Von Neumann Algebras Form a Model for the Quantum Lambda Calculus

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We present ...

A denotational model for

the Quantum Lambda Calculus

[Selinger & Valiron 2000s]

by

von Neumann Algebras

[von Neumann (with Murray) ’30s–’40s]

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generalisation of matrix algebras e.g. $\mathcal{M}_n = \mathbb{C}^{n \times n}$

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- Studied extensively by Selinger and Valiron in 2000s

Syntax of Quantum Lambda Calculus

We follow [Selinger & Valiron '06, '09] (with $\oplus$ type, without recursion)

$$\begin{aligned} \textit{Type} \quad & A, B ::= \top \mid \texttt{qbit} \mid\, !A \mid A \multimap B \mid A \otimes B \mid A \oplus B \\ \textit{Term} \quad & M, N, L ::= x \mid * \mid \texttt{new} \mid \texttt{meas} \mid U \mid \lambda x.M \mid M N \\ & \qquad \mid \texttt{let}\ \langle x, y \rangle = N\ \texttt{in}\ M \\ & \qquad \mid \langle M, N \rangle \mid \texttt{inl}(M) \mid \texttt{inr}(N) \\ & \qquad \mid \texttt{match}\ L\ \texttt{with}\ (x \mapsto M \mid y \mapsto N) \end{aligned}$$

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A (denotational/categorical) *model* of a language consists of a category $\mathbf{C}$ and an interpretation $[\![-]\!]$:

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- Two other models (both accommodate recursion)
  - [Hasuo & Hoshino, LICS’11], via GoI
  - [Pagani, Selinger & Valiron, POPL’14], applying quantitative semantics

Previous and our approaches

> One reason that designing such a semantics [of QLC] is difficult is that quantum computation is inherently defined on finite dimensional Hilbert spaces, whereas the semantics of higher-order functional programming languages [...] is inherently infinitary. [Pagani, Selinger & Valiron '14]

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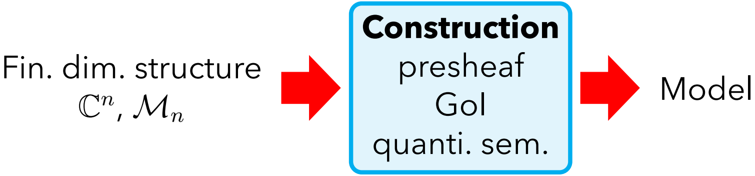

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$\mathbb{C}^n$, $\mathcal{M}_n$

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presheaf
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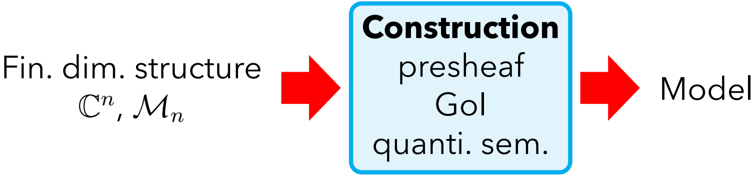

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Model

Our approach: simply use von Neumann algebras, an infinite dimensional generalisation of matrix algebras

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- A *von Neumann algebra* (aka. W^**-algebra*) is a $*$-algebra ('ring') of operators on a Hilbert space which is closed in a suitable topology
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> [The theory of von Neumann algebras] generalizes many familiar facts about finite-dimensional algebra, and is currently one of the most powerful tools in the study of quantum physics. [P. R. Halmos 1973]

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How does this work?

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- and certain conditions (e.g. L preserves $\otimes$,$\oplus$)

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- $\mathbf{vN}$: v.N. algebras and normal unital $*$-homomorphisms (aka. normal MIU-maps)
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Theorem (Kornell 2012)**.** *The SMC* $(\mathbf{vN}^{\mathrm{op}}, \otimes, \mathbb{C})$ *is* ***closed****. Namely: for any v.N. alg.* $\mathscr{A}, \mathscr{B}$ *there is* $\mathscr{B}^{*\mathscr{A}}$ *(called the free exponential) s.t.*

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Warning: we do not know a good description of the free exponential. (Even ${\mathcal{M}_2}^{*\mathcal{M}_2}$ is hard!)

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Theorem (Benton). *If we have a symm. mon. adjunction between a SMC and a cartesian monoidal category as in*

$$\begin{tikzcd}(\mathbf{B}, \times, 1) \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & (\mathbf{C}, \otimes, I) \arrow[l, bend left, "G"]\end{tikzcd}$$

then the comonad FG on $\mathbf{C}$ is linear exponential.

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Interpretation of types

$$\llbracket \top \rrbracket = \mathbb{C} \qquad \llbracket \mathtt{qbit} \rrbracket = \mathcal{M}_2$$

$$\llbracket A \otimes B \rrbracket = \llbracket A \rrbracket \otimes \llbracket B \rrbracket \qquad \llbracket A \oplus B \rrbracket = \llbracket A \rrbracket \oplus \llbracket B \rrbracket$$

$$\llbracket A \multimap B \rrbracket = (\mathcal{F}\mathcal{J}\llbracket B \rrbracket)^{*\llbracket A \rrbracket} \qquad \llbracket !A \rrbracket = \ell^{\infty}(\mathbf{vN}(\llbracket A \rrbracket, \mathbb{C}))$$

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Conclusions

Von Neumann algebras are powerful enough to interpret Selinger & Valiron's Quantum Lambda Calculus, via the adjunctions:

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Thank you!