Abstract Interpretation with Infinitesimals

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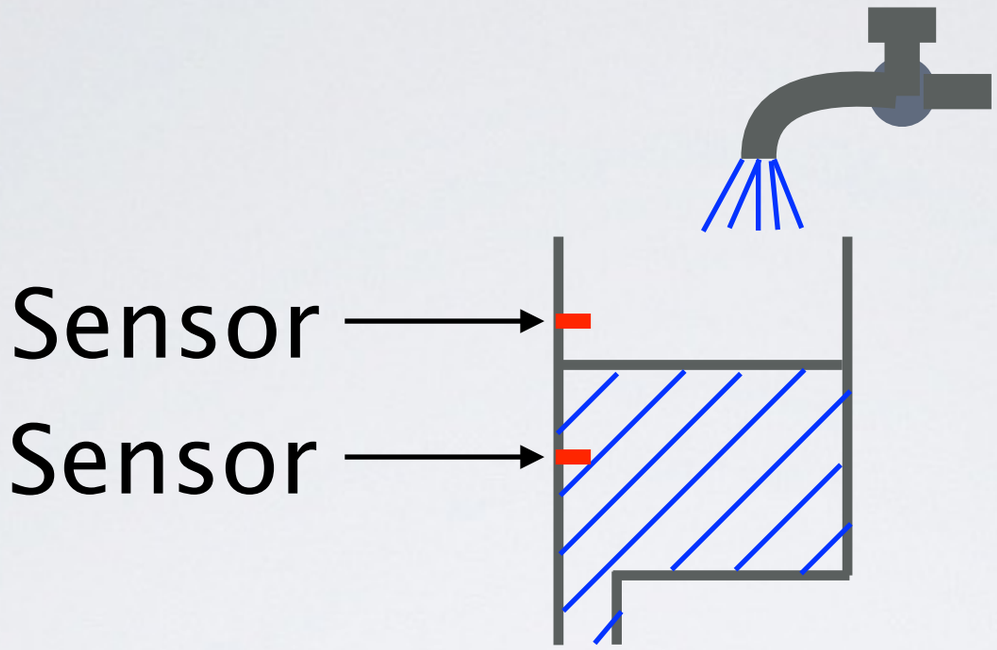
- Example of analysis
- Semantics of WHILE^{dt}
- Abstract interpretation with infinitesimals
- Implementation

Example: Water-level Monitor

[Alur et al. TCS 95]



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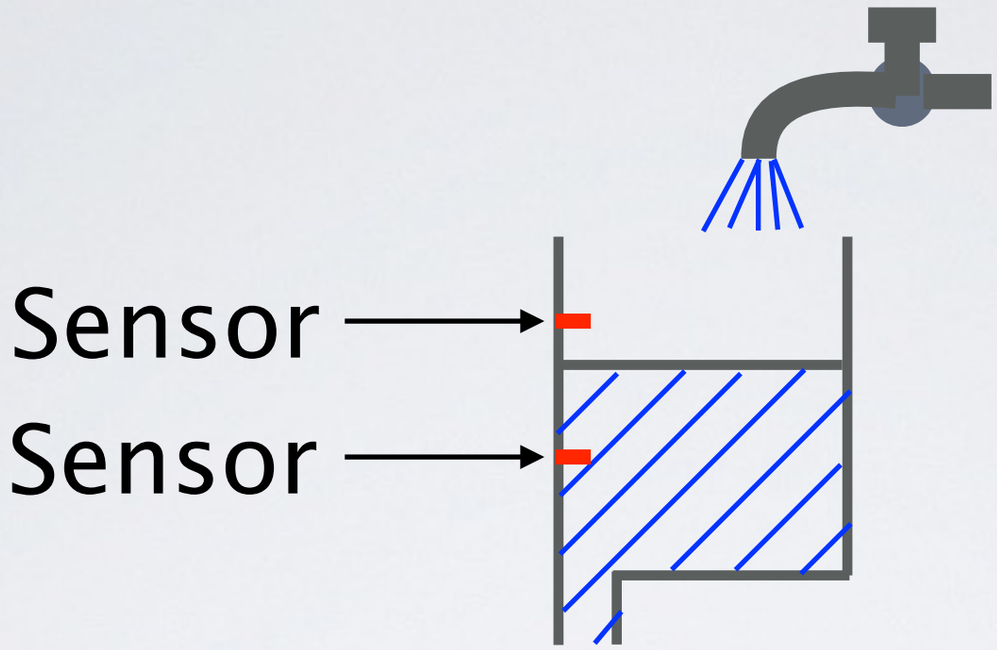

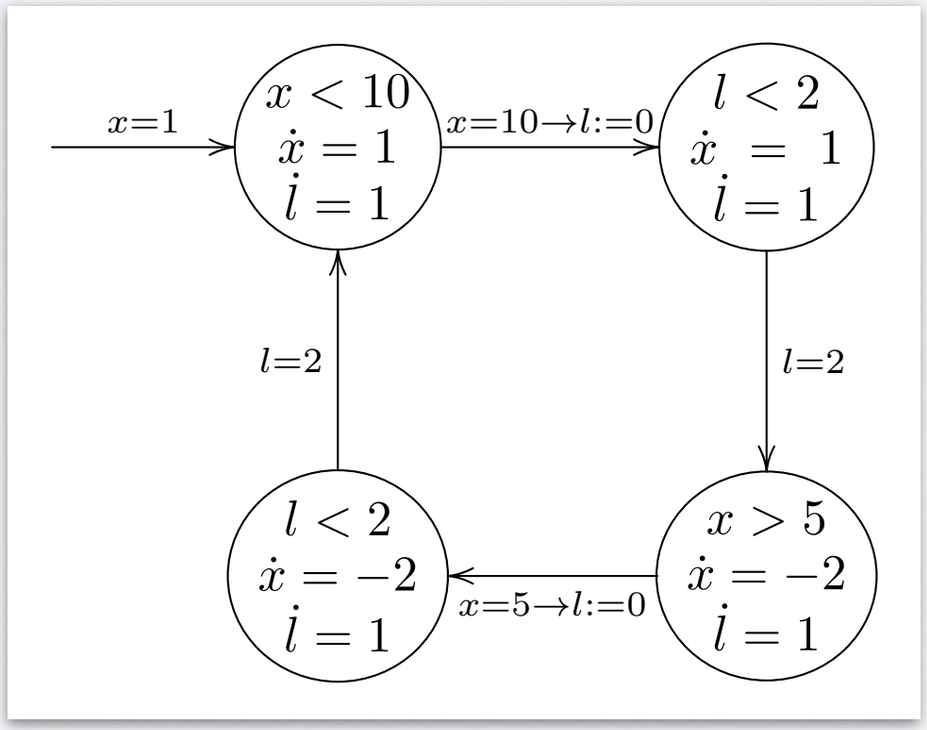

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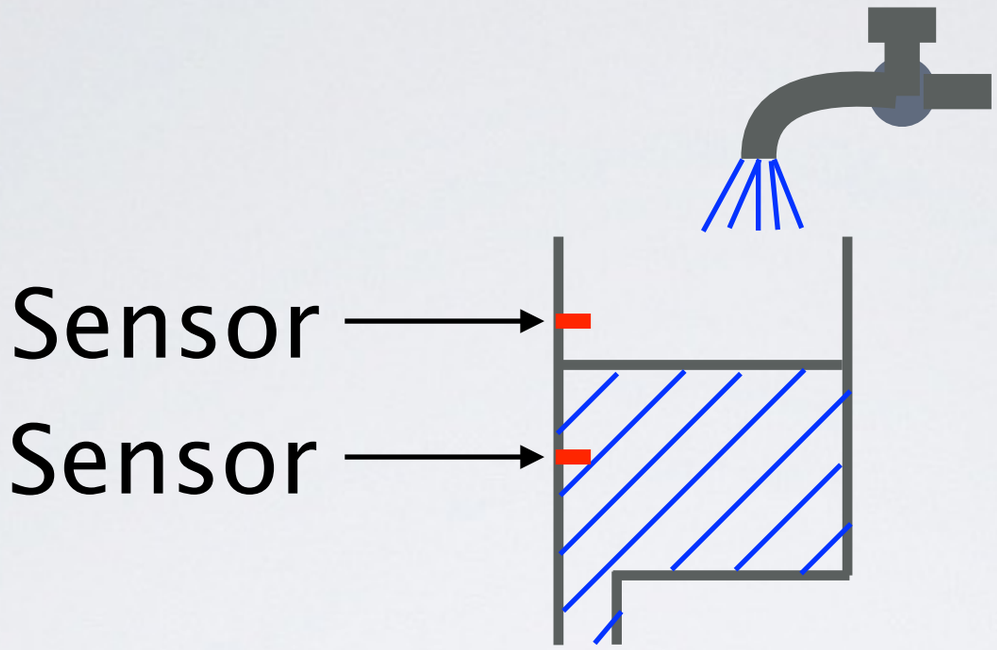

