Coalgebraic Fixed Point Logics in a Fibration

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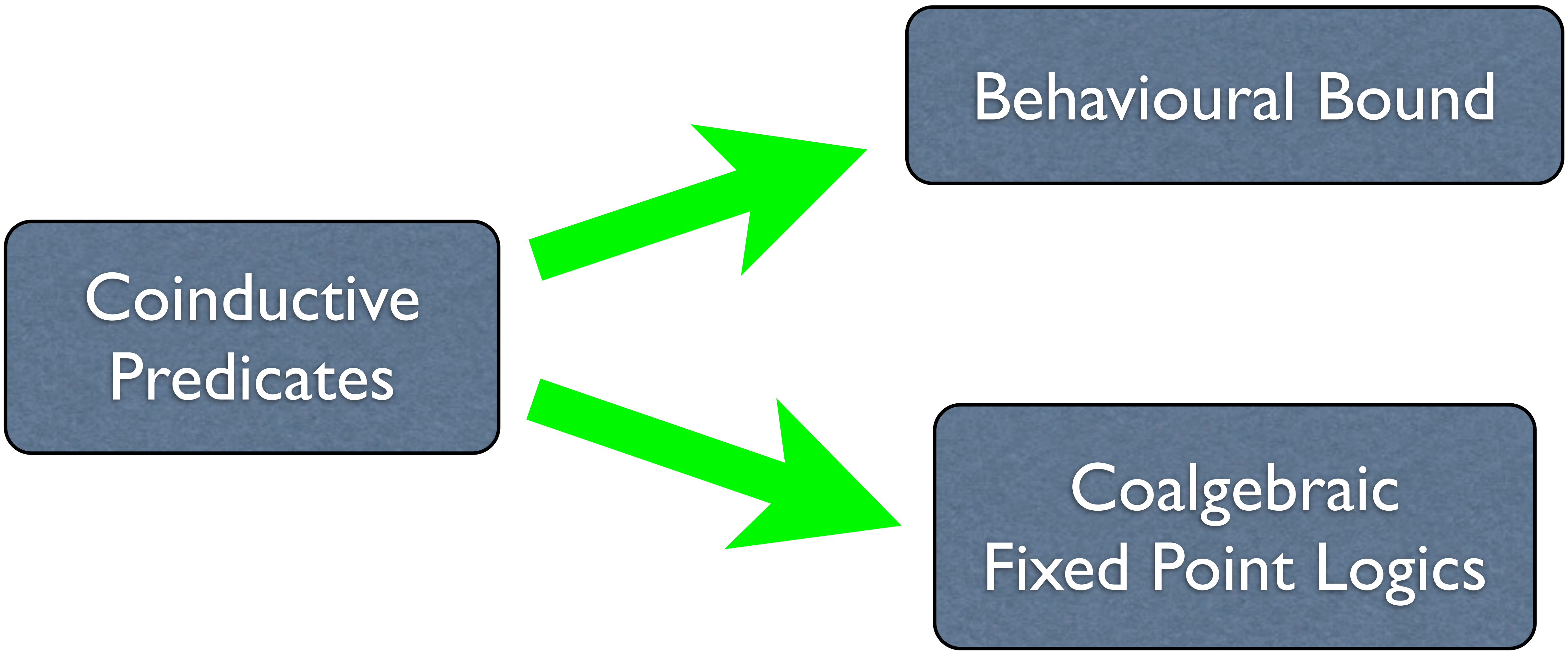
CALCO Early Ideas 2013

2 September 2013

Our work:
Fibrational approaches to



Our work:
Fibrational approaches to

fibration $\begin{array}{c}\mathbb{P}\\ \downarrow{p}\\ \mathbb{C}\end{array}$:
categorical model of predicate logic

Coinductive Predicates

Behavioural Bound

Coalgebraic Fixed Point Logics

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[MFPS 2013]

Behavioural Bound

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work in progress

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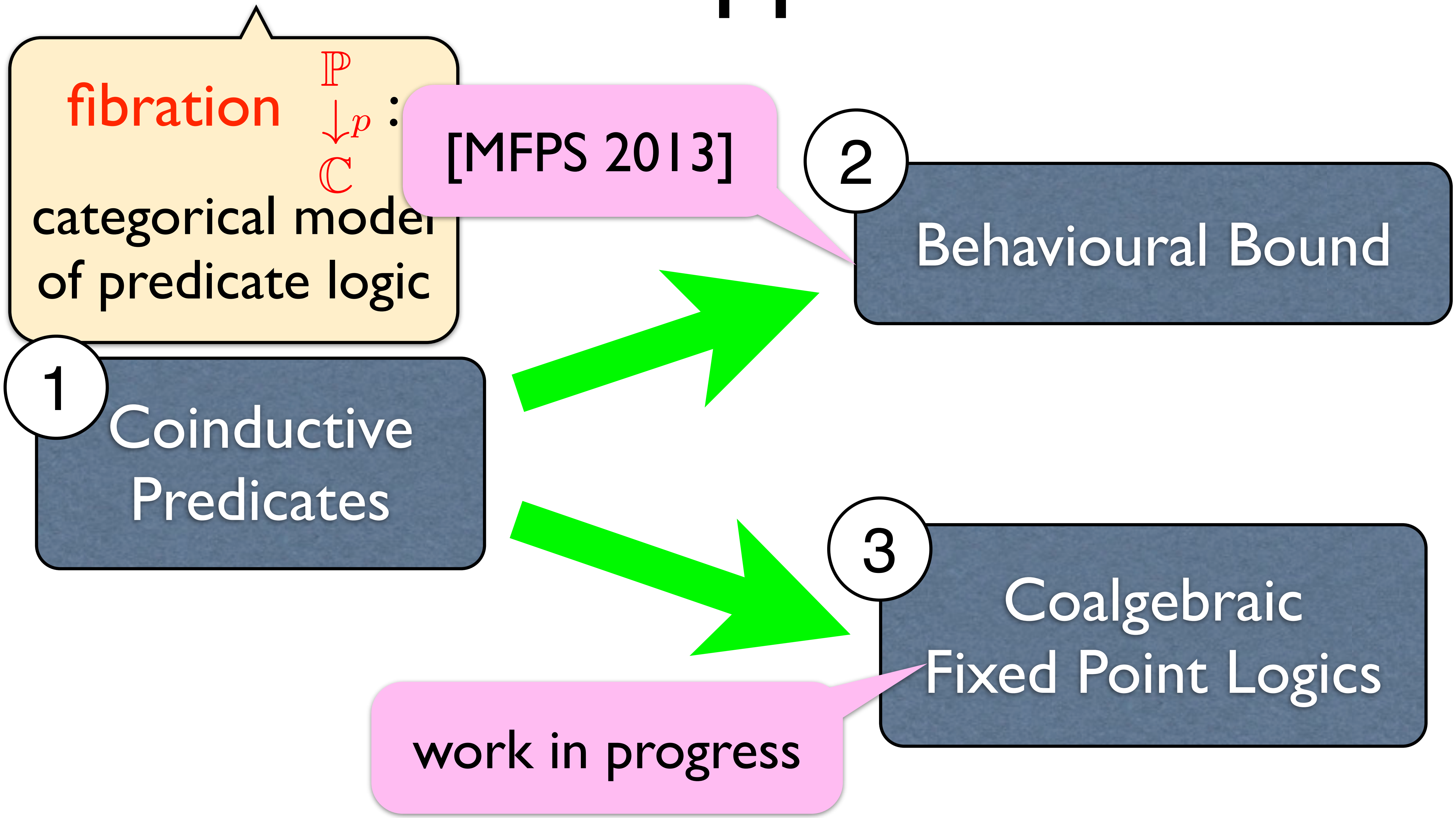

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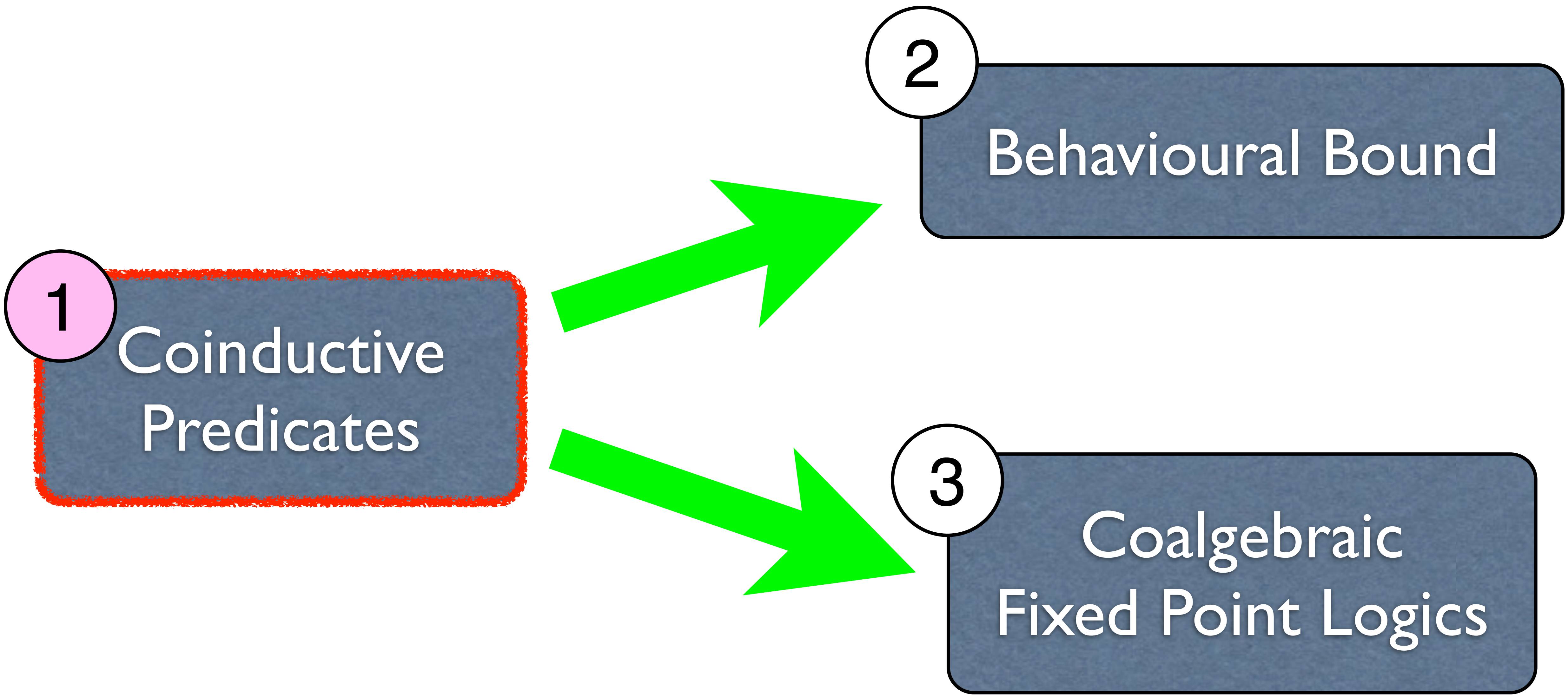

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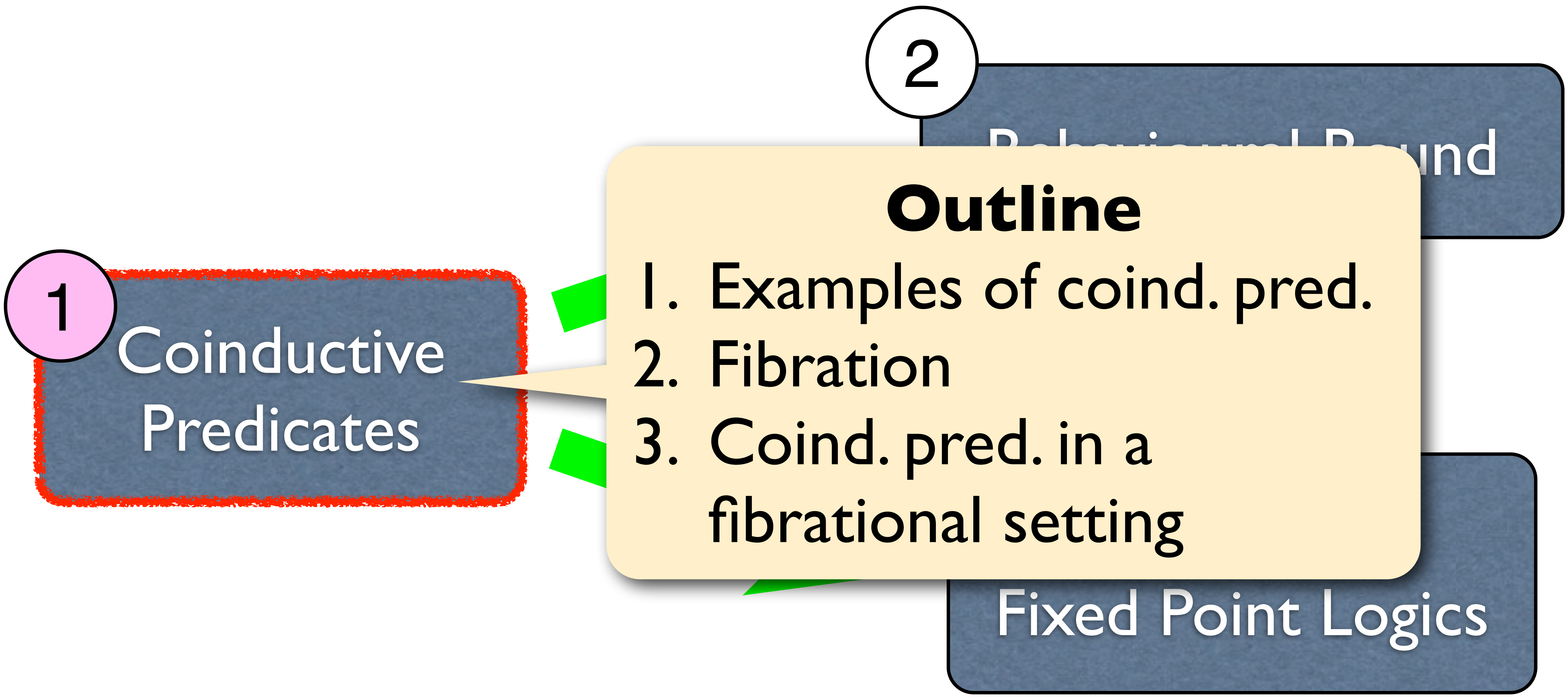


Coinductive Predicates

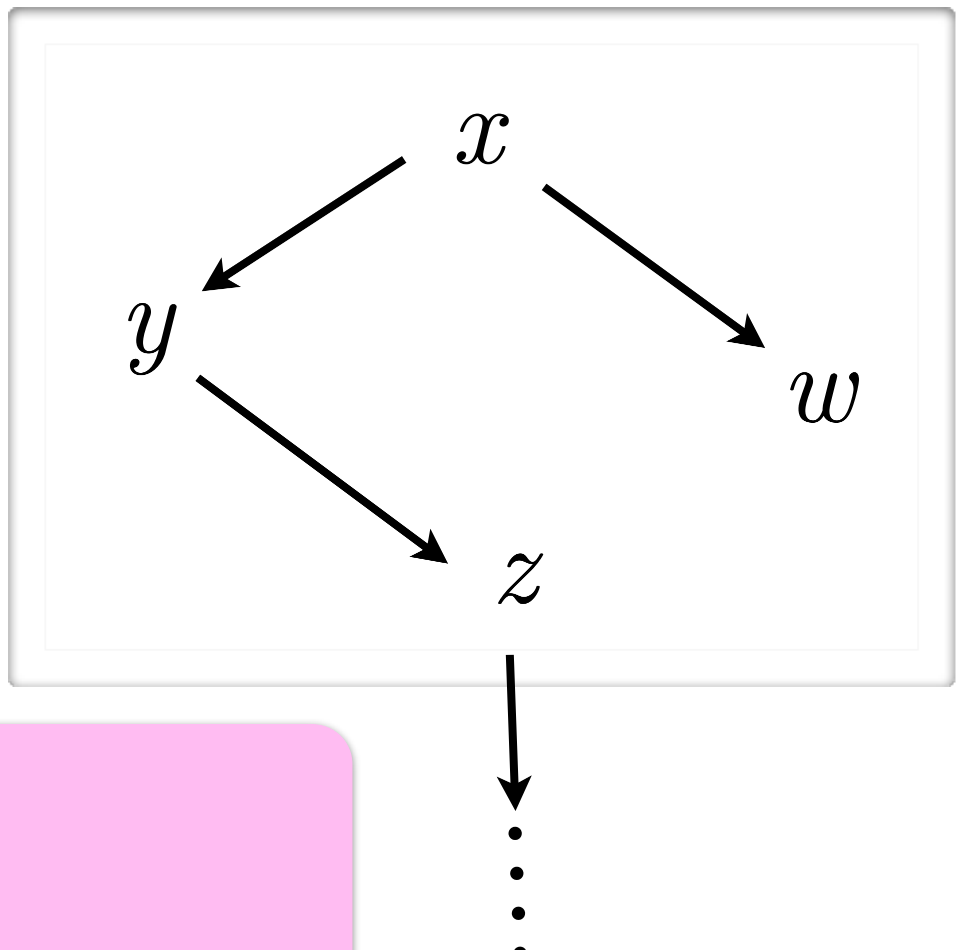
- Persisting predicates in transition/dynamical systems
  - Safety property
  - ν in fixed point logics
  - G in LTL/CTL

