P}_{FX} \xrightarrow{c^*} \mathbb{P}_X$. More fibrationally, it turns out to be the fibred final object of the lifted fibration $\begin{smallmatrix}\mathbf{Coalg}(\varphi)\\ \downarrow\overline{p}\\ \mathbf{Coalg}(F)\end{smallmatrix}$.

The main result in [6] is the following theorem.

Theorem 3. *For suitably "small"* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ *and* F*, the final* φ*-sequence stabilizes after* ω *steps, and it yields the* φ*-coinductive predicate.*

4 Coalgebraic fixed point logic

To interpret "full" fixed point logics, we consider the fibration $\begin{smallmatrix}\mathbb{P}_F\\ \downarrow U^*(p)\\ \mathbf{Coalg}(F)\end{smallmatrix}$ obtained by the change-of-base along the forgetful functor U.

$$\begin{tikzcd} \mathbb{P}_F \arrow[r] \arrow[d, "U^*(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P} \arrow[d, "p"] \\ \mathbf{Coalg}(F) \arrow[r, "U"'] & \mathbb{C} \end{tikzcd}$$

Definition 4. An *interpretation* of a formula α with n free variables in $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ and F is a morphism

$$\left(\begin{smallmatrix}\mathbb{P}_F\\ \downarrow U^*(p)\\ \mathbf{Coalg}(F)\end{smallmatrix}\right)^n \xrightarrow{[\![\alpha]\!]} \left(\begin{smallmatrix}\mathbb{P}_F\\ \downarrow U^*(p)\\ \mathbf{Coalg}(F)\end{smallmatrix}\right) \quad \text{in } \mathbf{Fib}(\mathbf{Coalg}(F)) \ ,$$

i.e. a fibred functor over $\mathbf{Coalg}(F)$. It is defined inductively on formulas α. For a modal operator $[\varphi]$ associated with a predicate lifting φ, an interpretation of a formula $[\varphi]\alpha$ is given by

$$[\![[\varphi]\alpha]\!] = \overline{\varphi} \circ [\![\alpha]\!] \ ,$$

where $\overline{\varphi} : U^*(p) \to U^*(p)$ is a fibred functor induced by φ. Fixed point operators μ and ν are interpreted respectively via *fibred initial algebras and fibred final coalgebras of fibred functors.*

Fibred functors are compatible with reindexing (along coalgebra homomorphisms in this case). Therefore,

Theorem 5. *The interpretation in Def. 4 is bisimulation invariant.*

5 Future work

We gave interpretations of coalgebraic fixed point logics in a fibration, but it is itself not very useful. Following [2, 11], where coalgebraic fixed point logics (on **Set**) are studied from the automata and game theoretic viewpoint, we are now studying *automata and games in a fibrational setting.* For example, we realized that for a predicate lifting φ, φ-coalgebras can be seen as successful run trees of the (trivial) automaton corresponding to the φ-coinductive predicate. We are trying to extend it to the relationship between fixed point logics and parity automata. We wish eventually to clarify the relationship between fixed point logics, automata and games at the fibrational level of abstraction.

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