$t := 0; l := 0; x := 1; p := 1; s := 0;$
while $t < t_{max}$ **do**
  $t := t +$ dt;
  if $p = 1$ **then** $x := x +$ dt; **else** $x := x - 2$dt;
  if $(x \leq 5 \wedge p = 0) \vee (x \geq 10 \wedge p = 1)$ **then** $s := 1$ **else** $s := 0$;
  if $s = 1$ **then** $l := l +$ dt;
  if $s = 1 \wedge l \geq 2$ **then** $p := 1 - p; s := 0; l := 0$

WHILE^{dt} [Suenaga & Hasuo ICALP 11]

$$\begin{aligned}
\mathbf{AExp} \ni a &::= x \mid r \mid a_1 \ \mathtt{aop}\ a_2 \mid \mathtt{dt} \mid \infty \\
&\text{where } \ x \in \mathbf{Var}, r \in \mathbb{R} \ \text{ and } \ \mathtt{aop} \in \{+, -, \cdot, \hat{}\ \} \\
\mathbf{BExp} \ni b &::= \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
\mathbf{Cmd} \ni c &::= \mathtt{skip} \mid x := a \mid c_1; c_2 \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c
\end{aligned}$$

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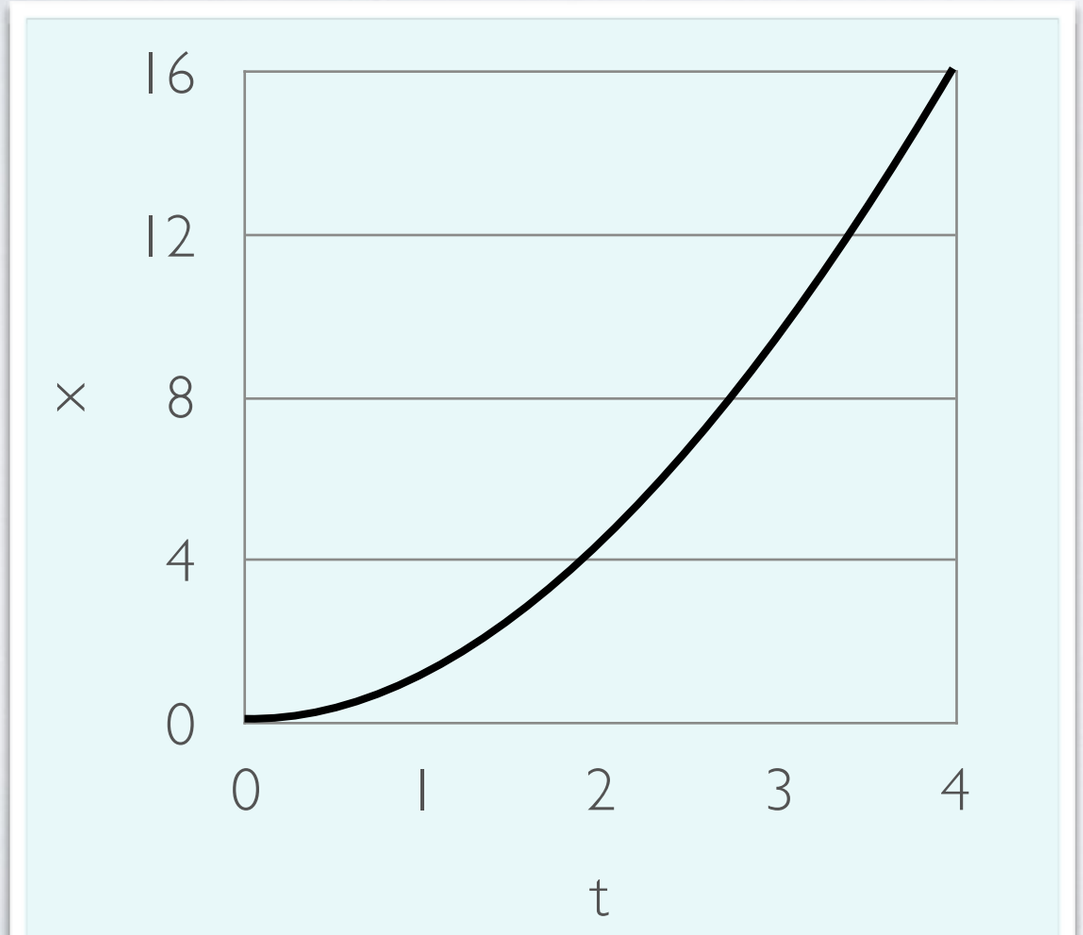

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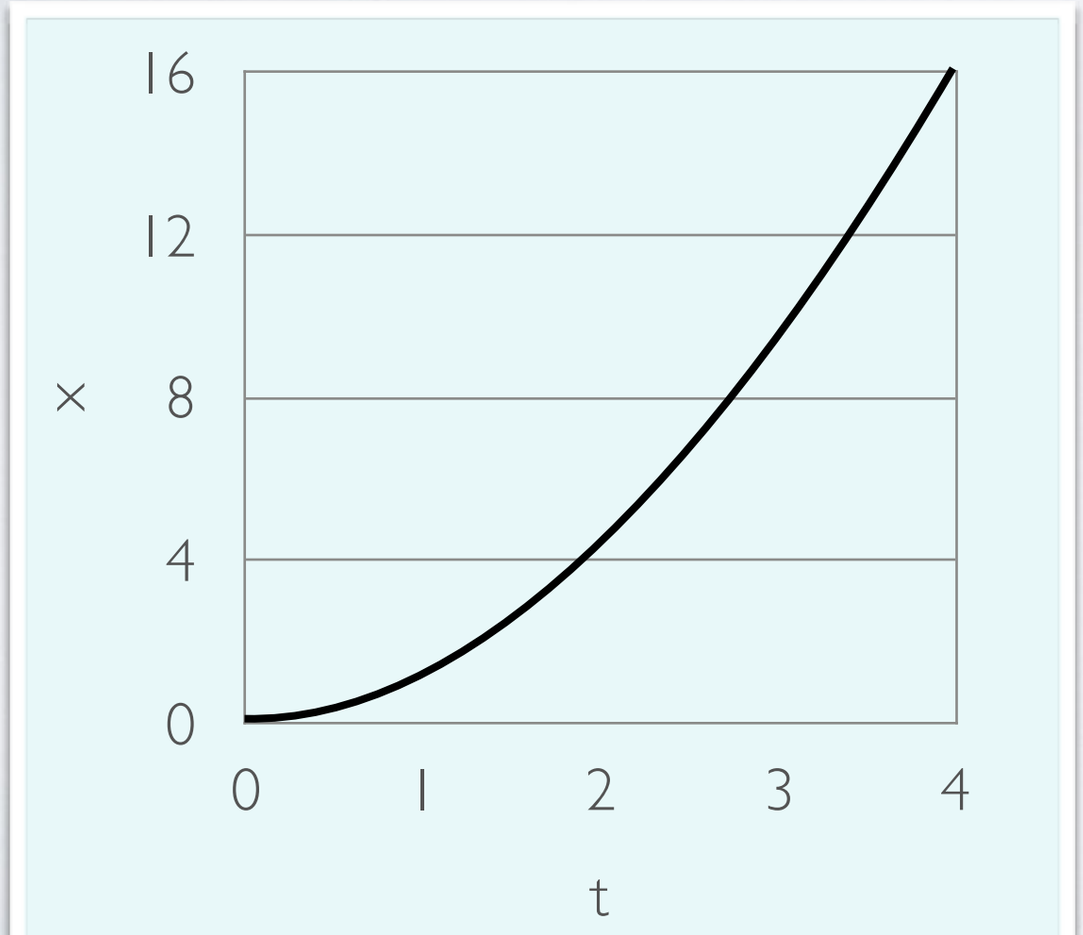
\end{aligned}$$




```
x := 0; t := 0;
while (x ≤ 10) {
   t := t + dt;
   x := x + 2t·dt
}
```