Example of coinductive predicates

$$\nu u.\,\Diamond u$$

Interpreted on a Kripke frame

$$\frac{c : X \to \mathcal{P}X}{\to_c \subseteq X \times X}$$



Note

$$x \models \Diamond\alpha \quad \stackrel{\text{def}}{\iff}$$

$$x \to_c \exists x' \text{ s.t. } x' \models \alpha$$

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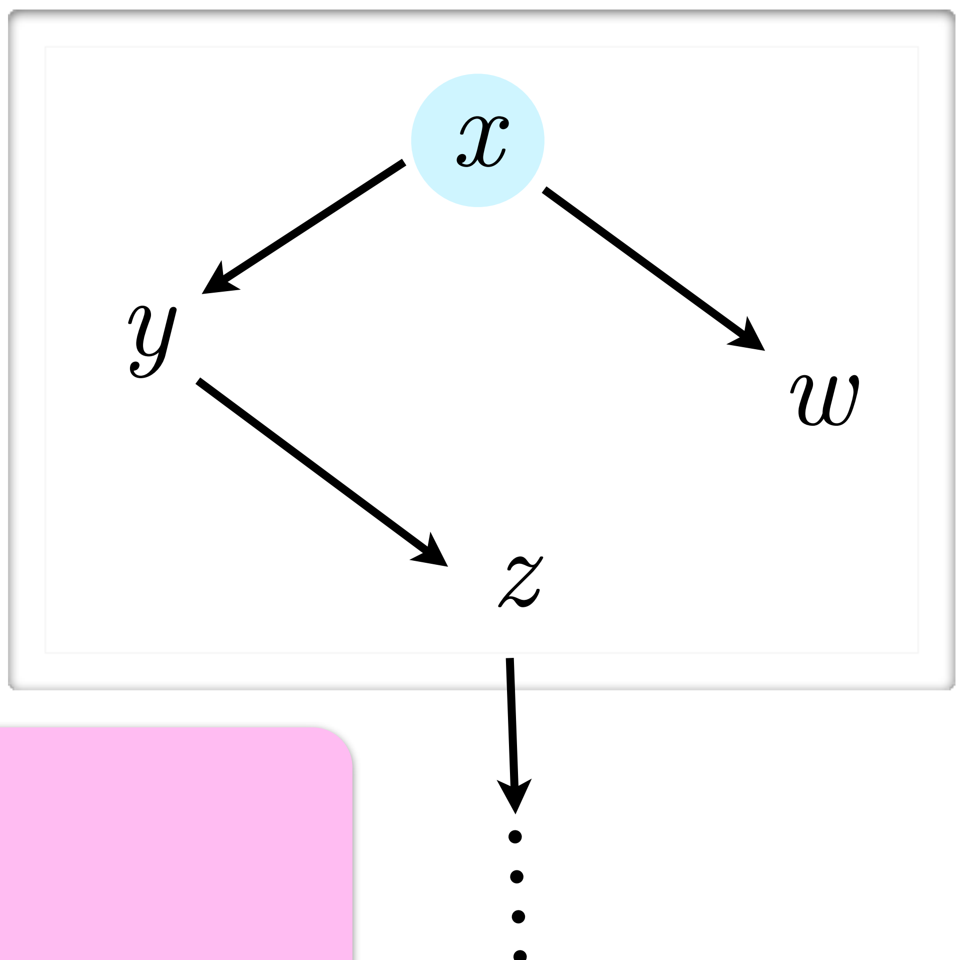

$$x \models \nu u.\, \Diamond u$$

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$$x \models \Diamond\alpha \quad \stackrel{\text{def}}{\Longleftrightarrow}$$

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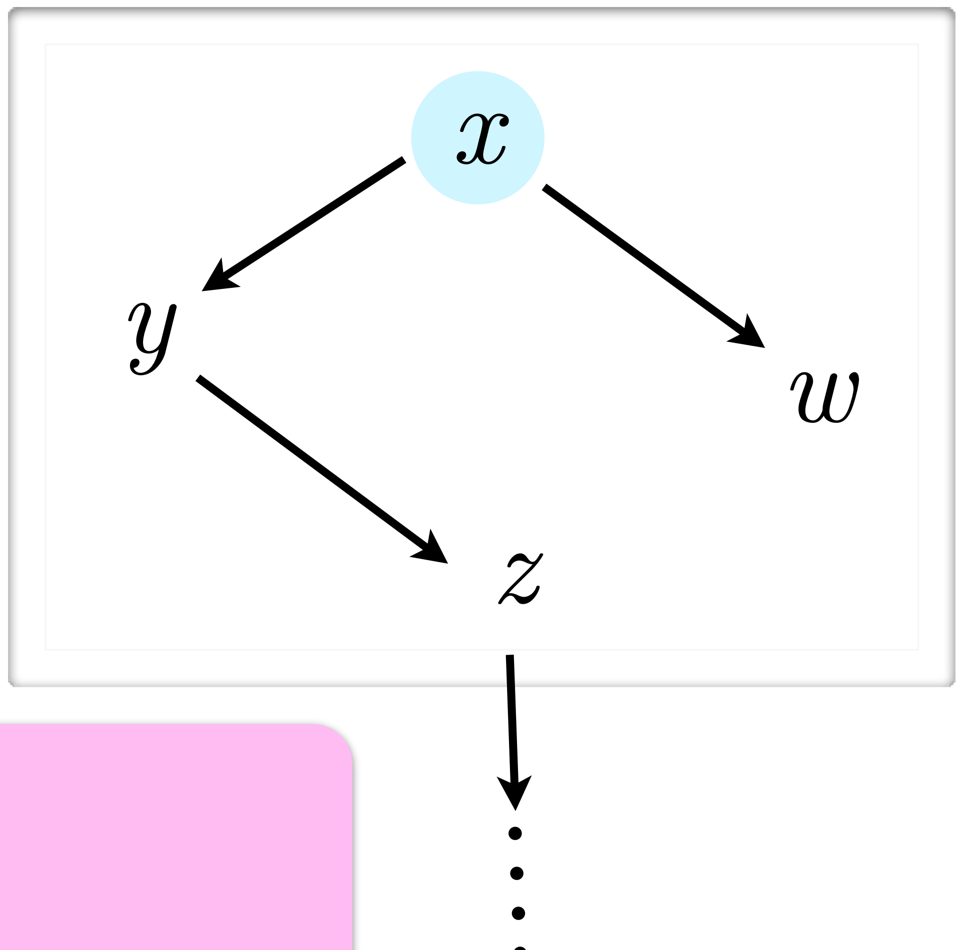

$$x \models \nu u.\, \Diamond u$$
$$\approx \Diamond(\nu u.\, \Diamond u)$$

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$$x \models \Diamond \alpha \quad \stackrel{\text{def}}{\iff}$$

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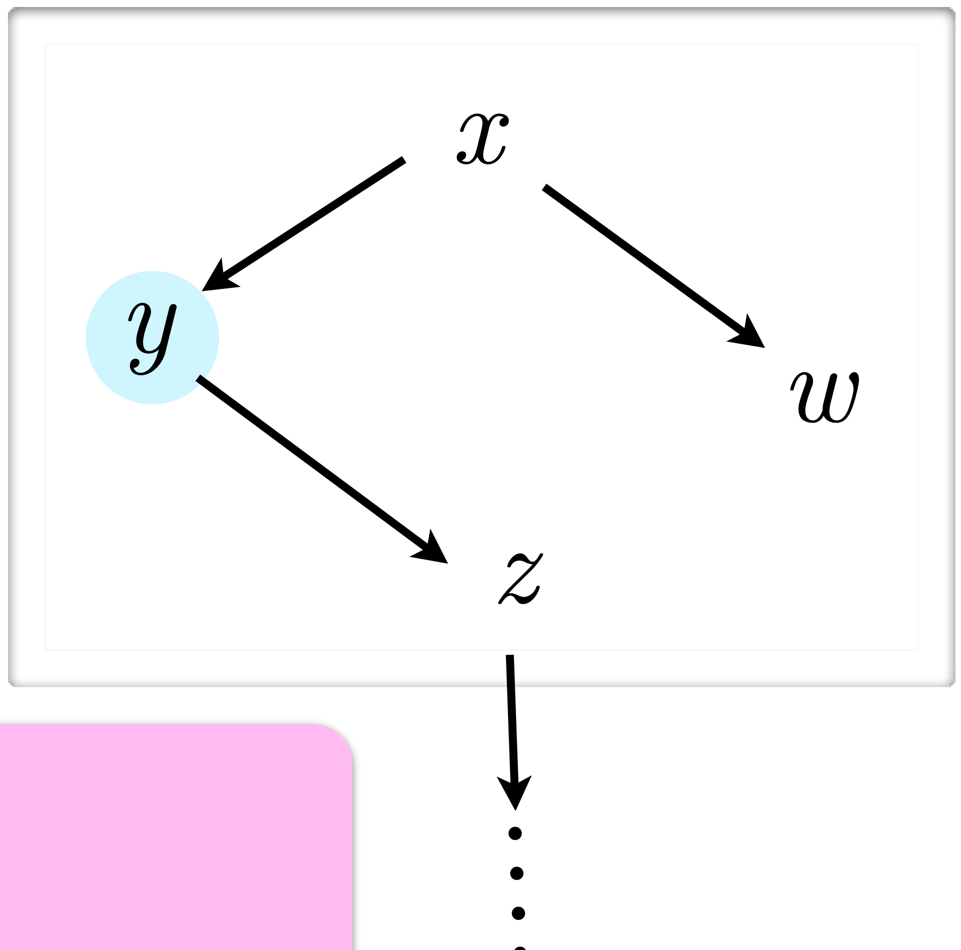

$$x \vDash \nu u.\, \Diamond u$$
$$\approx \Diamond(\nu u.\, \Diamond u)$$
$$y \vDash \nu u.\, \Diamond u$$

Note

$$x \vDash \Diamond \alpha \quad \stackrel{\text{def}}{\Longleftrightarrow}$$

$$x \to_c \exists x' \text{ s.t. } x' \vDash \alpha$$

Example of coinductive predicates

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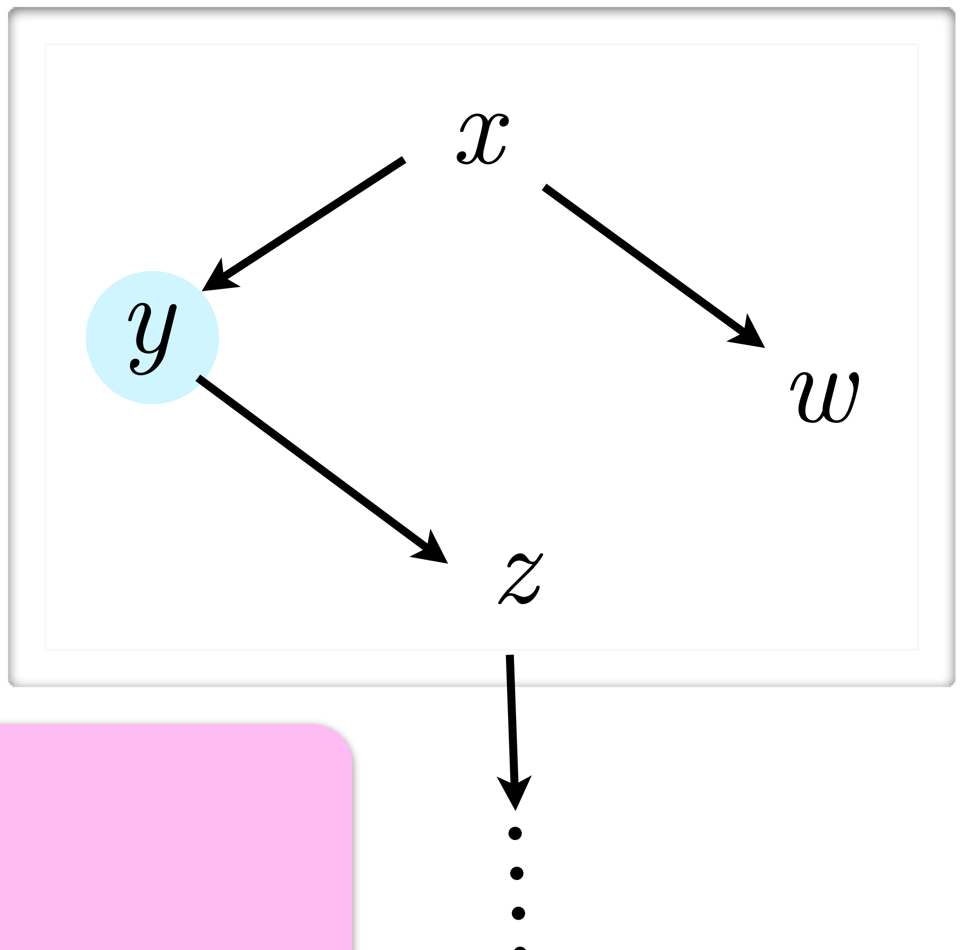

$$x \models \nu u.\, \Diamond u \approx \Diamond(\nu u.\, \Diamond u)$$

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Example of coinductive predicates

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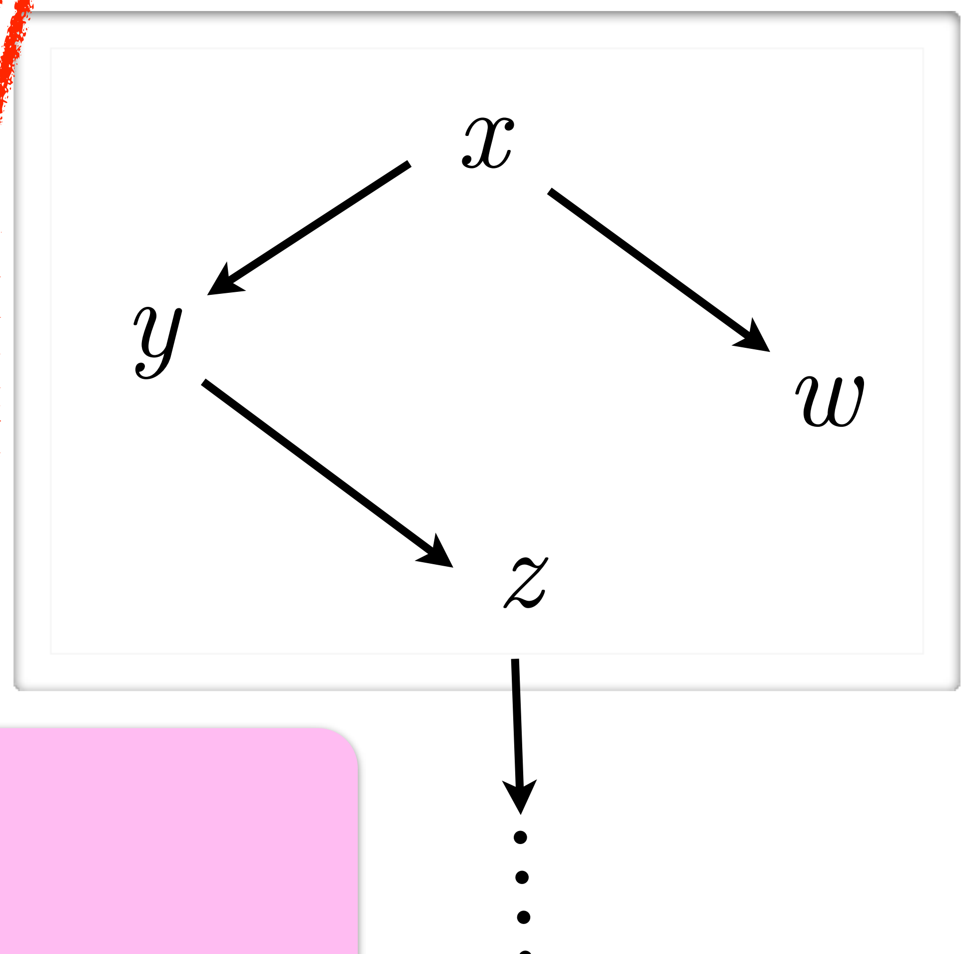
$$\frac{c : X \to \mathcal{P}X}{\to_c \subseteq X \times X}$$

There is an infinite path!



$$x \models \nu u.\, \Diamond u \approx \Diamond(\nu u.\, \Diamond u)$$

$$y \models \nu u.\, \Diamond u \approx \Diamond(\nu u.\, \Diamond u)$$

$$z \models \nu u.\, \Diamond u \approx \Diamond(\nu u.\, \Diamond u)$$

$$\vdots$$

Note

$$x \models \Diamond \alpha \quad \stackrel{\text{def}}{\iff} \quad x \to_c \exists x' \text{ s.t. } x' \models \alpha$$

Formal definition

"(Co)recursive definition"

$$[\![\nu u.\, \Diamond u]\!]_{\substack{\mathcal{P}X \\ \uparrow c \\ X}} = \mathrm{gfp}\Big(2^X \xrightarrow{\varphi_\Diamond} 2^{\mathcal{P}X} \xrightarrow{c^{-1}} 2^X \Big)$$

$$P \overset{\varphi_\Diamond}{\mapsto} \{U \in \mathcal{P}X \mid U \cap P \neq \varnothing\} \overset{c^{-1}}{\mapsto} \{x \in X \mid c(x) \cap P \neq \varnothing\}$$

Greatest fixed point

The gfp exists by the Knaster-Tarski theorem

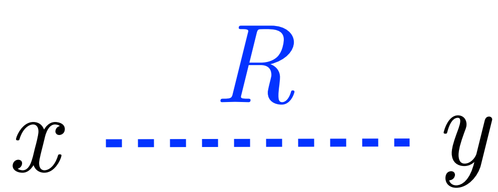
Another example: Bisimilarity
(on Kripke frames)

Let $X \overset{c}{\to} \mathcal{P}X$ and $Y \overset{d}{\to} \mathcal{P}Y$ be Kripke frames.

on c and d

Def. A *bisimulation* is a relation $R \subseteq X \times Y$ s.t.

$$xRy \implies \begin{cases} \forall x' \in c(x).\, \exists y' \in d(y).\ x'Ry' \ \& \\ \forall y' \in d(y).\, \exists x' \in c(x).\ x'Ry' \ . \end{cases}$$