$t := 0; l := 0; x := 1; p := 1; s := 0;$
while $t < t_{max}$ **do**
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if $p = 1$ **then** $x := x +$ dt; **else** $x := x - 2$dt;
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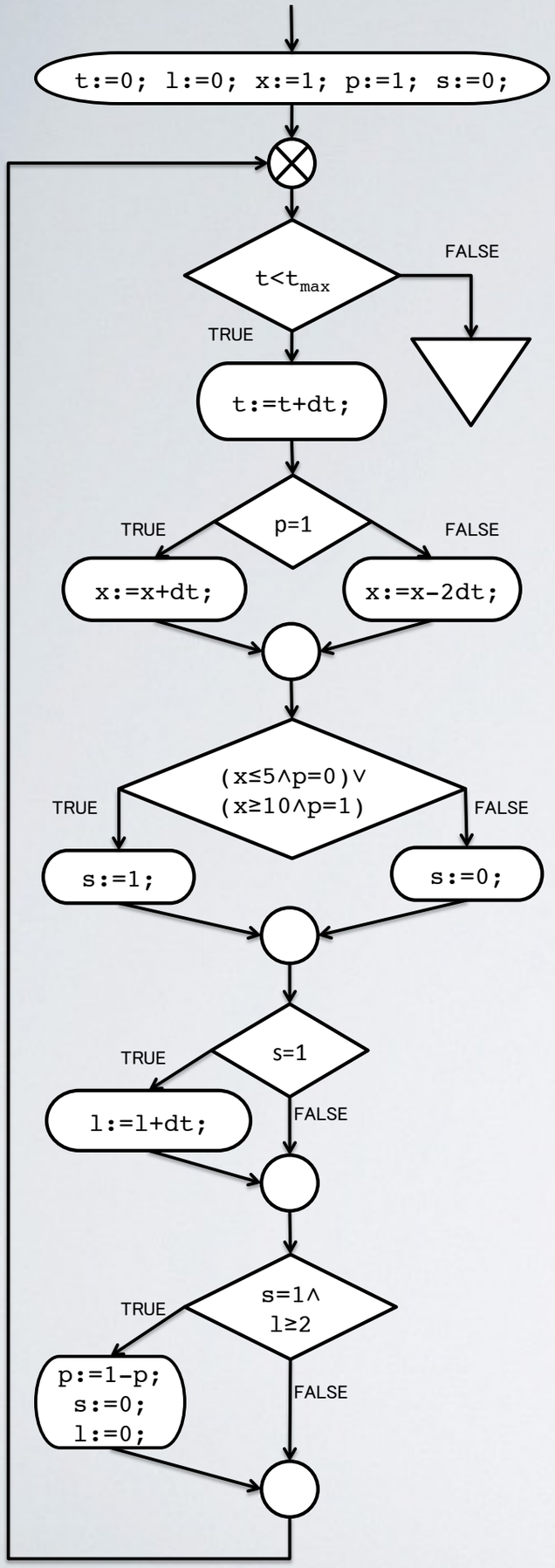
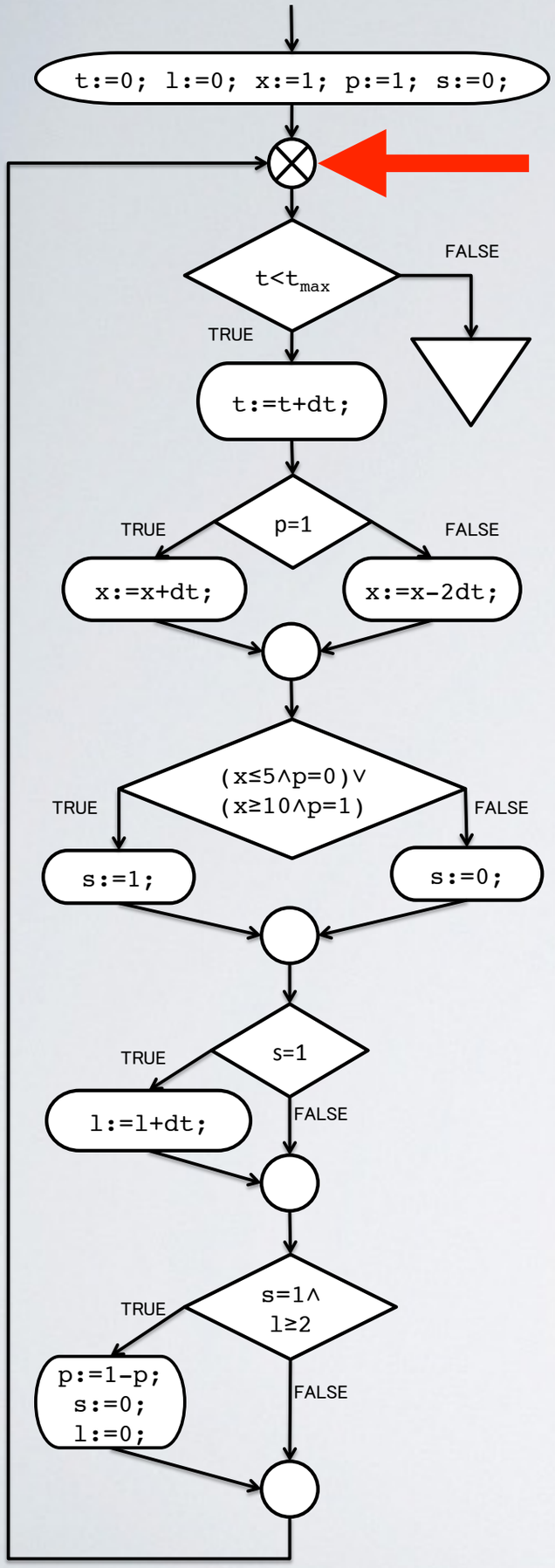
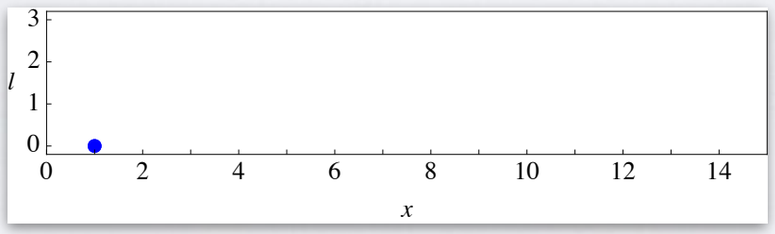
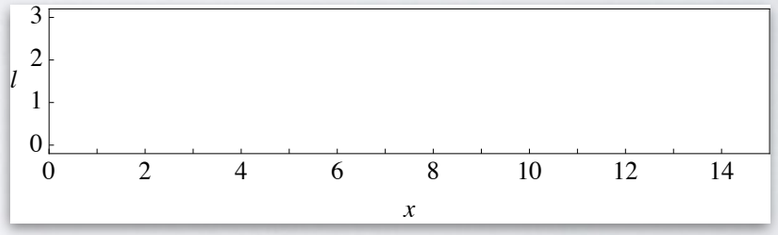

t:=0; l:=0; x:=1; p:=1; s:=0;
t<t_max
FALSE
TRUE
t:=t+dt;
p=1
TRUE
FALSE
x:=x+dt;
x:=x-2dt;
(x≤5∧p=0)∨
(x≥10∧p=1)
TRUE
FALSE
s:=1;
s:=0;
s=1
TRUE
FALSE
l:=l+dt;
s=1∧
l≥2
TRUE
FALSE
p:=1-p;
s:=0;
l:=0;

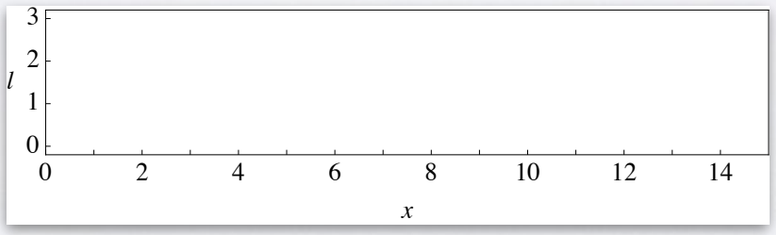


p=1, s=0

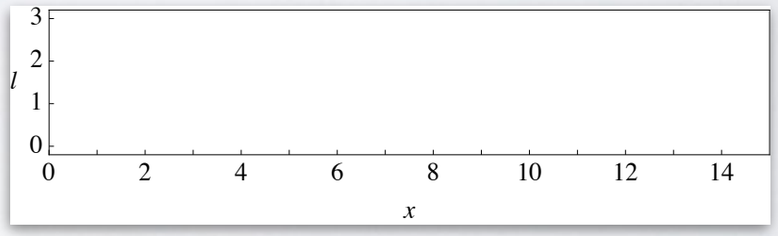


p=1, s=1

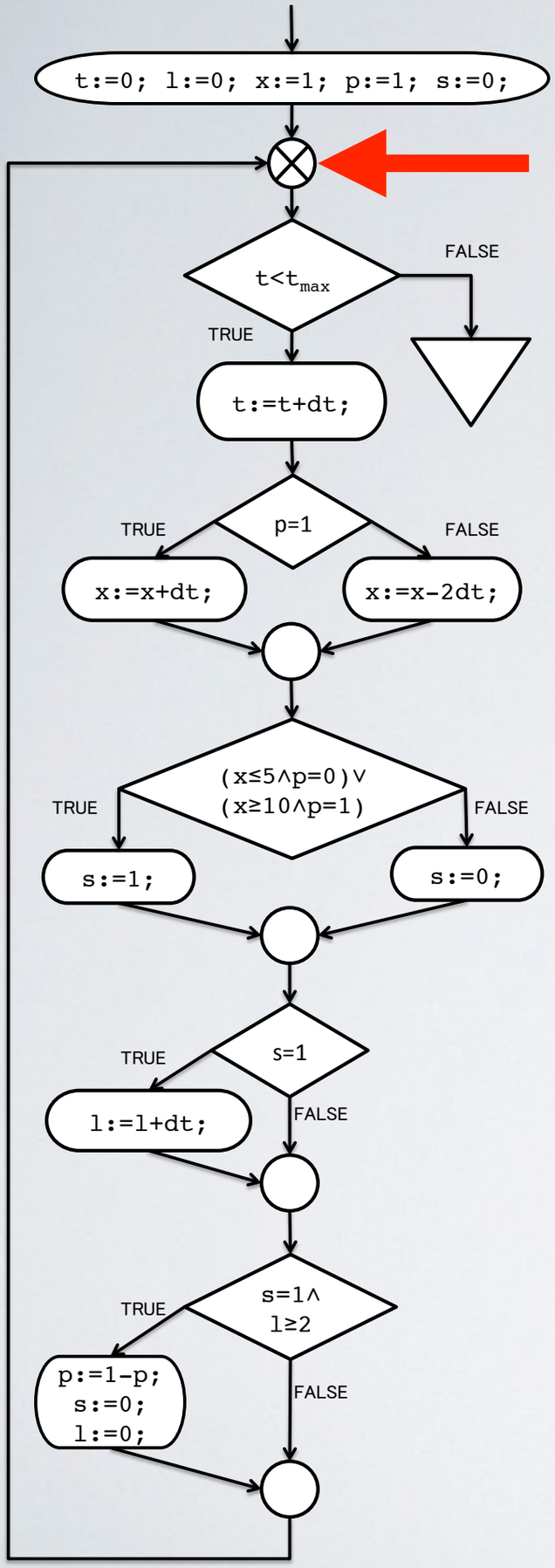


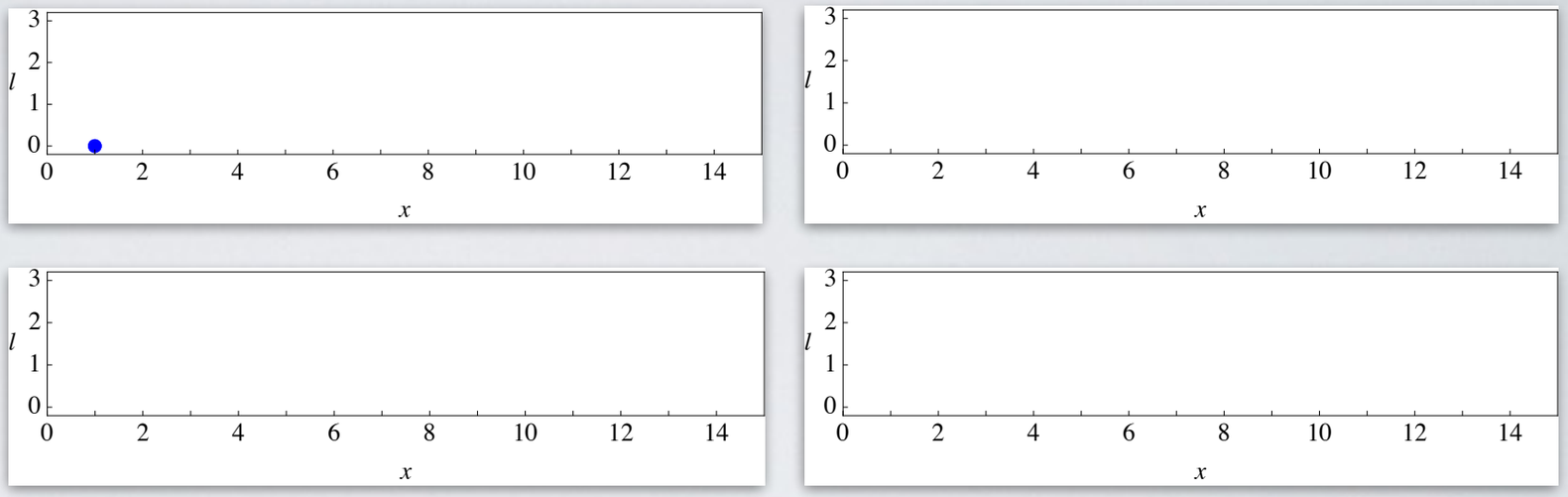


p=0, s=0



p=0, s=1





p=1, s=0