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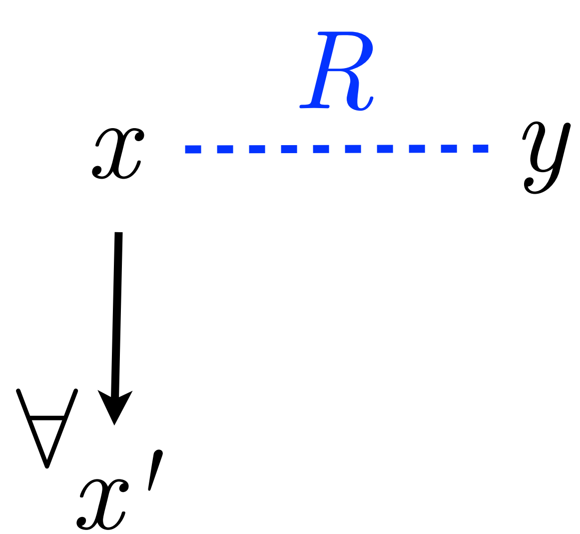

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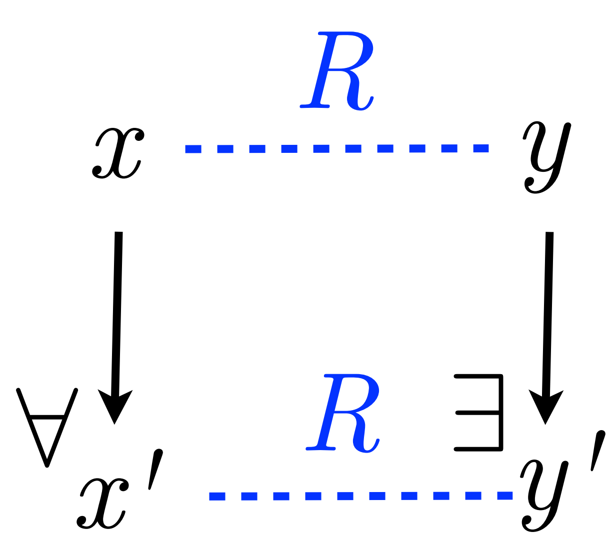

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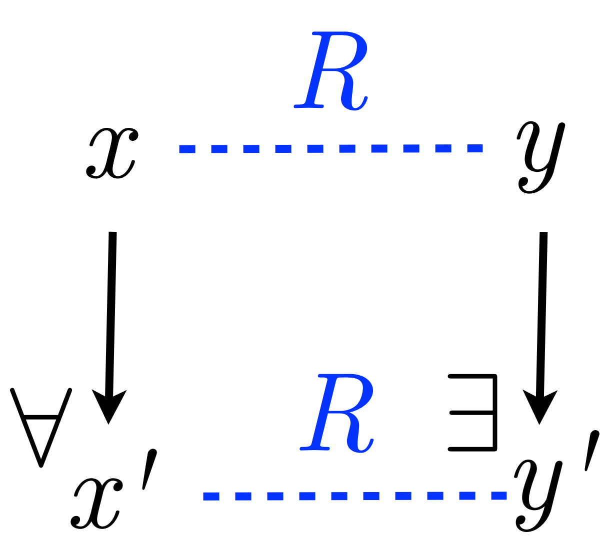

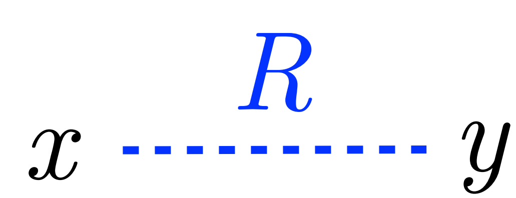

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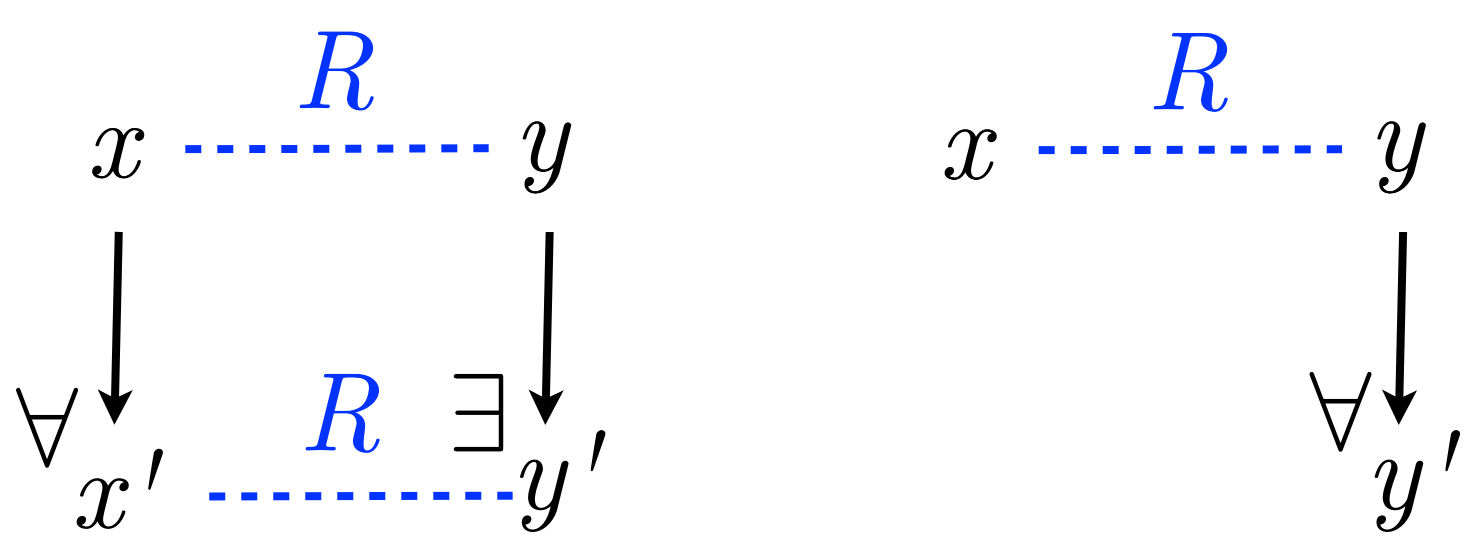

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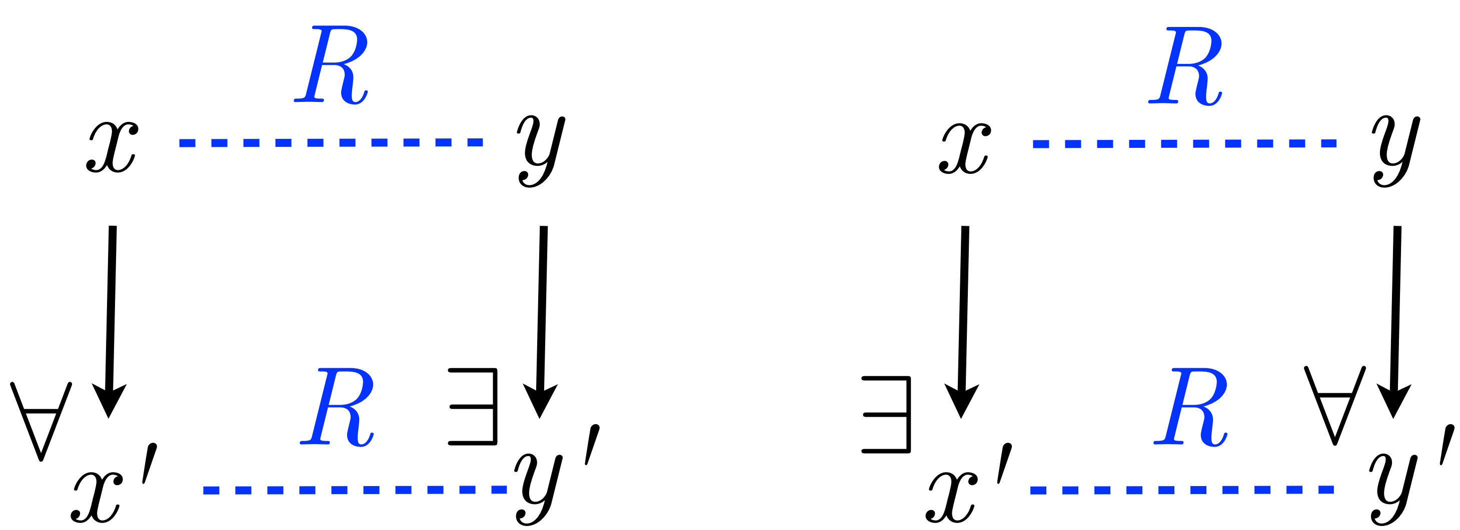

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on c and d

Def. $x \in X$ and $y \in Y$ are *bisimilar*, written $x \sim_{c,d} y$, if $\exists$bisimulation R s.t. xRy. Explicitly,

$$\sim_{c,d} = \bigcup \{ \text{ a bisimulation on } c \text{ and } d \ \} \ .$$

Bisimilarity as the gfp

Prop.

$$\sim_{c,d} = \mathrm{gfp}\Big(2^{X\times Y} \xrightarrow{\rho} 2^{\mathcal{P}X\times\mathcal{P}Y} \xrightarrow{(c\times d)^{-1}} 2^{X\times Y} \Big)$$

$$R \overset{\rho}{\mapsto} \begin{array}{l}\{(U,V)\in\mathcal{P}X\times\mathcal{P}Y \mid \\ \forall x\in U.\,\exists y\in V.\,xRy\ \&\\ \forall y\in V.\,\exists x\in U.\,xRy \quad \}\end{array} \overset{(c\times d)^{-1}}{\mapsto} \begin{array}{l}\{(x,y)\in X\times Y \mid \\ \forall c(x)\in U.\,\exists d(y)\in V.\,xRy\ \&\\ \forall d(y)\in V.\,\exists c(x)\in U.\,xRy \quad \}\end{array}$$

It follows from:

$$\mathrm{gfp}((c\times d)^{-1}\circ\rho) = \bigcup\{R\in 2^{X\times Y} \mid R\subseteq ((c\times d)^{-1}\circ\rho)(R)\}$$

by Knaster-Tarski, and

$$R\subseteq ((c\times d)^{-1}\circ\rho)(R) \iff R \text{ is a bisimulation on } c \text{ and } d$$

Bisimilarity as the gfp

Prop.

$$\sim_{c,d} = \mathrm{gfp}\Big(\; 2^{X\times Y} \xrightarrow{\;\rho\;} 2^{\mathcal{P}X\times\mathcal{P}Y} \xrightarrow{(c\times d)^{-1}} 2^{X\times Y} \;\Big)$$

$$R \overset{\rho}{\mapsto} \begin{array}{l}\{(U,V)\in\mathcal{P}X\times\mathcal{P}Y \mid \\ \quad \forall x\in U.\,\exists y\in V.\,xRy \;\&\; \\ \quad \forall y\in V.\,\exists x\in U.\,xRy \quad\}\end{array} \overset{\times d)^{-1}}{\mapsto} \begin{array}{l}\{(x,y)\in X\times Y \mid \\ \quad \forall c(x)\in U.\,\exists d(y)\in V.\,xRy \;\&\; \\ \quad \forall d(y)\in V.\,\exists c(x)\in U.\,xRy \quad\}\end{array}$$

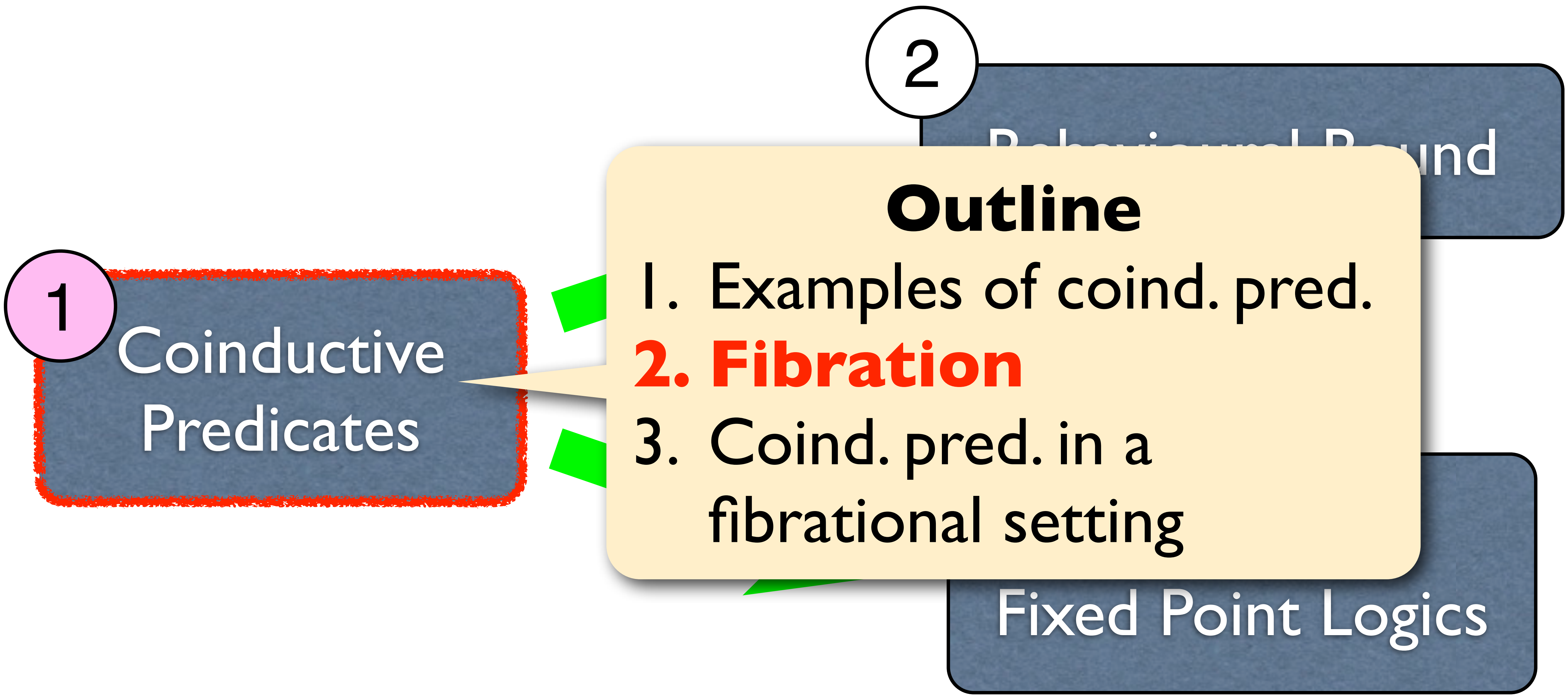
It

by

Much the same as

$$[\![\nu u.\,\Diamond u]\!]_{\substack{\mathcal{P}X\\ \uparrow c\\ X}} = \mathrm{gfp}\Big(\; 2^{X} \xrightarrow{\;\varphi_\Diamond\;} 2^{\mathcal{P}X} \xrightarrow{\;c^{-1}\;} 2^{X} \;\Big)$$