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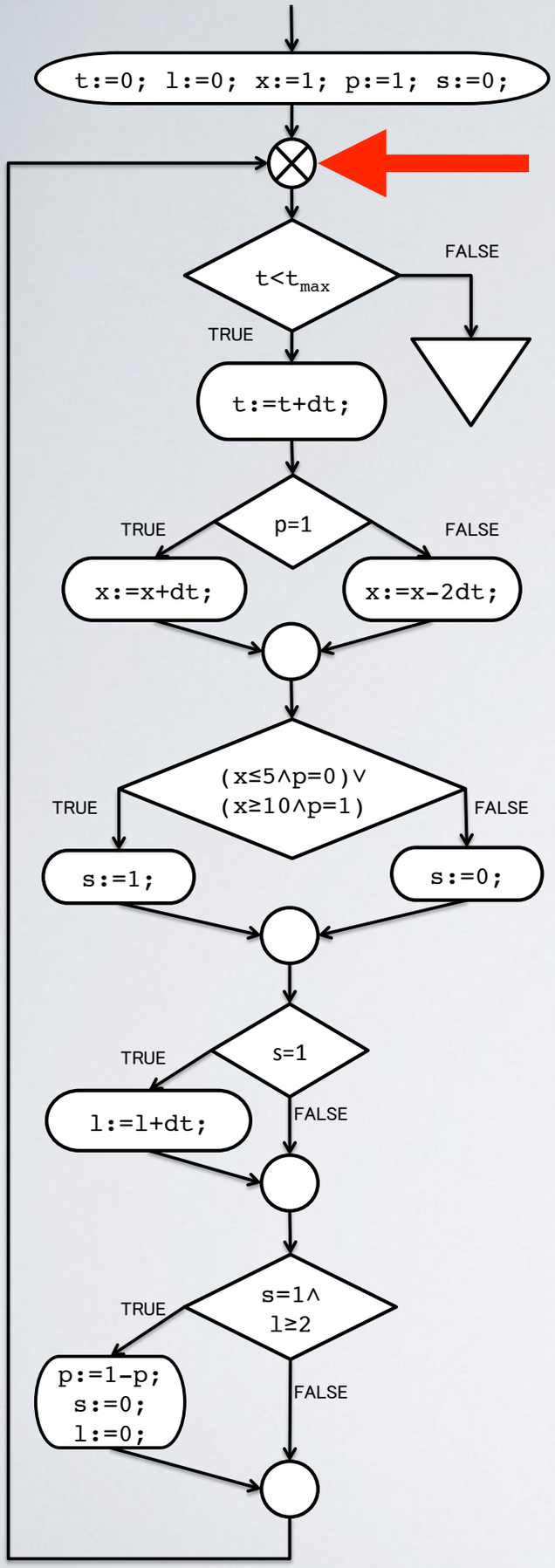

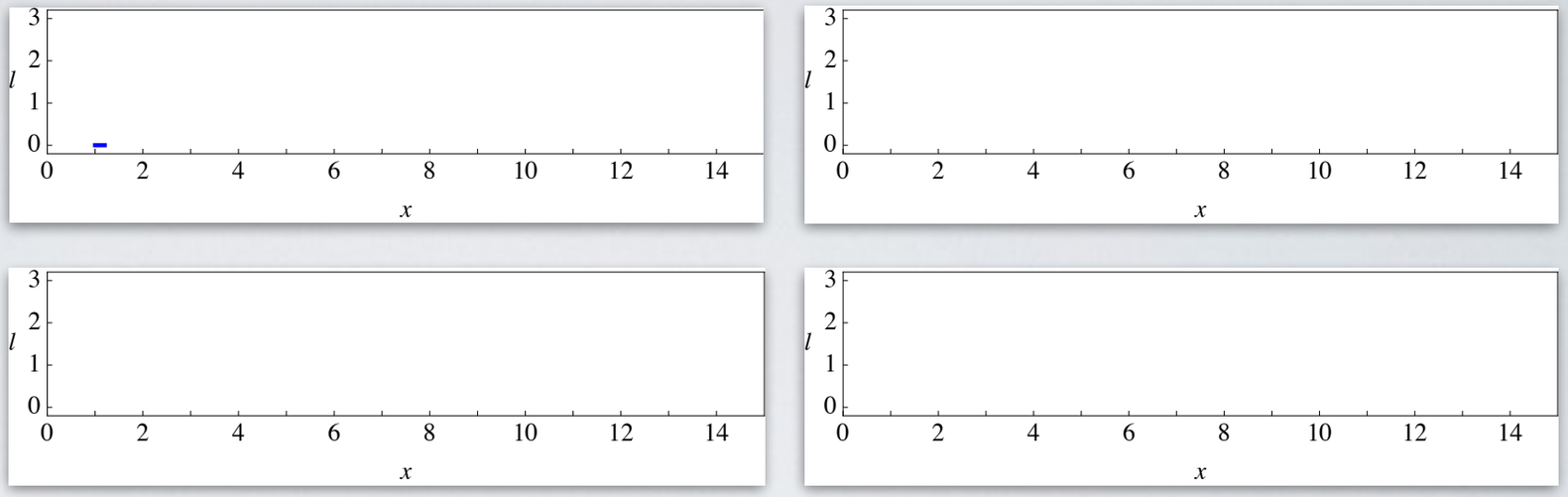

p=1, s=0

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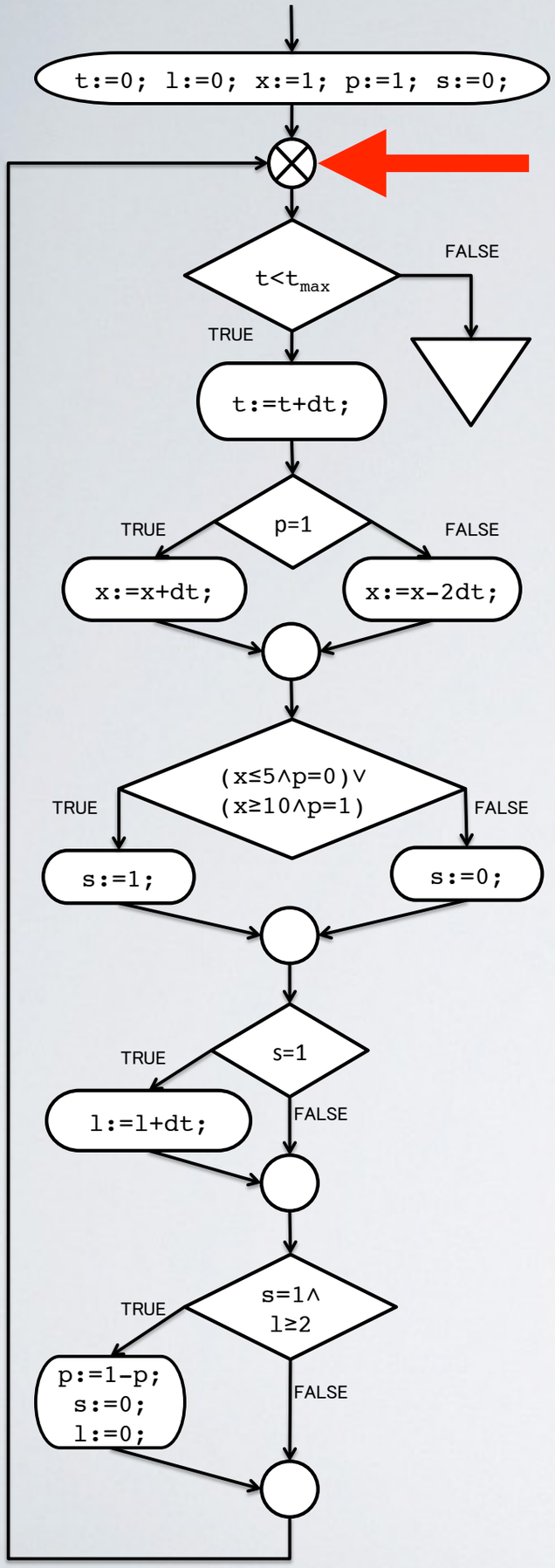