Our work:
Fibrational approaches to

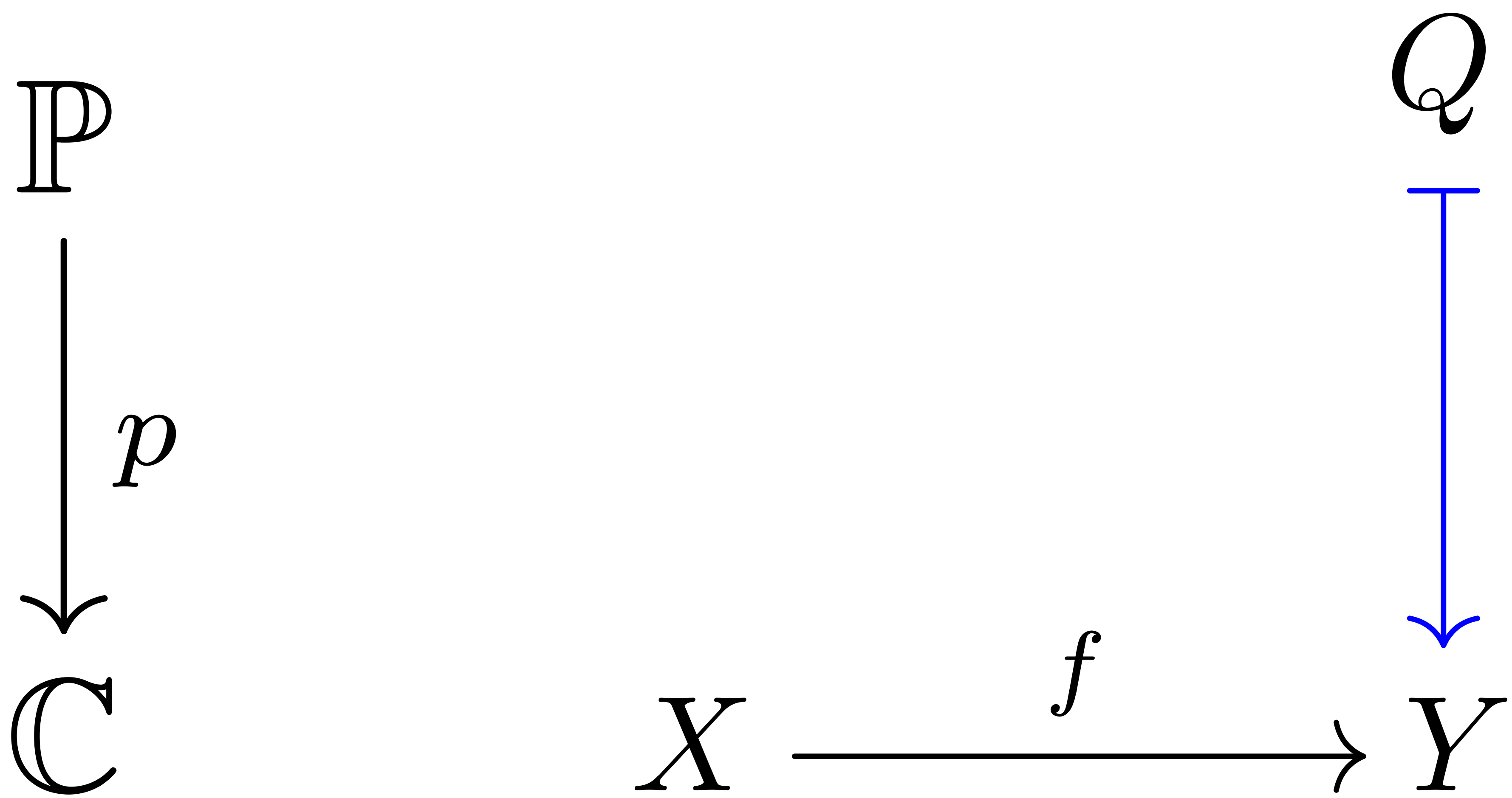


Fibration

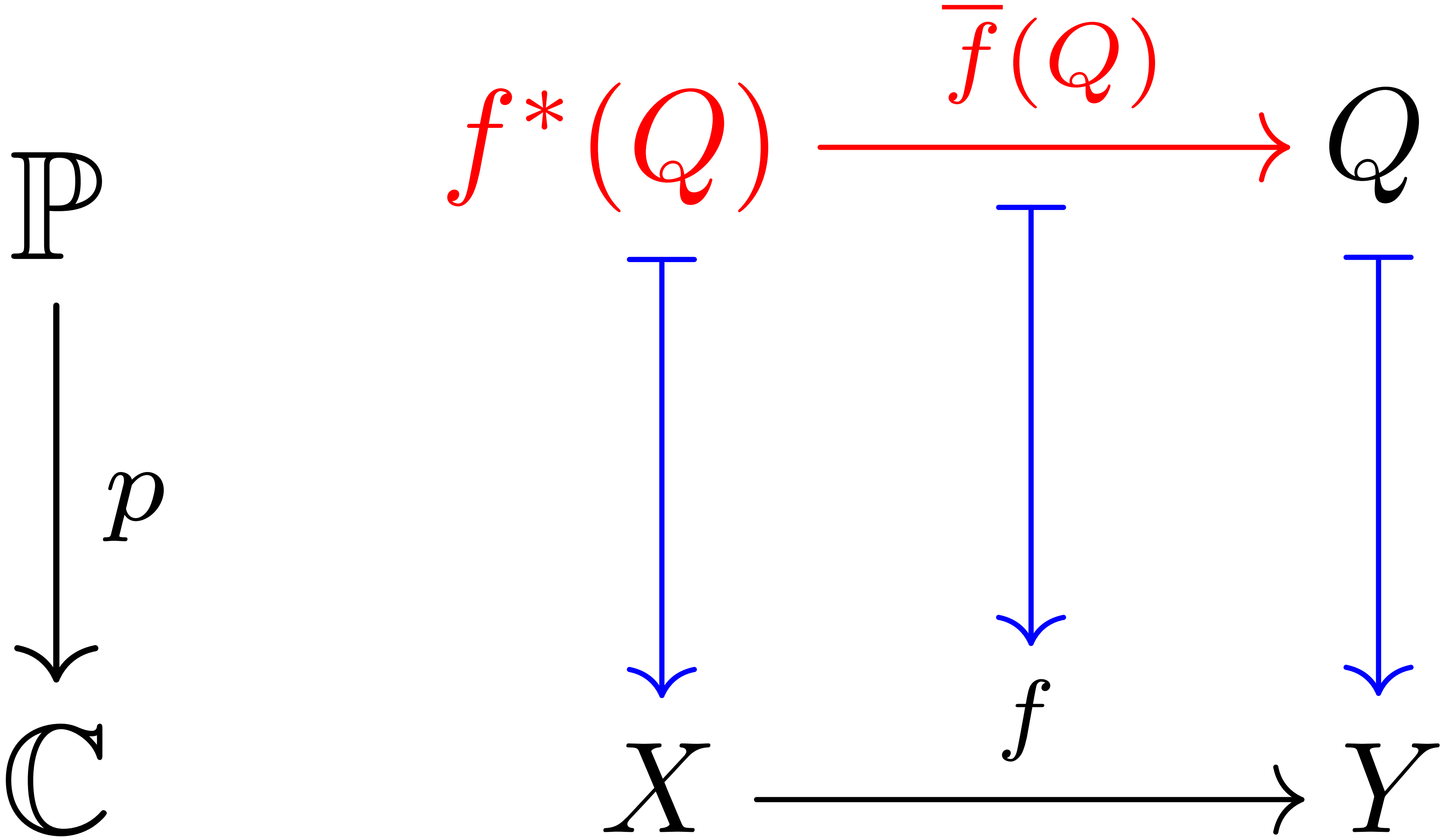
- Categorical way of organizing indexed entities
- Categorical model of predicate logics

Def. A *fibration* is a functor $p : \mathbb{P} \to \mathbb{C}$ that has Cartesian liftings.

Cartesian lifting

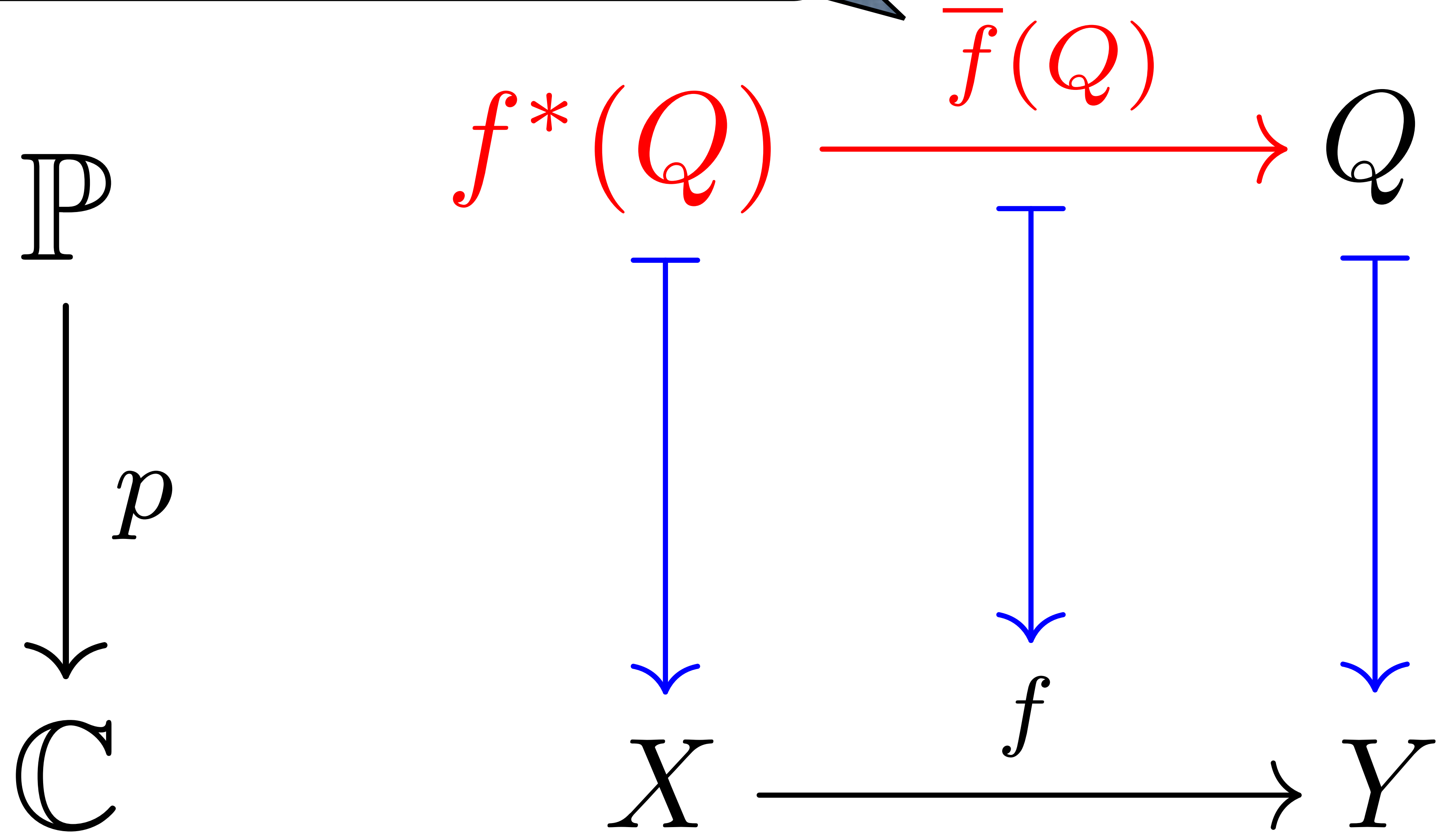


Cartesian lifting



Cartesian lifting

satisfy certain universality



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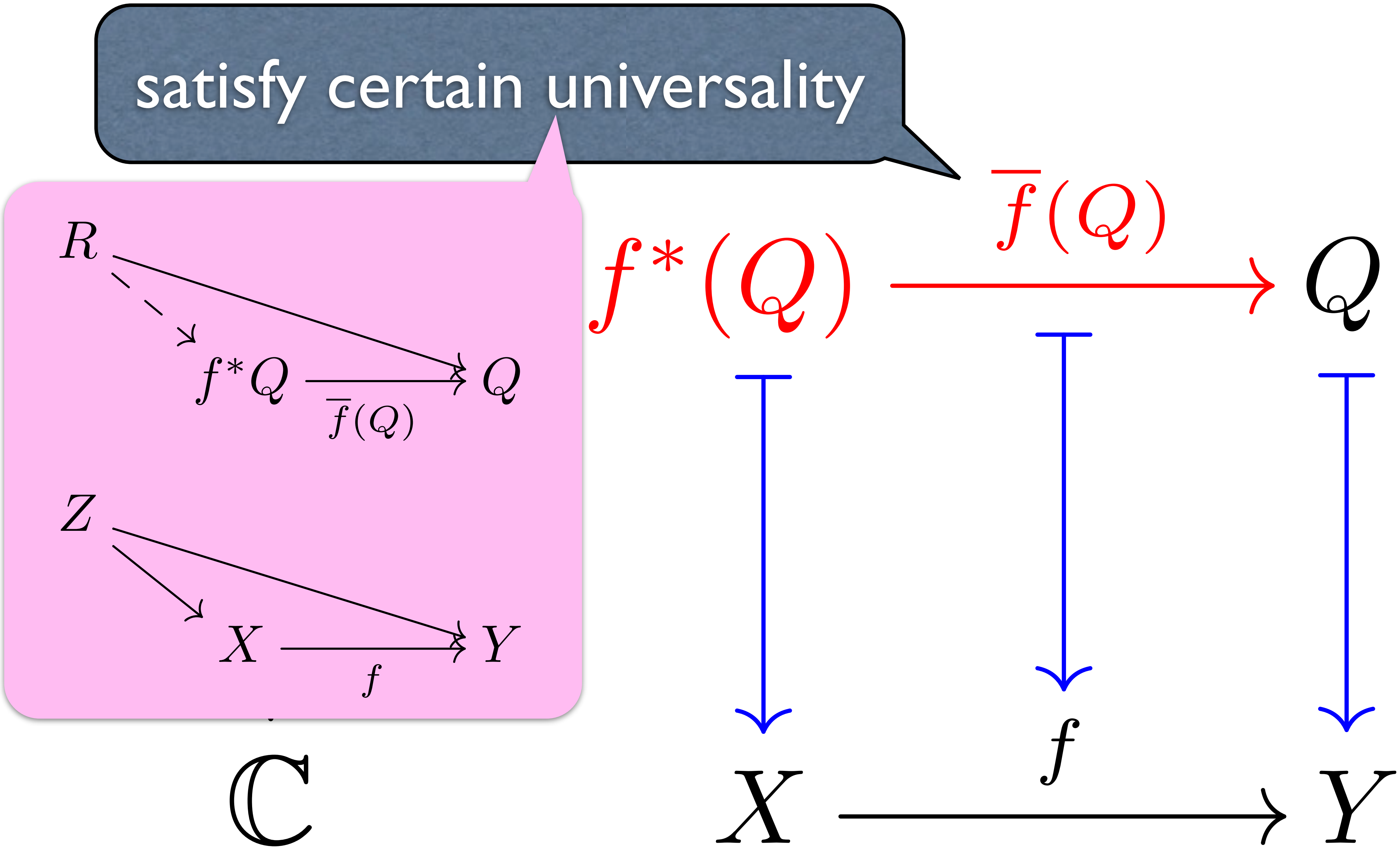
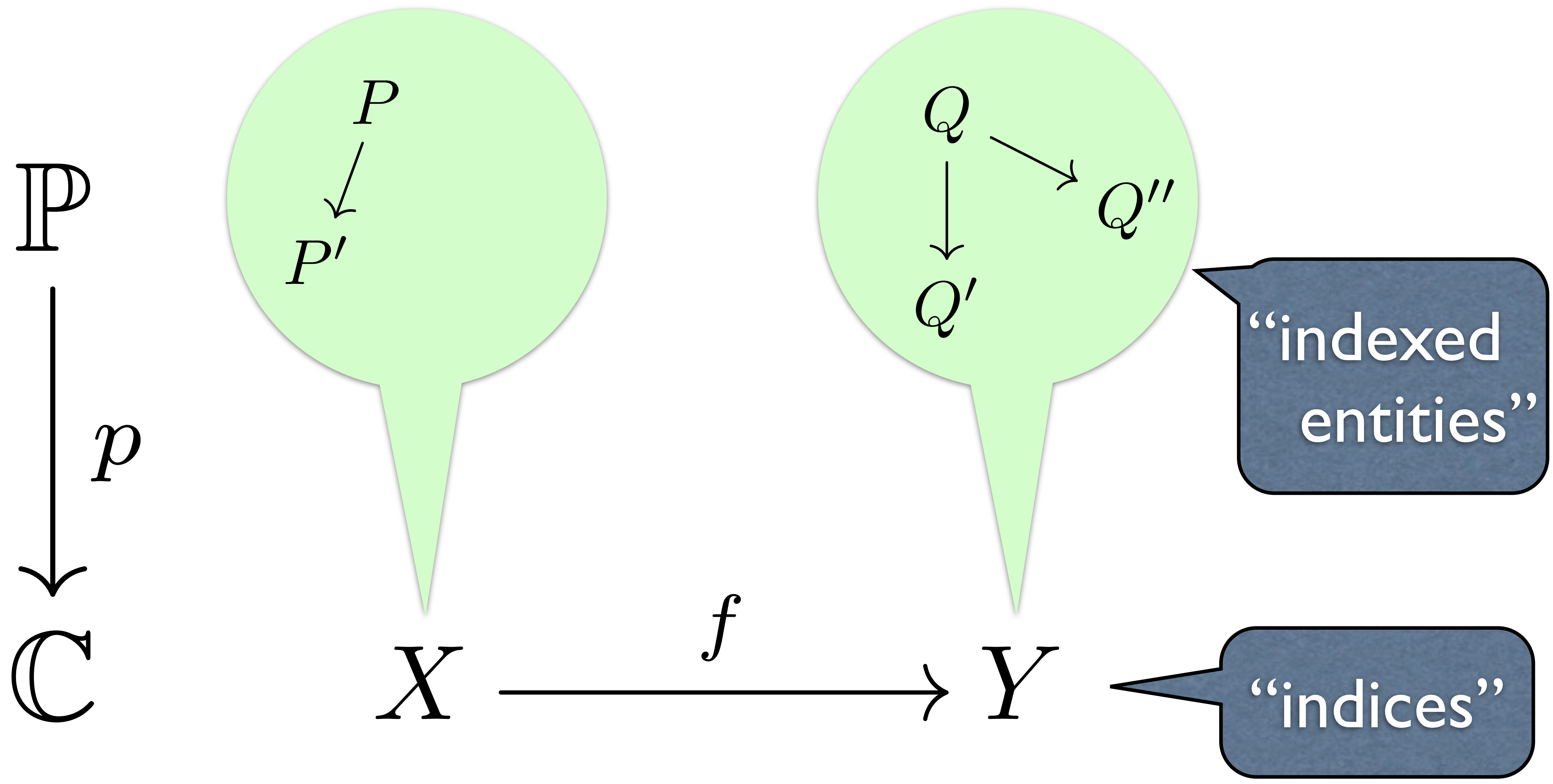