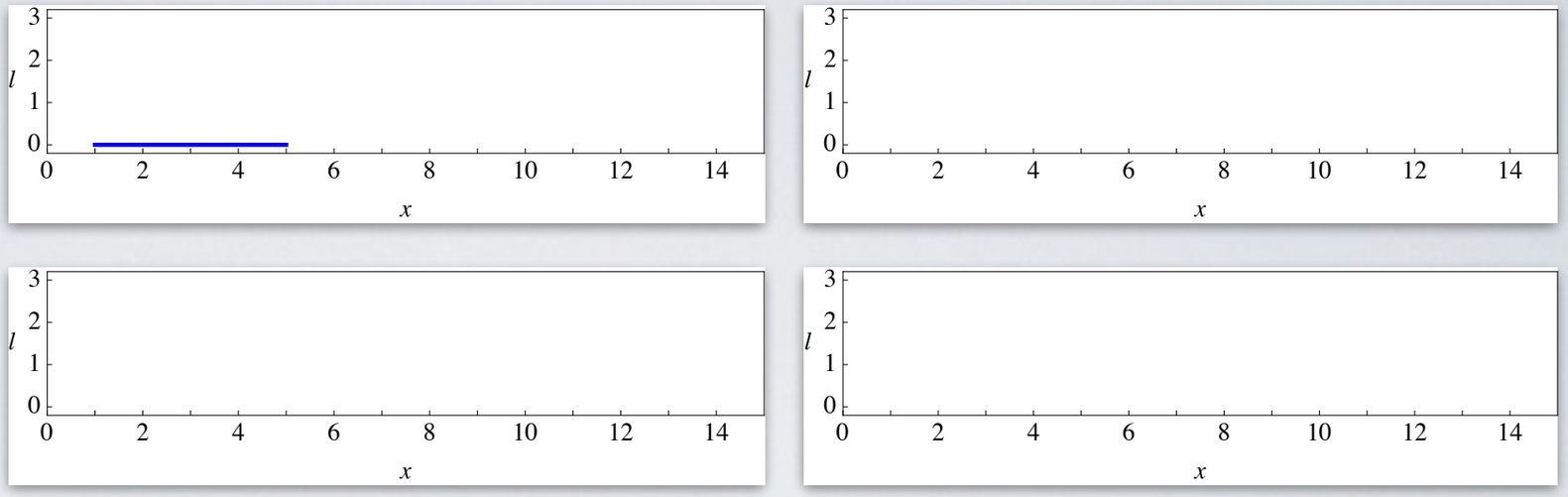

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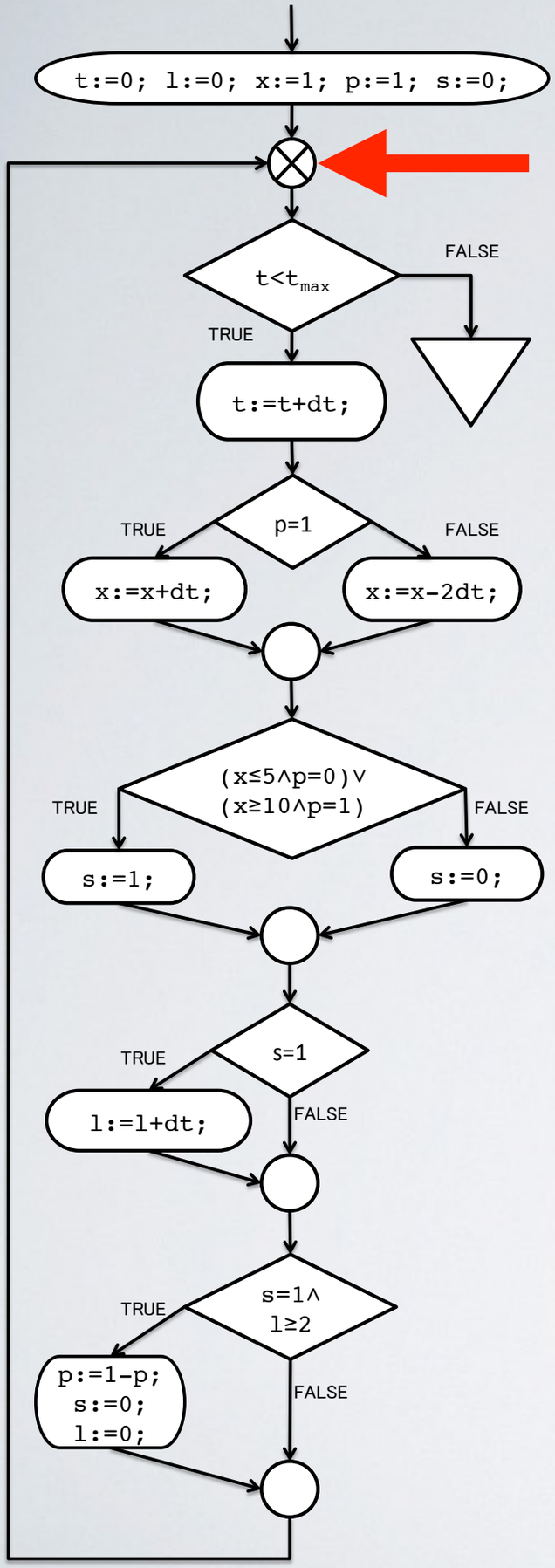
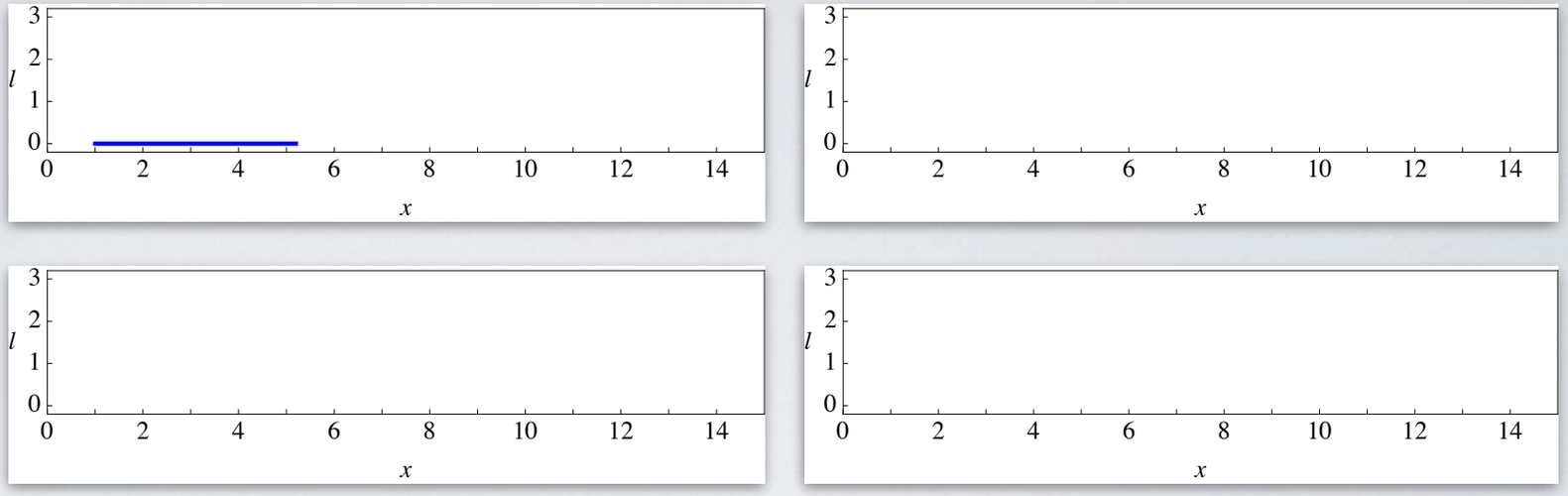
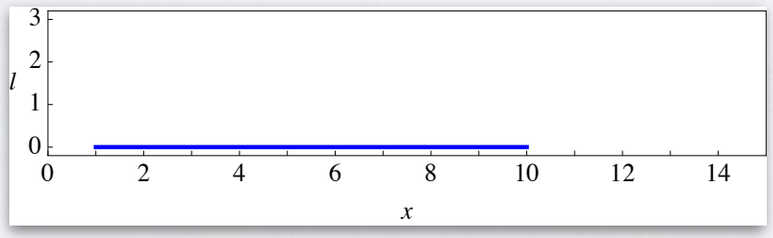




p=1, s=0



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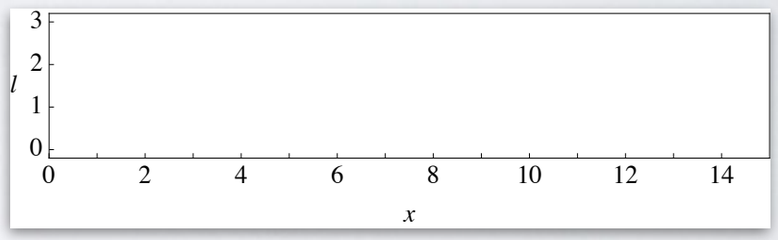