Fibration organizes indexed entities



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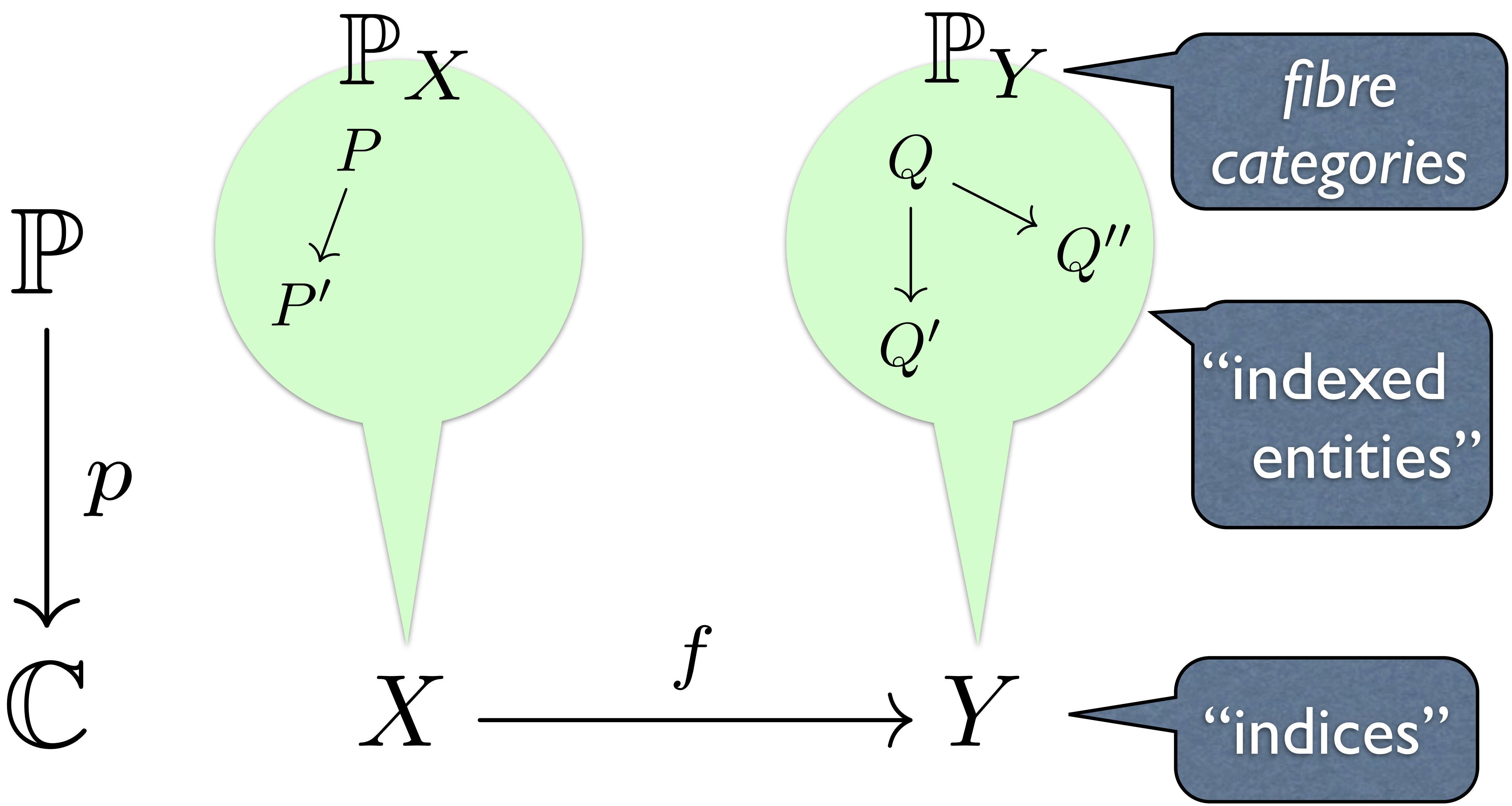

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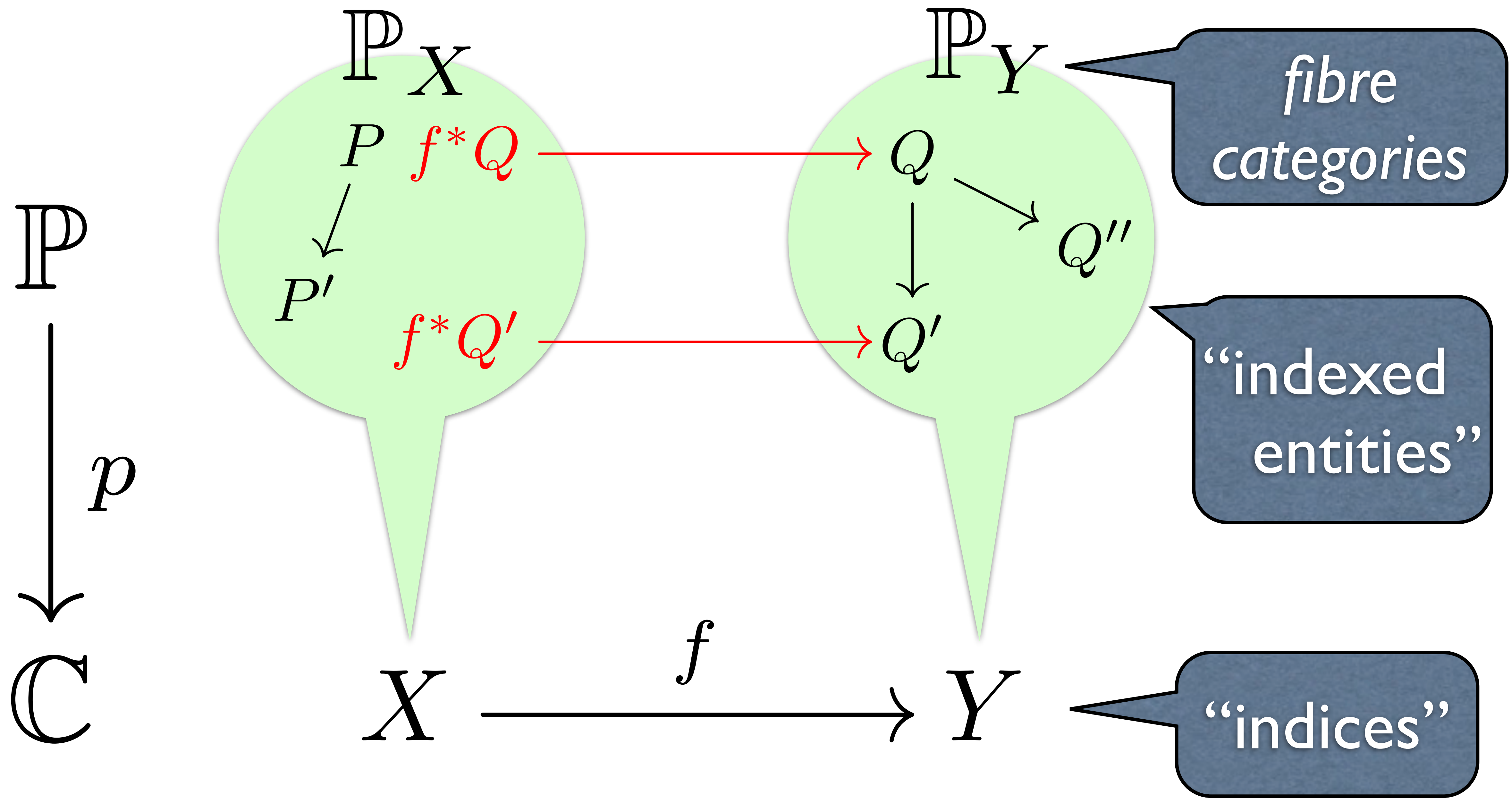

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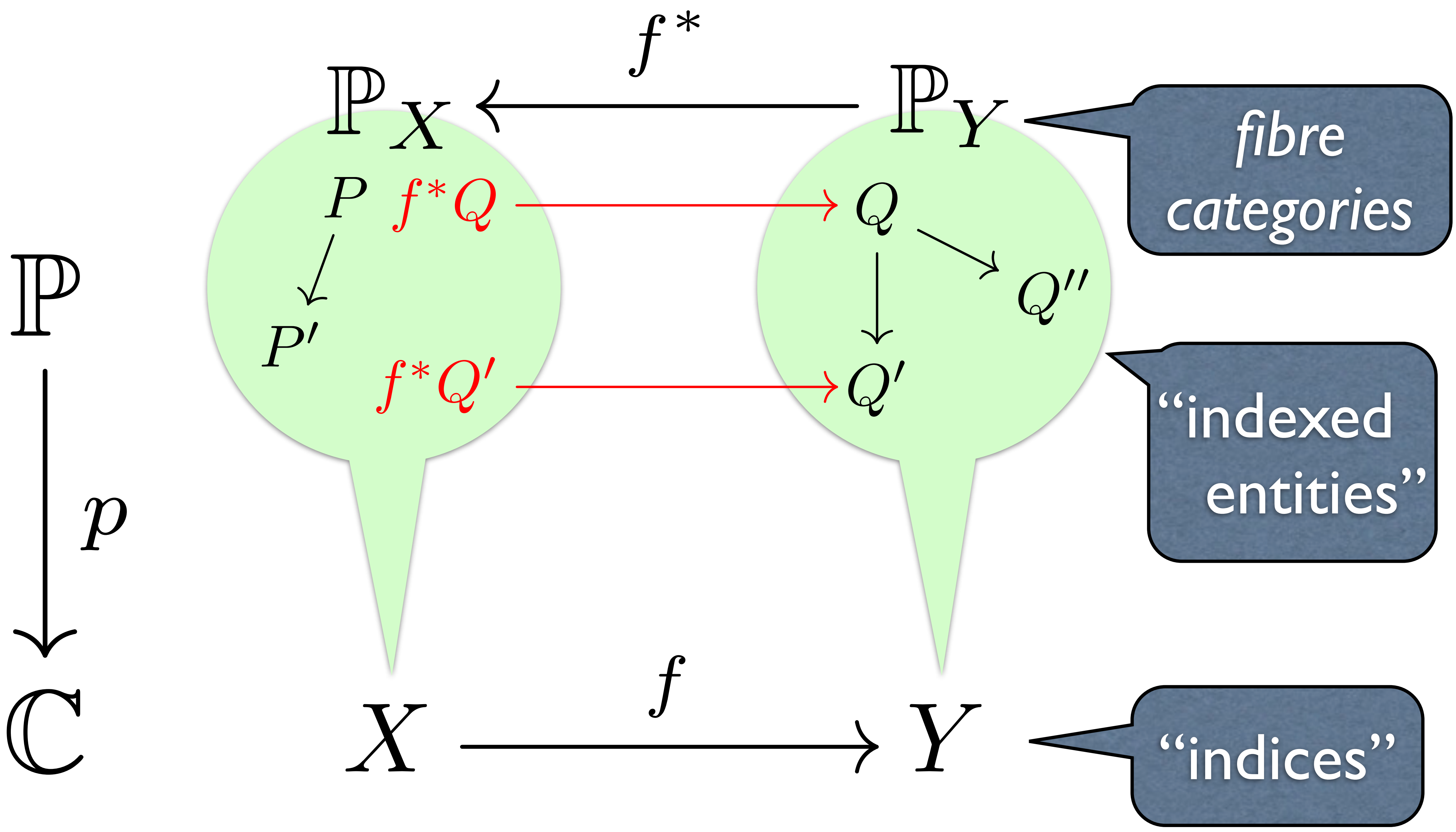

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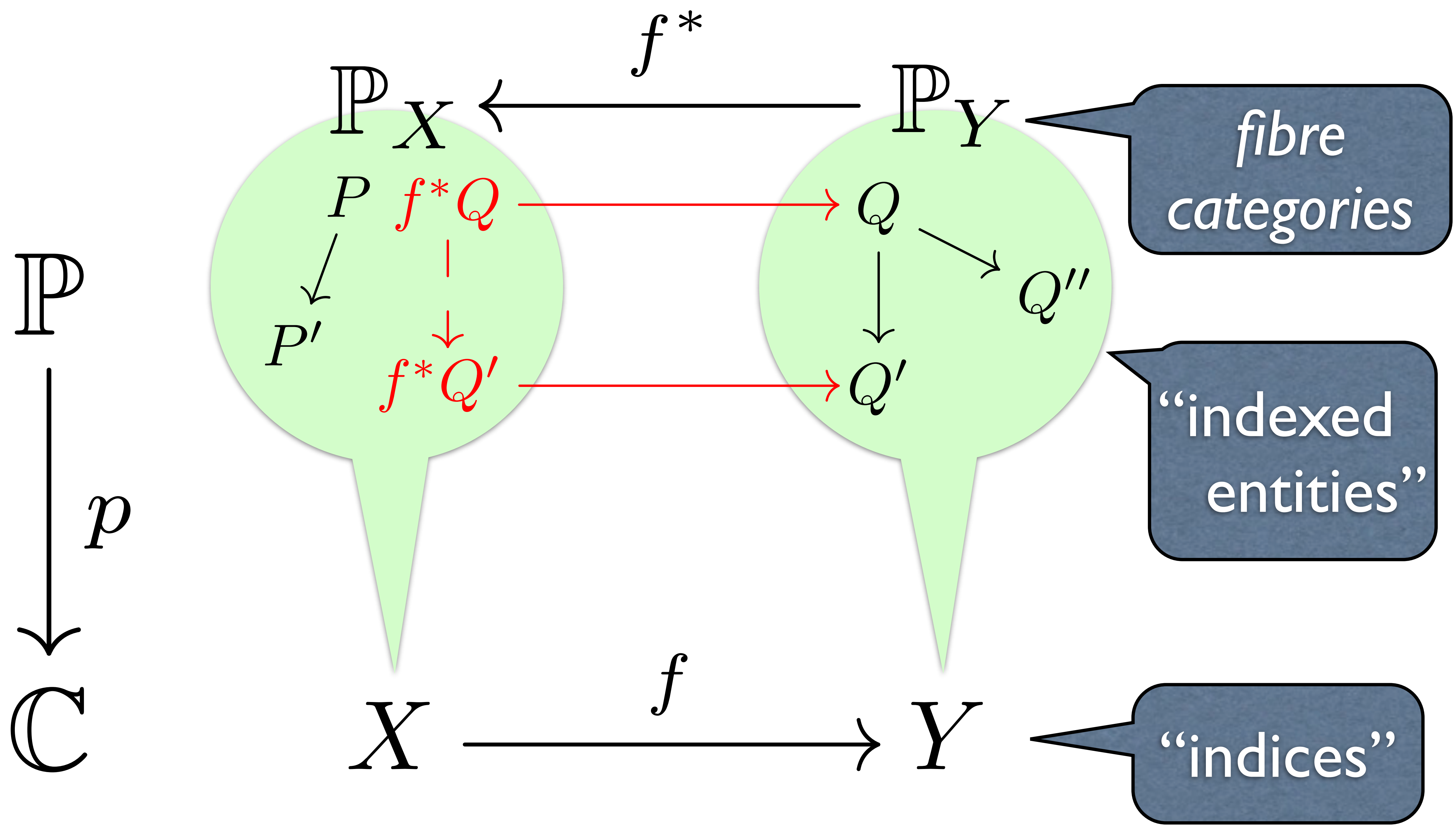

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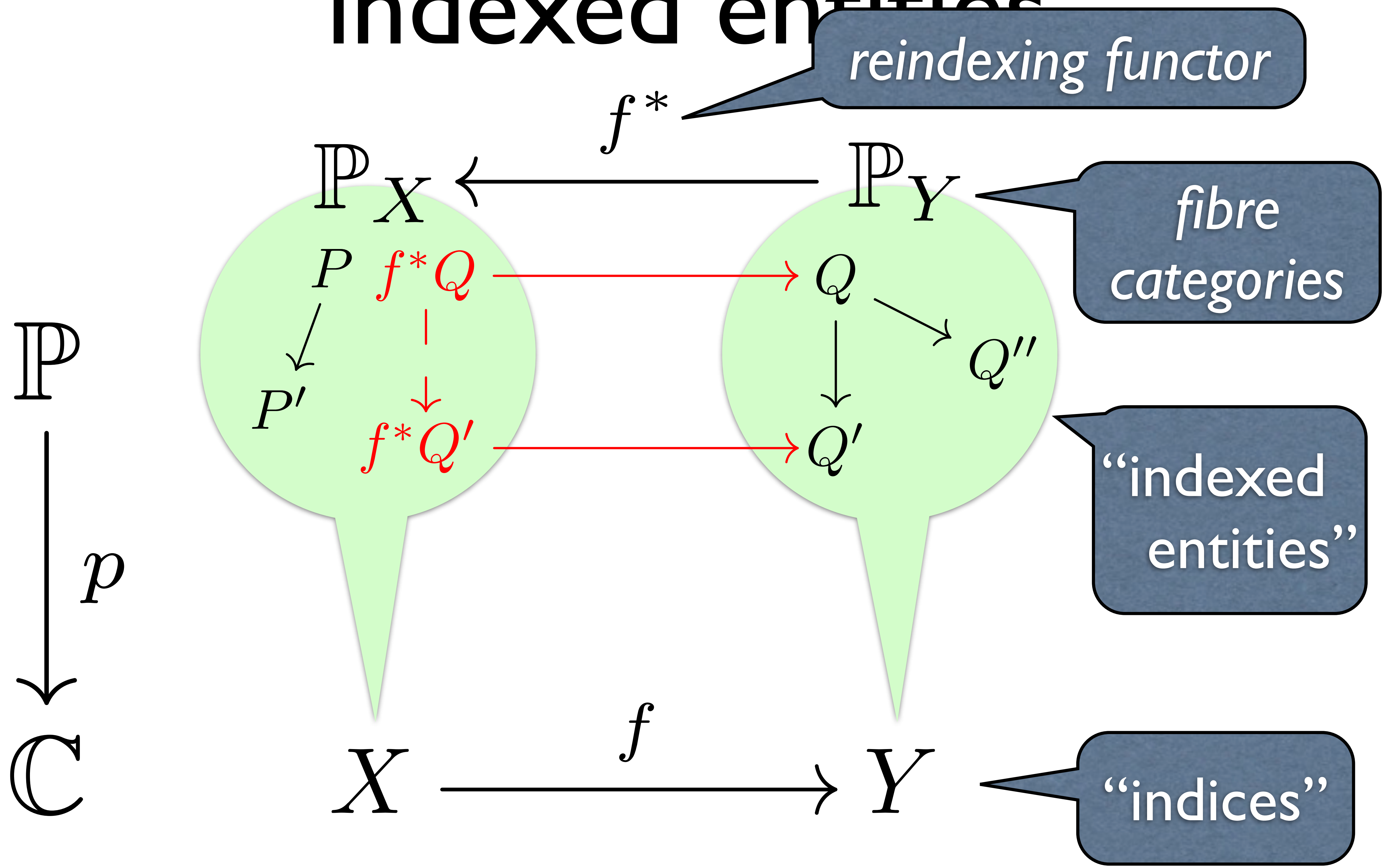
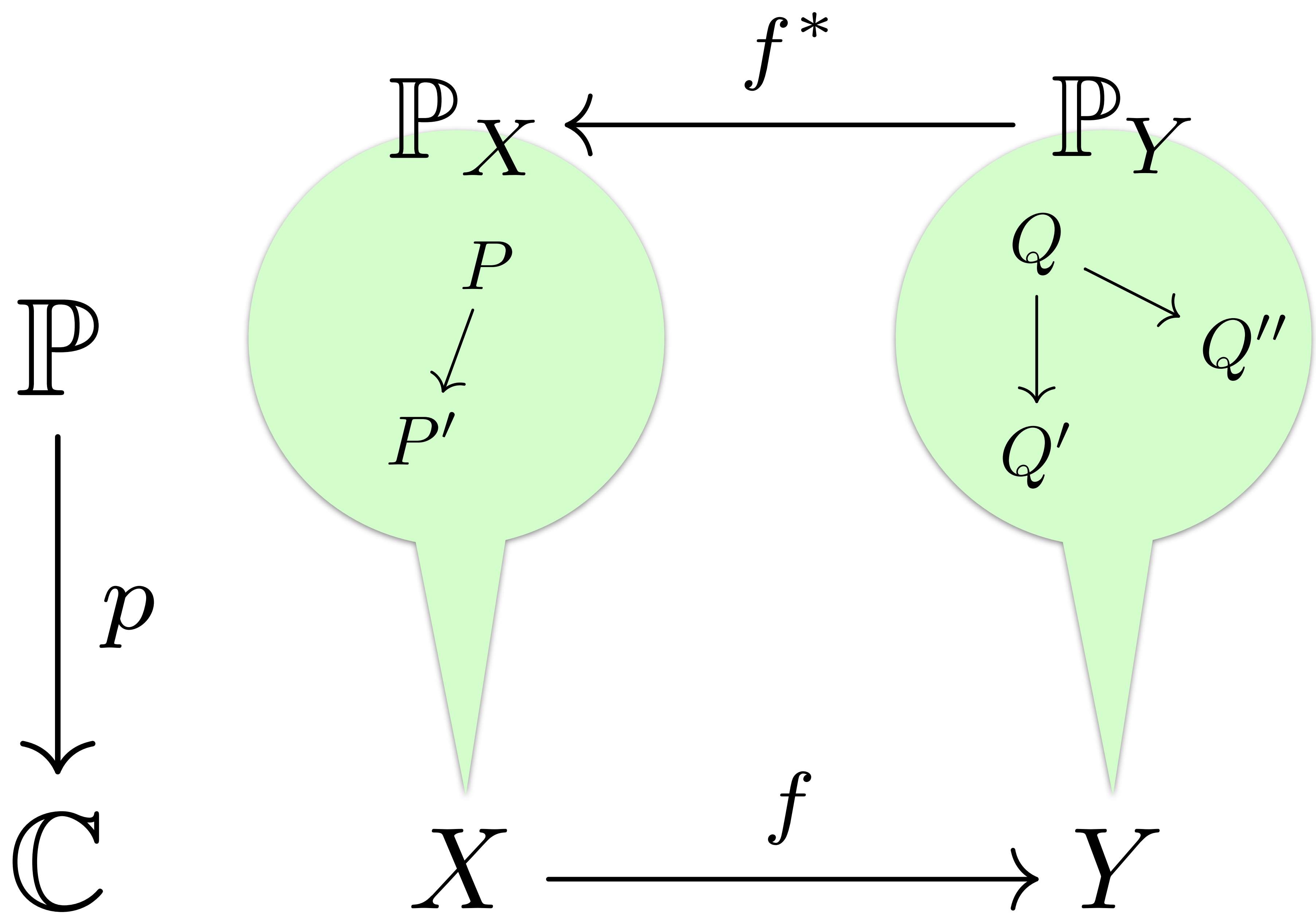


Logical view



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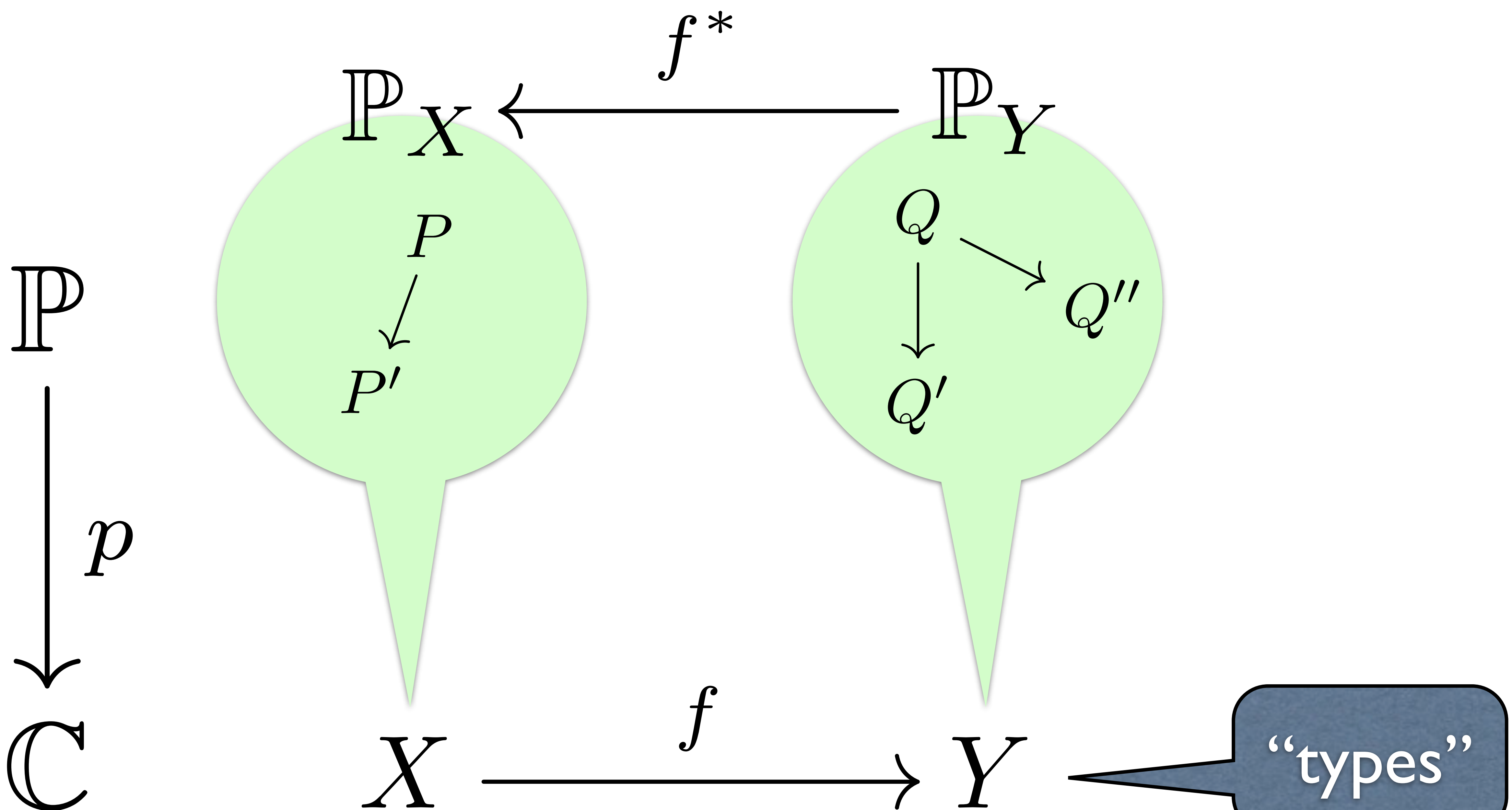

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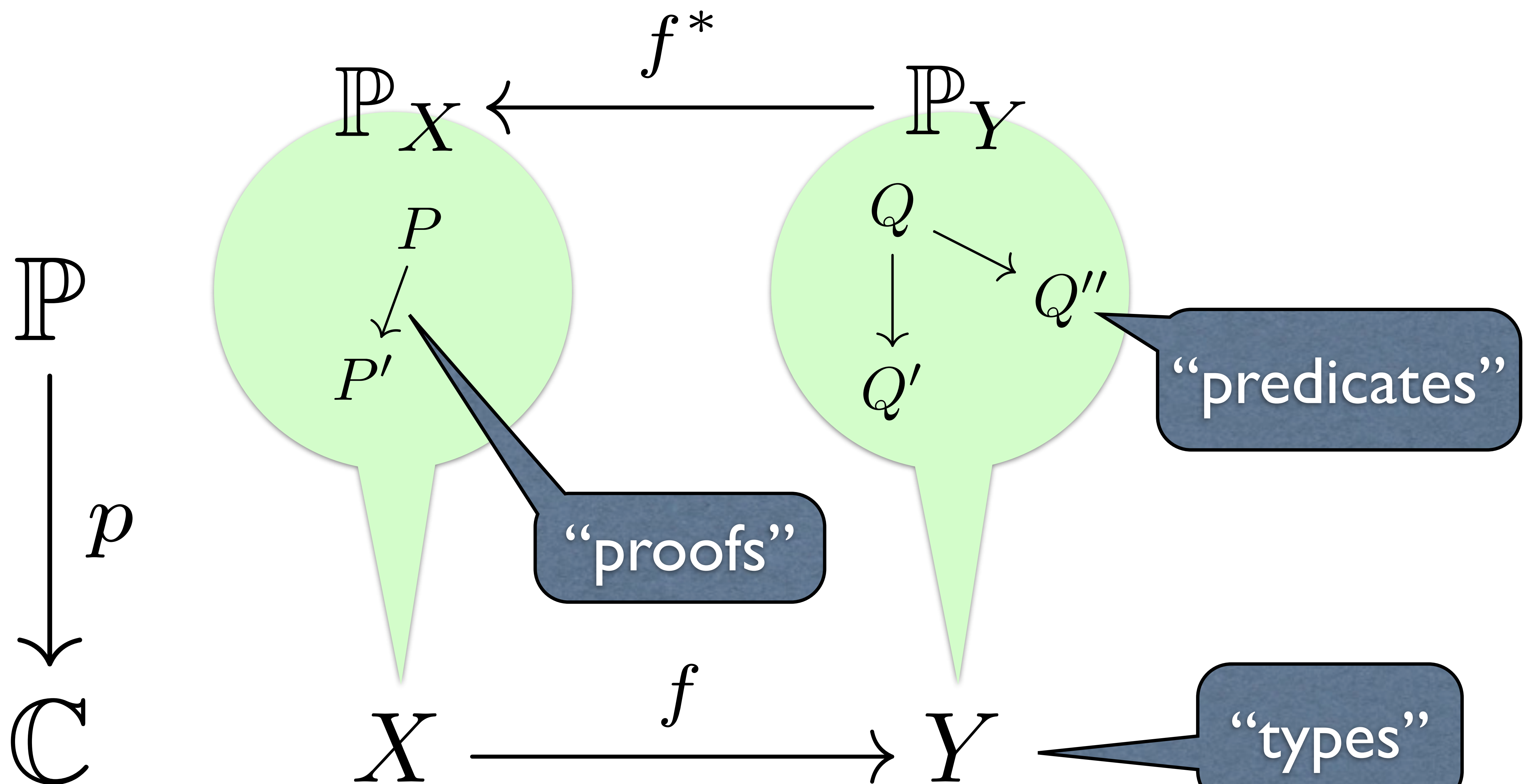
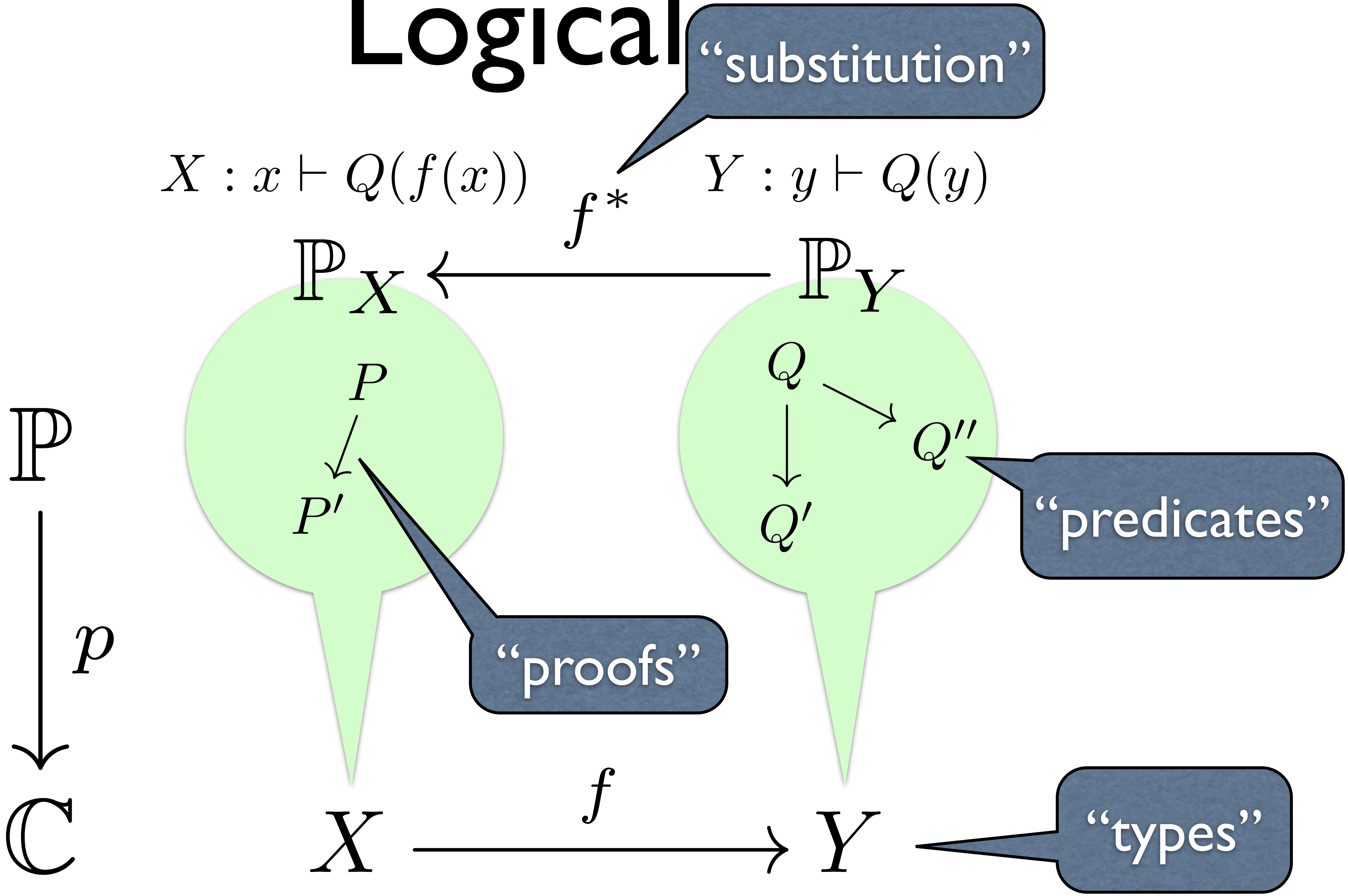
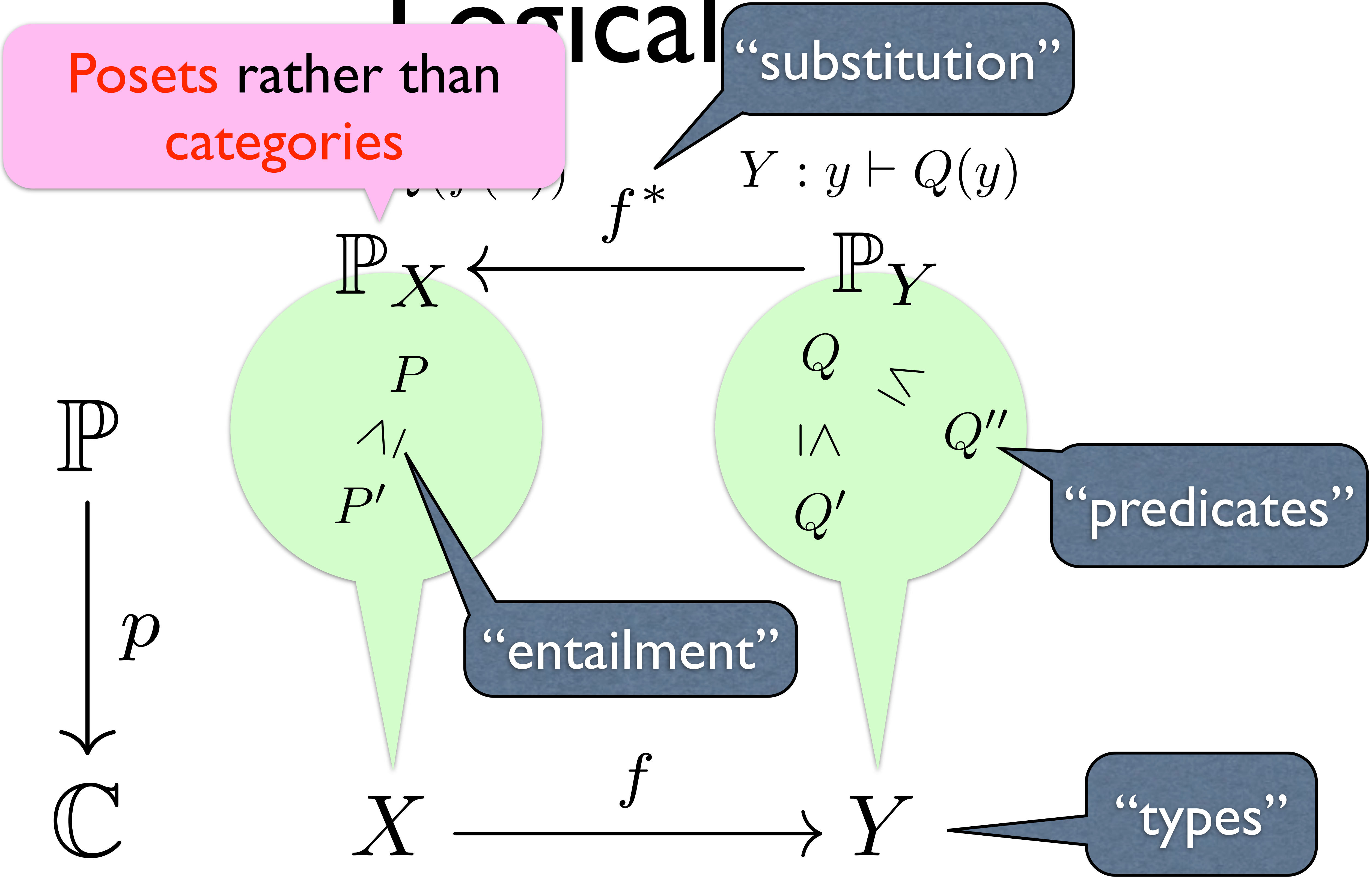