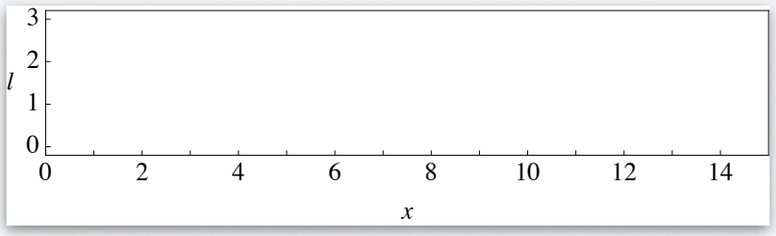

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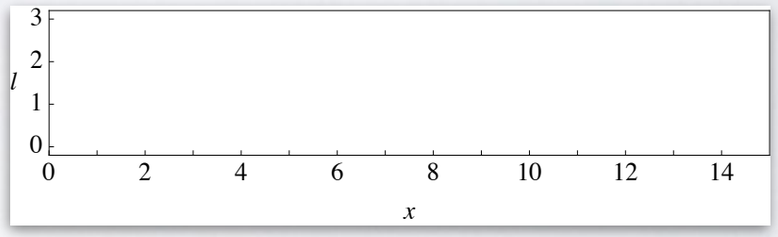

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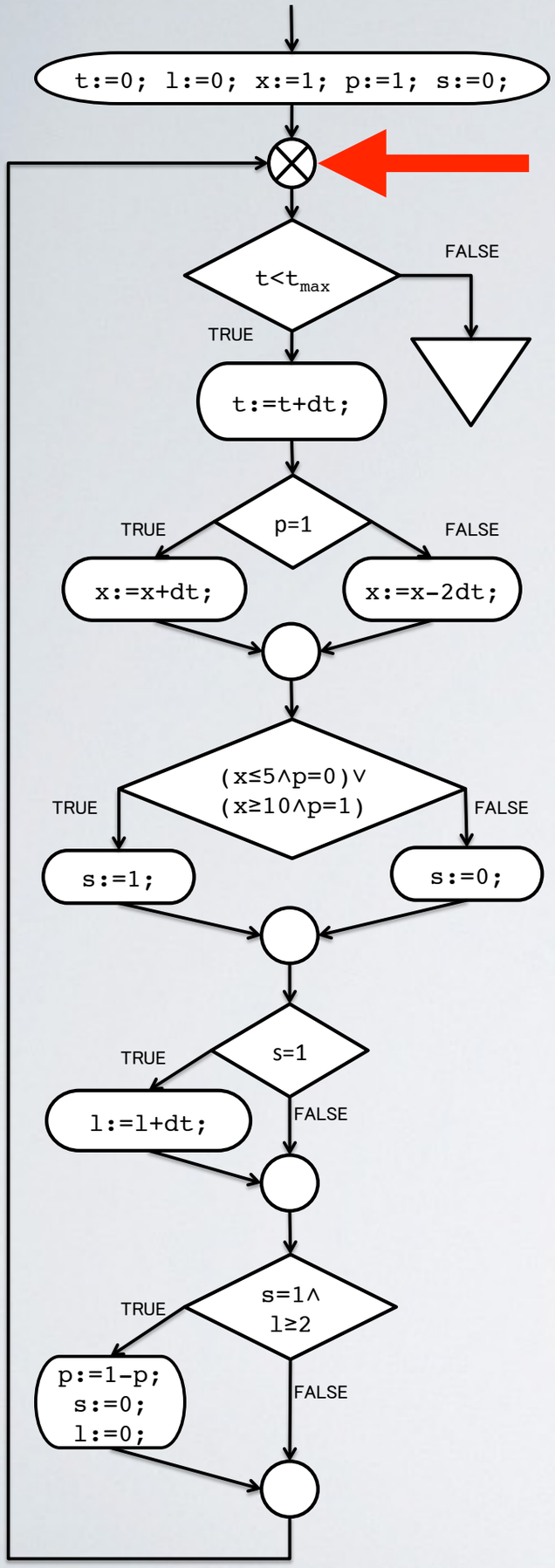