Logical


"substitution"
$X : x \vdash Q(f(x))$
f^*
$Y : y \vdash Q(y)$
$\mathbb{P}_X$
$\mathbb{P}_Y$
P
P'
Q
Q'
Q''
"predicates"
"proofs"
$\mathbb{P}$
p
$\mathbb{C}$
X
f
Y
"types"

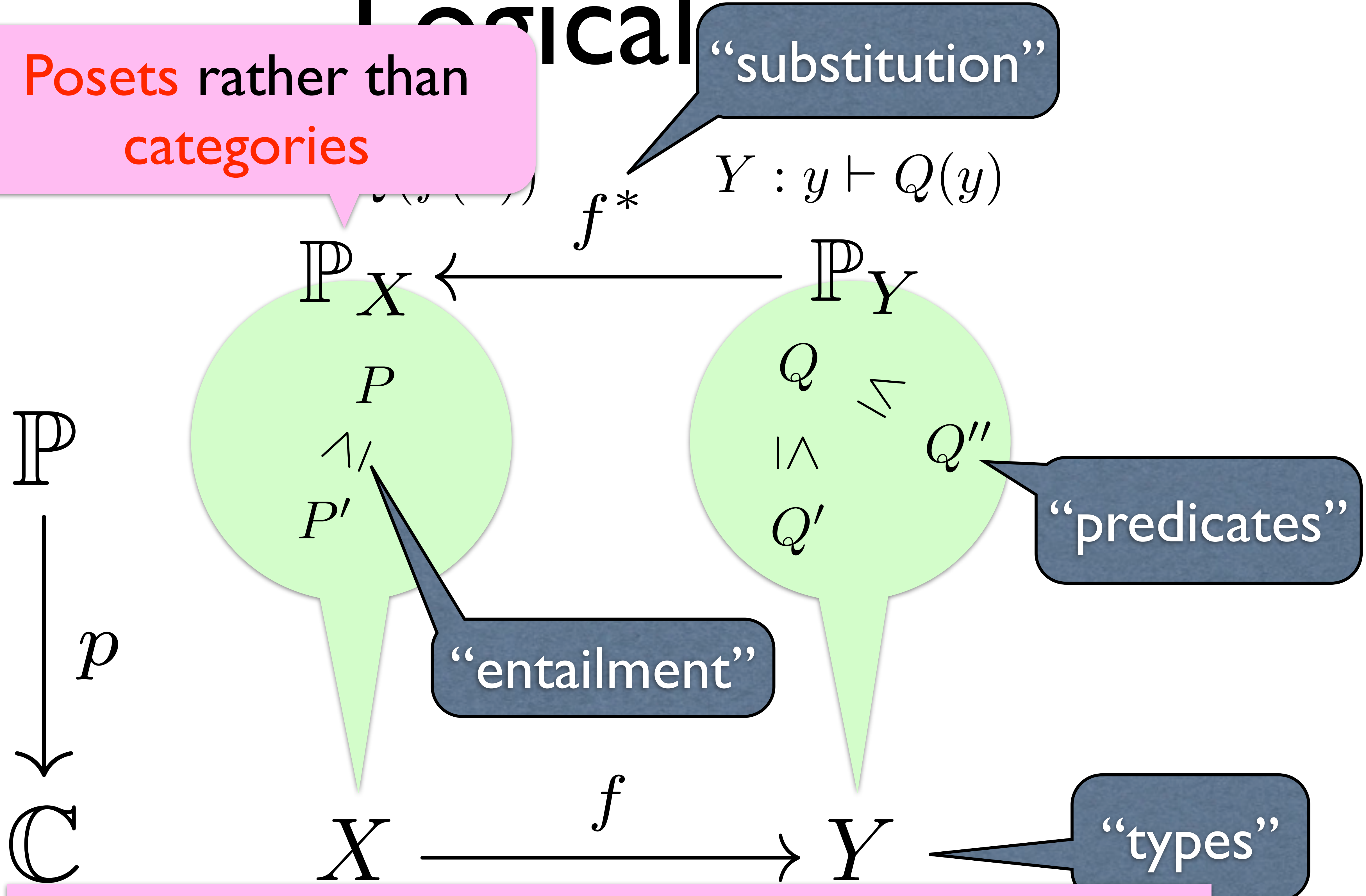
Logical



Posets rather than categories
“substitution”
$Y : y \vdash Q(y)$
f^*
$\mathbb{P}_X$
$\mathbb{P}_Y$
P
P'
Q
Q'
Q''
“entailment”
“predicates”
$\mathbb{P}$
p
$\mathbb{C}$
X
f
Y
“types”

Logical

Posets rather than categories



From now on, we consider only poset fibrations

Examples of fibrations

conventional

$$2^X \xleftarrow{f^{-1}} 2^Y$$

$$\begin{array}{ccc} \mathbf{Pred} & & (f^{-1}Q \subseteq X) \longrightarrow (Q \subseteq Y) \\ \downarrow & & \\ \mathbf{Set} & & X \xrightarrow{\quad f \quad} Y \end{array}$$

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relational

$$2^{X \times X'} \xleftarrow{(f \times f')^{-1}} 2^{Y \times Y'}$$

$$\begin{array}{ccc} \mathbf{Rel} & & ((f \times f')^{-1}R \subseteq X \times X') \longrightarrow (R \subseteq Y \times Y') \\ \downarrow & & \\ \mathbf{Set} \times \mathbf{Set} & & (X, X') \xrightarrow{\quad (f, f') \quad} (Y, Y') \end{array}$$

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relational

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topos (constructive)

$$\mathrm{Sub}(X) \xleftarrow{\;f^*\;} \mathrm{Sub}(Y)$$

$$\begin{array}{ccc} f^*Q & \longrightarrow & Q \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\;f\;} & Y \end{array}$$

$$\begin{array}{ccc} & \mathrm{Sub}(\mathbb{C}) & \\ & \downarrow & \\ (\mathbb{C}\text{: topos}) & \mathbb{C} & X \xrightarrow{\;f\;} Y \end{array}$$

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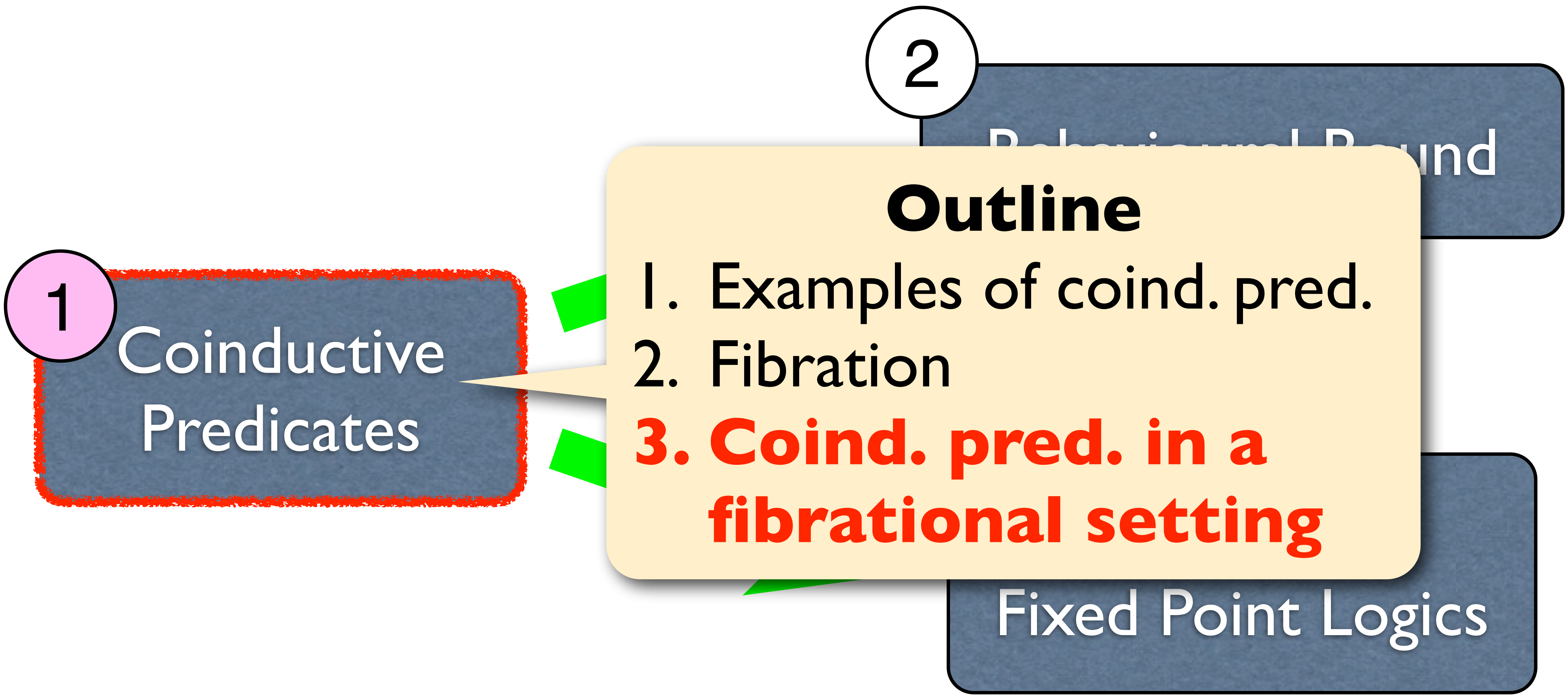

Coinductive predicates in a fibrational setting

Type of systems: functor $F : \mathbb{C} \to \mathbb{C}$

A system: coalgebra $c : X \to FX$

Underlying logic: fibration $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$

(Co)recursive definition: predicate lifting $\varphi : \mathbb{P} \to \mathbb{P}$

Coinductive predicates in a fibrational setting

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(Co)recursive definition: predicate lifting $\varphi : \mathbb{P} \to \mathbb{P}$

Def. A *predicate lifting* of $F : \mathbb{C} \to \mathbb{C}$ along $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$ is a functor $\varphi : \mathbb{P} \to \mathbb{P}$ that makes the diagram

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\varphi} & \mathbb{P} \\ \downarrow p & & \downarrow p \\ \mathbb{C} & \xrightarrow{F} & \mathbb{C} \end{array}$$

commute and preserves Cartesian maps.

Hence, $\varphi_X : \mathbb{P}_X \to \mathbb{P}_{FX}$

Coinductive predicates in a fibrational setting

predicate lifting

fibration

Def. Consider:

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\varphi} & \mathbb{P} \\ \downarrow p & & \downarrow p \\ \mathbb{C} & \xrightarrow{F} & \mathbb{C} \end{array}$$

and a coalgebra $c : X \to FX$. Then the *coinductive predicate* for φ in c is the greatest fixed point of a functor

$$\mathbb{P}_X \xrightarrow{\varphi} \mathbb{P}_{FX} \xrightarrow{c^*} \mathbb{P}_X \quad ,$$

and written as

$$[\![\nu\varphi]\!]_c := \mathrm{gfp}(c^* \circ \varphi) \quad .$$

Examples again

Example 1 $(\nu u.\, \Diamond u)$ Consider:

$$(P \subseteq X) \longmapsto (\{U \mid U \cap P \neq \varnothing\} \subseteq \mathcal{P}X)$$

$$\begin{array}{ccc} \mathbf{Pred} & \xrightarrow{\varphi_\Diamond} & \mathbf{Pred} \\ \downarrow & & \downarrow \\ \mathbf{Set} & \xrightarrow{\mathcal{P}} & \mathbf{Set} \end{array}$$

and $c : X \to \mathcal{P}X$

Then

$$[\![\nu u.\, \Diamond u]\!]_c = [\![\nu \varphi_\Diamond]\!] := \mathrm{gfp}\Big(\, 2^X \xrightarrow{\varphi_\Diamond} 2^{\mathcal{P}X} \xrightarrow{c^{-1}} 2^X \,\Big)$$

Examples again

Example 2 (bisimilarity) Consider:

$$(R \subseteq X \times Y) \longmapsto \begin{array}{l}(\{(U,V) \mid \\ \quad \forall x \in U.\, \exists y \in V.\, xRy \;\&\; \\ \quad \forall y \in V.\, \exists x \in U.\, xRy \quad \} \\ \qquad\qquad \subseteq \mathcal{P}X \times \mathcal{P}Y)\end{array}$$

$$\begin{array}{ccc} \mathbf{Rel} & \xrightarrow{\rho} & \mathbf{Rel} \\ \downarrow & & \downarrow \\ \mathbf{Set} \times \mathbf{Set} & \xrightarrow{\mathcal{P}\times\mathcal{P}} & \mathbf{Set} \times \mathbf{Set} \end{array}$$

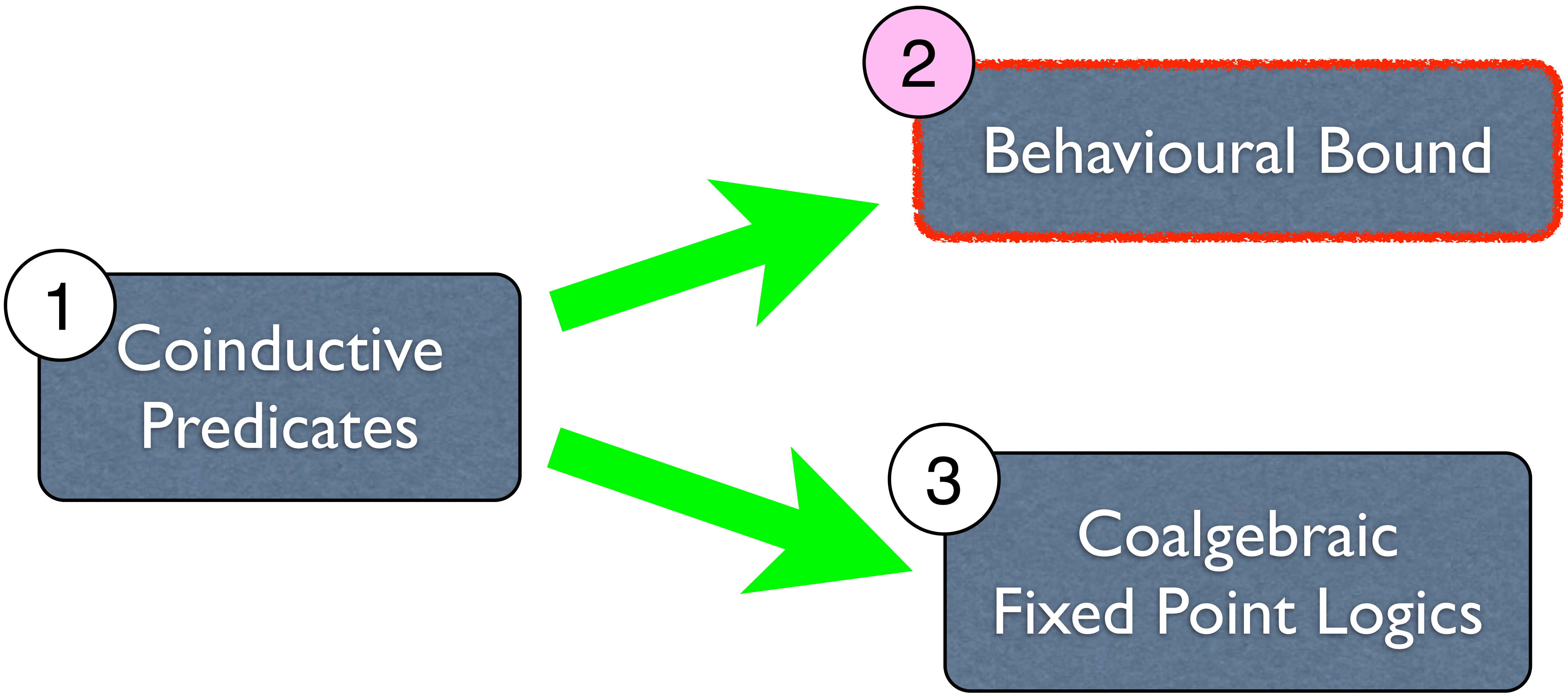
and

$$c : X \to \mathcal{P}X$$
$$d : Y \to \mathcal{P}Y$$

Then

$$\sim_{c,d} = [\![\nu\rho]\!]_{(c,d)} := \mathrm{gfp}\Big(\, 2^{X\times Y} \xrightarrow{\rho} 2^{\mathcal{P}X\times\mathcal{P}Y} \xrightarrow{(c\times d)^{-1}} 2^{X\times Y} \,\Big)$$

Our work: Fibrational approaches to



Our work:
Fibrational approaches to

Main result of a paper:

Ichiro Hasuo, Kenta Cho, Toshiki Kataoka, and Bart Jacobs. **Coinductive Predicates and Final Sequences in a Fibration**. MFPS XXIX, June 2013.

Look it over quickly!

2 Behavioural Bound

3 Coalgebraic Fixed Point Logics

Cousot & Cousot's construction of gfp

Recall the example:

$$[\![\nu u.\, \Diamond u]\!]_c = [\![\nu \varphi_\Diamond]\!]_c = \mathrm{gfp}\Big(2^X \xrightarrow{\varphi_\Diamond} 2^{\mathcal{P}X} \xrightarrow{c^{-1}} 2^X \Big)$$

Cousot & Cousot's construction of gfp

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whole space

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$$X \supseteq (c^{-1} \circ \varphi_\Diamond)(X) \supseteq (c^{-1} \circ \varphi_\Diamond)^2(X) \supseteq \cdots$$

(transfinitely)

stabilizes, yielding the greatest fixed point!

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- When?
  - |X| steps in general
  - ω steps if a Kripke frame c is finitely branching

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Behavioural bound

Behavioural bound in a fibrational setting

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\varphi} & \mathbb{P} \\ \downarrow p & & \downarrow p \\ \mathbb{C} & \xrightarrow[F]{} & \mathbb{C} \end{array}$$

Behavioural bound in a fibrational setting

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“finitely branching” system

Behavioural bound in a fibrational setting

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"finitely branching" system

$\mathbb{C}$: locally finitely presentable category

F: finitary functor

Behavioural bound in a fibrational setting

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p: "well-founded" fibration

compatibility with LFP structure, and certain well-foundedness

"finitely branching" system

$\mathbb{C}$: locally finitely presentable category

F: finitary functor

Theorem. Assume:

- $F : \mathbb{C} \to \mathbb{C}$ a finitary functor on an LFP category
- $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$ a "well-founded" fibration
- $\begin{array}{ccc}\mathbb{P} & \xrightarrow{\varphi} & \mathbb{P}\\ \downarrow p & & \downarrow p\\ \mathbb{C} & \xrightarrow{F} & \mathbb{C}\end{array}$ a predicate lifting
- $c : X \to FX$ a coalgebra

Then, the sequence

$$\top_X \geq (c^* \circ \varphi)(\top_X) \geq (c^* \circ \varphi)^2(\top_X) \geq \cdots$$

stabilizes after ω steps, yielding the coinductive predicate $[\![\nu\varphi]\!]_c := \mathrm{gfp}(c^* \circ \varphi)$.

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Cousot & Cousot's constr.

greatest in $\mathbb{P}_X$

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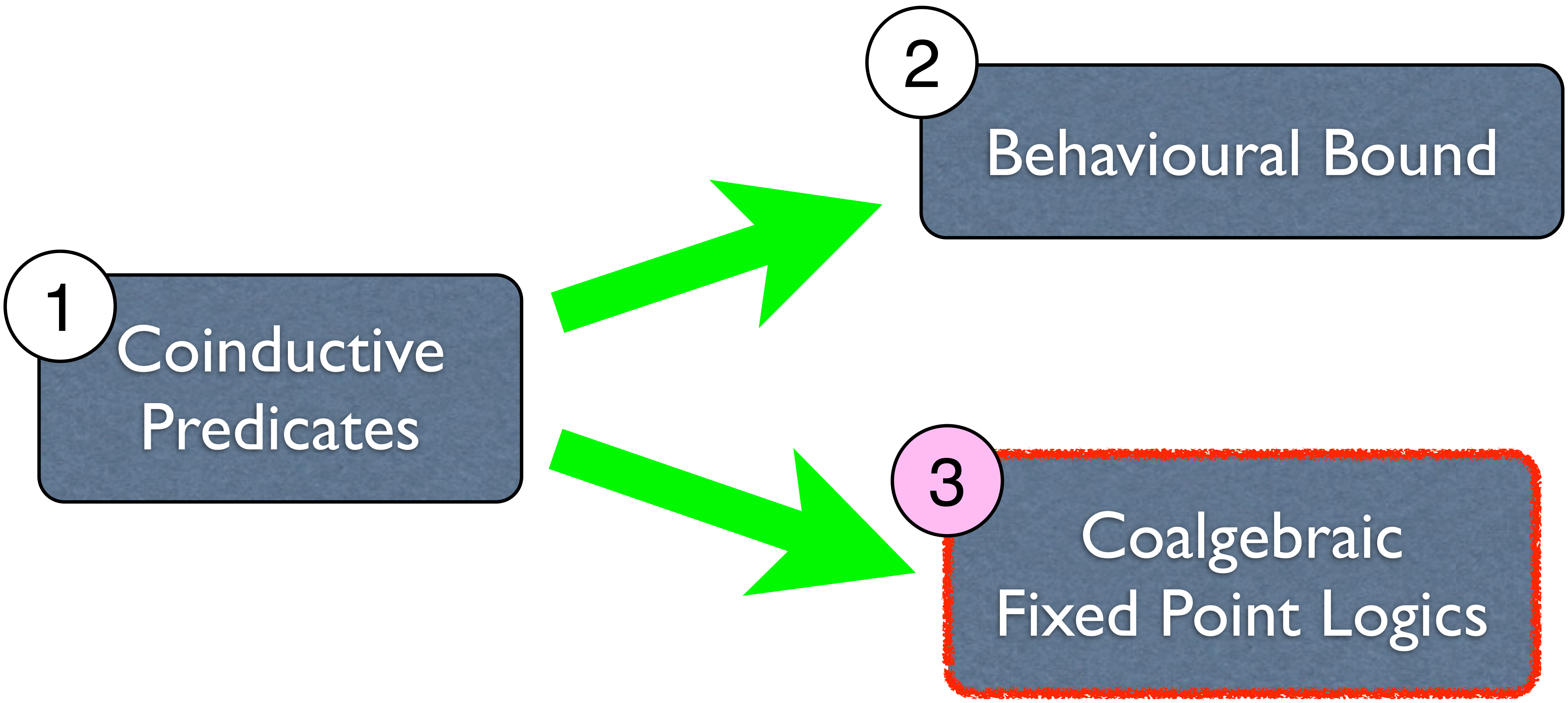

Predicate lifting as modality

- A predicate lifting can be also seen as a modality in modal logic

Example

$$\begin{array}{ccc} \mathbf{Pred} & \overset{\varphi_\Diamond}{\underset{\varphi_\Box}{\rightrightarrows}} & \mathbf{Pred} \\ \downarrow & & \downarrow \\ \mathbf{Set} & \xrightarrow[\mathcal{P}]{} & \mathbf{Set} \end{array}$$

$$\varphi_\Diamond(P \subseteq X) = (\{U \mid U \cap P \neq \emptyset\} \subseteq \mathcal{P}X)$$

$$\varphi_\Box(P \subseteq X) = (\{U \mid U \subseteq P\} \subseteq \mathcal{P}X)$$

For $c : X \to \mathcal{P}X$,

$$[\![\Diamond\alpha]\!]_c = c^{-1}(\varphi_\Diamond([\![\alpha]\!]_c)), \quad [\![\Box\alpha]\!]_c = c^{-1}(\varphi_\Box([\![\alpha]\!]_c))$$

Rem. Predicate liftings of $F : \mathbf{Set} \to \mathbf{Set}$ along $\begin{array}{c}\mathbf{Pred}\\ \downarrow \\ \mathbf{Set}\end{array}$ coincide with natural transformations $2^{-} \Rightarrow 2^{F-}$ s.t. each component is monotone.

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- The following fragment of the modal μ-calculus is coinductive predicate

$$\alpha ::= \Diamond u \mid \Box u \mid \alpha \wedge \alpha \mid \alpha \vee \alpha$$

(u: a fixed variable)

interpretation of $\nu u.\, \alpha$

$$[\![\nu u.\, \alpha]\!]_c = [\![\nu \varphi_\alpha]\!]_c \qquad \text{for } c : X \to \mathcal{P}X$$

coinductive predicate for φ_α

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$$\varphi_{\Diamond u} = \varphi_{\Diamond}$$

$$\varphi_{\Box u} = \varphi_{\Box}$$

$$\varphi_{\alpha \wedge \beta}(P \subseteq X) = (\varphi_\alpha(P) \cap \varphi_\beta(P) \subseteq \mathcal{P}X)$$

$$\varphi_{\alpha \vee \beta}(P \subseteq X) = (\varphi_\alpha(P) \cup \varphi_\beta(P) \subseteq \mathcal{P}X)$$

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Only one ν at the outermost position

licate

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coinductive predicate for φ_{α}

“Full” fixed point logic in a fibration

Mixture of ν and μ

Let Φ be a set of predicate liftings of F along p

$$
\begin{array}{ccc}
 & \Phi & \\
 & \rotatebox{90}{$\in$} & \\
 & \varphi & \\
\mathbb{P} & \longrightarrow & \mathbb{P} \\
\downarrow p & & \downarrow p \\
\mathbb{C} & \xrightarrow[F]{} & \mathbb{C}
\end{array}
$$

Formulas are defined inductively by

$$\alpha ::= u \mid \top \mid \bot \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid [\varphi]\alpha \mid \nu u.\, \alpha \mid \mu u.\, \alpha$$

$$(\varphi \in \Phi;\ \ u \in \mathsf{Var},\ \text{where } \mathsf{Var} \text{ is a set of variables})$$

Fibrewise interpretation

For a coalgebra $c : X \to FX$, fibrewisely in $\mathbb{P}_X$, we can interpret them in the usual way

$[\![\alpha]\!]_{c,V} \in \mathbb{P}_X$ is defined for a valuation $V : \mathsf{Var} \to \mathbb{P}_X$,

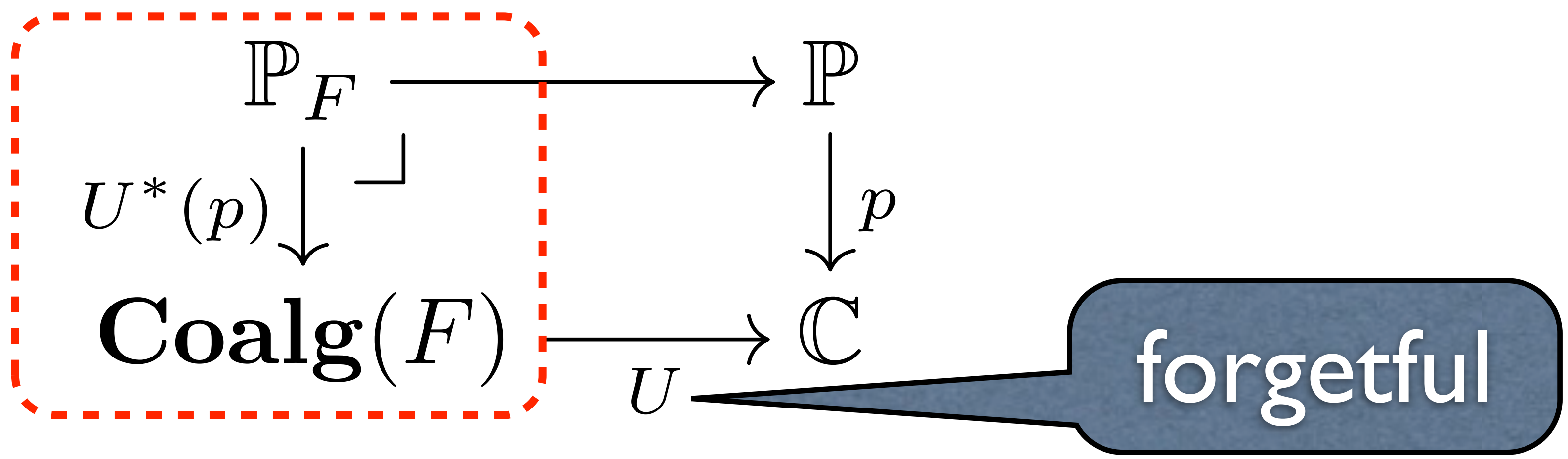
$$
\begin{aligned}
[\![u]\!]_{c,V} &= V(u) & [\![[\varphi]\alpha]\!]_{c,V} &= c^{-1}(\varphi([\![\alpha]\!]_{c,V})) \quad (\varphi \in \Phi)\\
[\![\top]\!]_{c,V} &= \top_X & [\![\nu u.\,\alpha]\!]_{c,V} &= \mathrm{gfp}(\lambda P.\,[\![\alpha]\!]_{c,V[P/u]})\\
[\![\bot]\!]_{c,V} &= \bot_X & [\![\mu u.\,\alpha]\!]_{c,V} &= \mathrm{lfp}(\lambda P.\,[\![\alpha]\!]_{c,V[P/u]})\\
[\![\alpha \wedge \beta]\!]_{c,V} &= [\![\alpha]\!]_{c,V} \wedge [\![\beta]\!]_{c,V}\\
[\![\alpha \vee \beta]\!]_{c,V} &= [\![\alpha]\!]_{c,V} \vee [\![\beta]\!]_{c,V}
\end{aligned}
$$

We assume $\mathbb{P}_X$ is a complete lattice

Interpretation as a fibred functor

- We can give interpretations "globally" in a fibration, rather than fibrewise

- The point is to consider the fibration obtained by "change-of-base"



- Then an interpretation of a formula α is a morphism (i.e. fibred functor)

with n free var.

$$\begin{pmatrix} \mathbb{P}_F \\ \downarrow U^*(p) \\ \mathbf{Coalg}(F) \end{pmatrix}^n \xrightarrow{[\![\alpha]\!]} \begin{pmatrix} \mathbb{P}_F \\ \downarrow U^*(p) \\ \mathbf{Coalg}(F) \end{pmatrix} \quad \text{in } \mathbf{Fib}(\mathbf{Coalg}(F))$$

Specifically

$$\left(\begin{array}{c}\mathbb{P}_F\\ \downarrow U^*(p)\\ \mathbf{Coalg}(F)\end{array}\right)^n \xrightarrow{[\![\alpha]\!]} \left(\begin{array}{c}\mathbb{P}_F\\ \downarrow U^*(p)\\ \mathbf{Coalg}(F)\end{array}\right) \quad \text{in } \mathbf{Fib}(\mathbf{Coalg}(F))$$

$$\begin{array}{ccc}(\mathbb{P}_F)^{\times_F n} & \xrightarrow{[\![\alpha]\!]} & \mathbb{P}_F\\ & \searrow \quad \swarrow & \\ & \mathbf{Coalg}(F) & \end{array} \quad \text{in } \mathbf{Cat}$$

preserves Cartesian maps

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$$\overline{\overline{}}$$

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preserves Cartesian maps

See it fibrewise,

$$[\![\alpha]\!]_c : (\mathbb{P}_X)^n \to \mathbb{P}_X \qquad \text{for each } X \xrightarrow{c} FX \text{ in } \mathbf{Coalg}(F)$$

How is it defined?

- fibrationally technical but straightforward, by composing fibred functors inductively on formulas

For example,

$$\left(\begin{array}{c}\mathbb{P}_F \\ \downarrow \\ \mathbf{Coalg}(F)\end{array}\right)^n \xrightarrow{[\![\alpha]\!]} \left(\begin{array}{c}\mathbb{P}_F \\ \downarrow \\ \mathbf{Coalg}(F)\end{array}\right) \xrightarrow{\overline{\varphi}} \left(\begin{array}{c}\mathbb{P}_F \\ \downarrow \\ \mathbf{Coalg}(F)\end{array}\right)$$

$$\underbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx}}_{[\![[\varphi]\alpha]\!]}$$

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induced by
$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\varphi} & \mathbb{P} \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow[F]{} & \mathbb{C} \end{array}$$

$$\left(\begin{array}{c}\mathbb{P}_F\\ \downarrow\\ \mathbf{Coalg}(F)\end{array}\right)^n \xrightarrow{[\![\alpha]\!]} \left(\begin{array}{c}\mathbb{P}_F\\ \downarrow\\ \mathbf{Coalg}(F)\end{array}\right) \xrightarrow{\overline{\varphi}} \left(\begin{array}{c}\mathbb{P}_F\\ \downarrow\\ \mathbf{Coalg}(F)\end{array}\right)$$

$$[\![[\varphi]\alpha]\!]$$

Fixed point operators

n+1 free var.

$$\begin{pmatrix} \mathbb{P}_F \\ \downarrow \\ \mathbf{Coalg}(F) \end{pmatrix}^n \times \begin{pmatrix} \mathbb{P}_F \\ \downarrow \\ \mathbf{Coalg}(F) \end{pmatrix} \xrightarrow{[\![\alpha]\!]} \begin{pmatrix} \mathbb{P}_F \\ \downarrow \\ \mathbf{Coalg}(F) \end{pmatrix}$$

$$\begin{pmatrix} \mathbb{P}_F \\ \downarrow \\ \mathbf{Coalg}(F) \end{pmatrix}^n \underset{[\![\mu u.\, \alpha]\!]}{\overset{[\![\nu u.\, \alpha]\!]}{\rightrightarrows}} \begin{pmatrix} \mathbb{P}_F \\ \downarrow \\ \mathbf{Coalg}(F) \end{pmatrix}$$

by taking (parameterized) fibred final coalgebras/initial algebras

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- Interpretations are compatible with reindexing along coalgebra morphisms

i.e. $[\![\alpha]\!]_c = f^*[\![\alpha]\!]_d$ for

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ {\scriptstyle c}\uparrow & & {\scriptstyle d}\uparrow \\ X & \xrightarrow{f} & Y \end{array}$$

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i.e. $[\![\alpha]\!]_c = f^* [\![\alpha]\!]_d \quad \text{for} \quad \begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ {\scriptstyle c}\uparrow & & {\scriptstyle d}\uparrow \\ X & \xrightarrow{f} & Y \end{array}$

- "Logic respects behaviour"

What is good about “global” interpretation?

- Fibred functors are, by definition,

- Perhaps not hard to show it directly
- Trivial in “globall” interpretation

i.e. $[\![\alpha]\!]_c = f^*[\![\alpha]\!]_d$ for $\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ c\uparrow & & d\uparrow \\ X & \xrightarrow{f} & Y \end{array}$

- “Logic respects behaviour”

Future work

- Fixed point logics in a fibration
  - More interesting results...
  - Investigate relations fibrationally betw. fixed point logics, automata and games

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