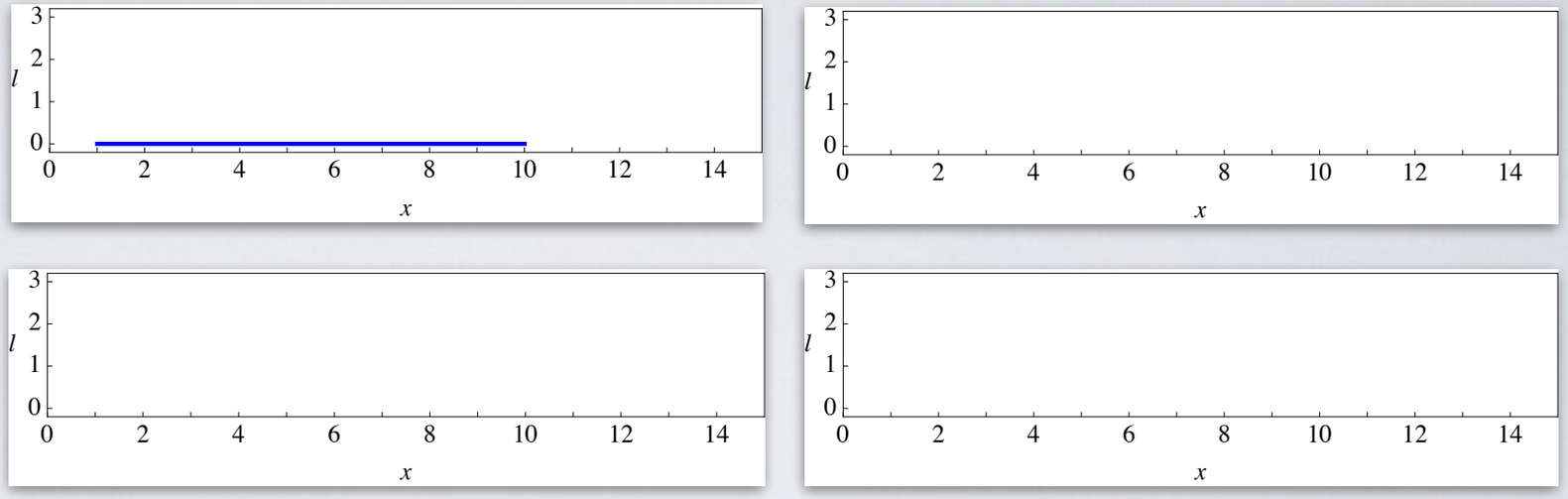

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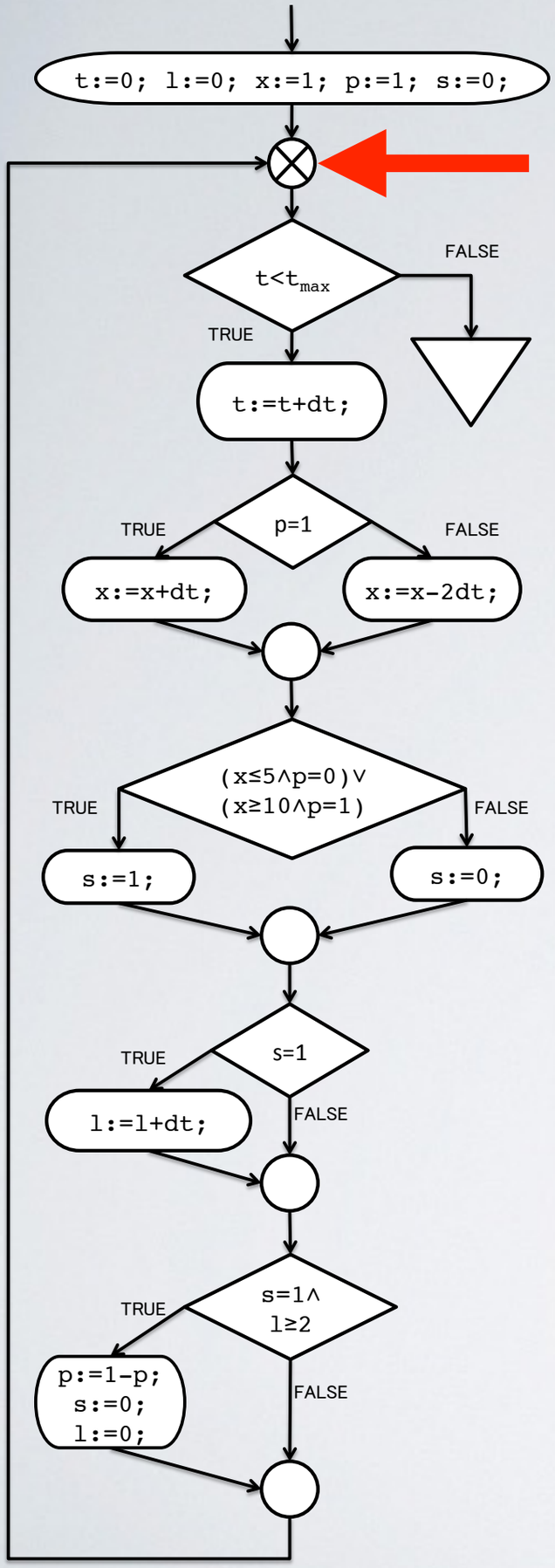
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FALSE
x:=x+dt;
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(x≤5∧p=0)∨
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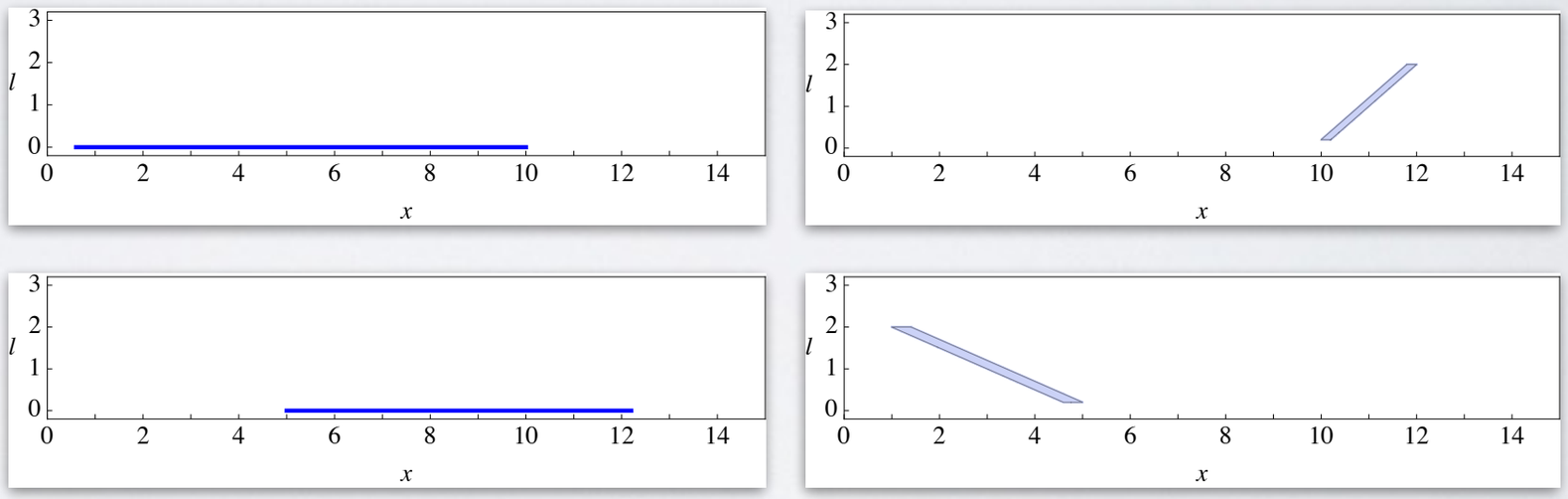
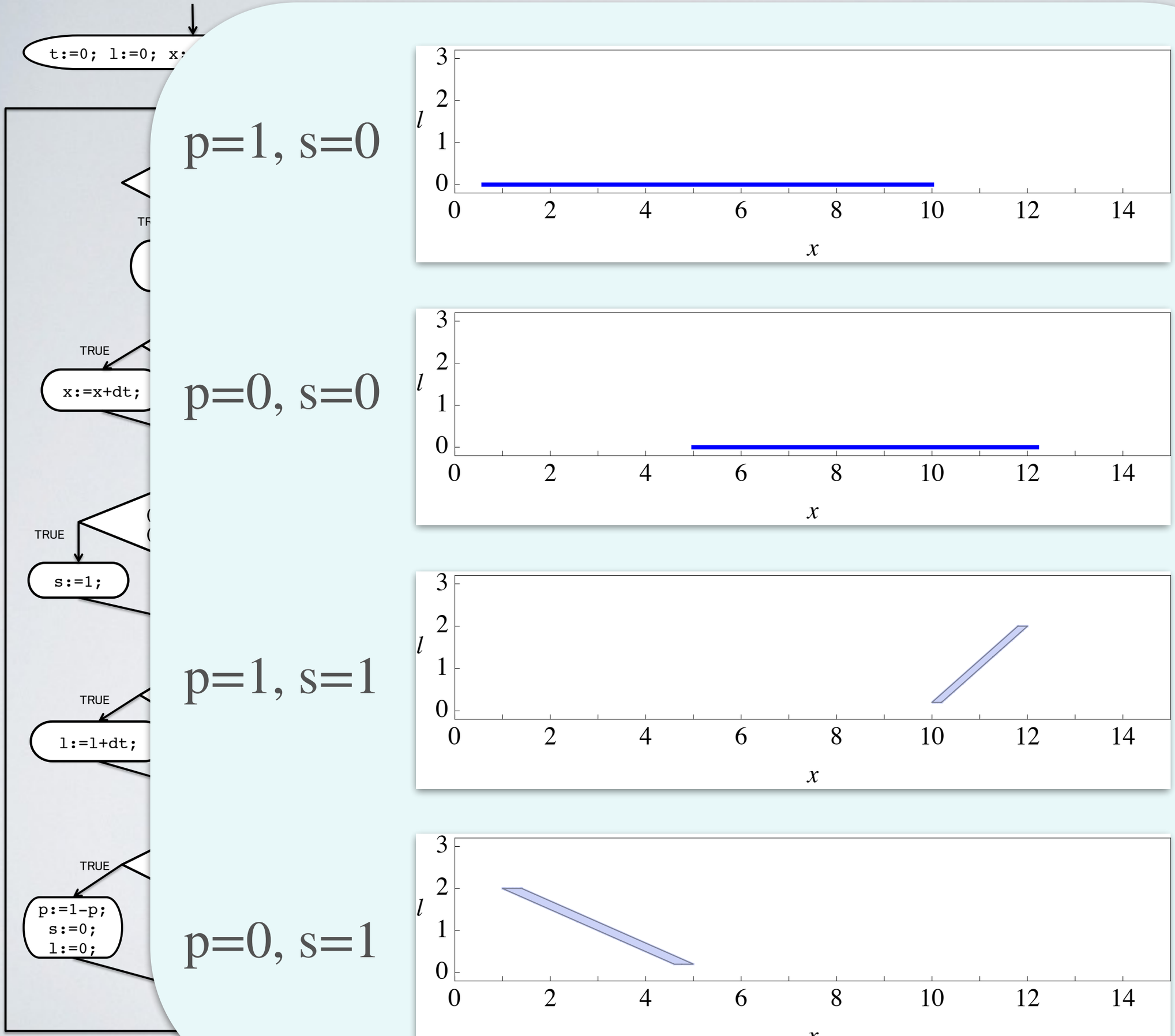
TRUE
FALSE
s:=1;
s:=0;
s=1
TRUE
FALSE
l:=l+dt;
s=1∧
l≥2
TRUE
FALSE
p:=1-p;
s:=0;
l:=0;


p=1, s=0
p=1, s=1
p=0, s=0
p=0, s=1
l
x



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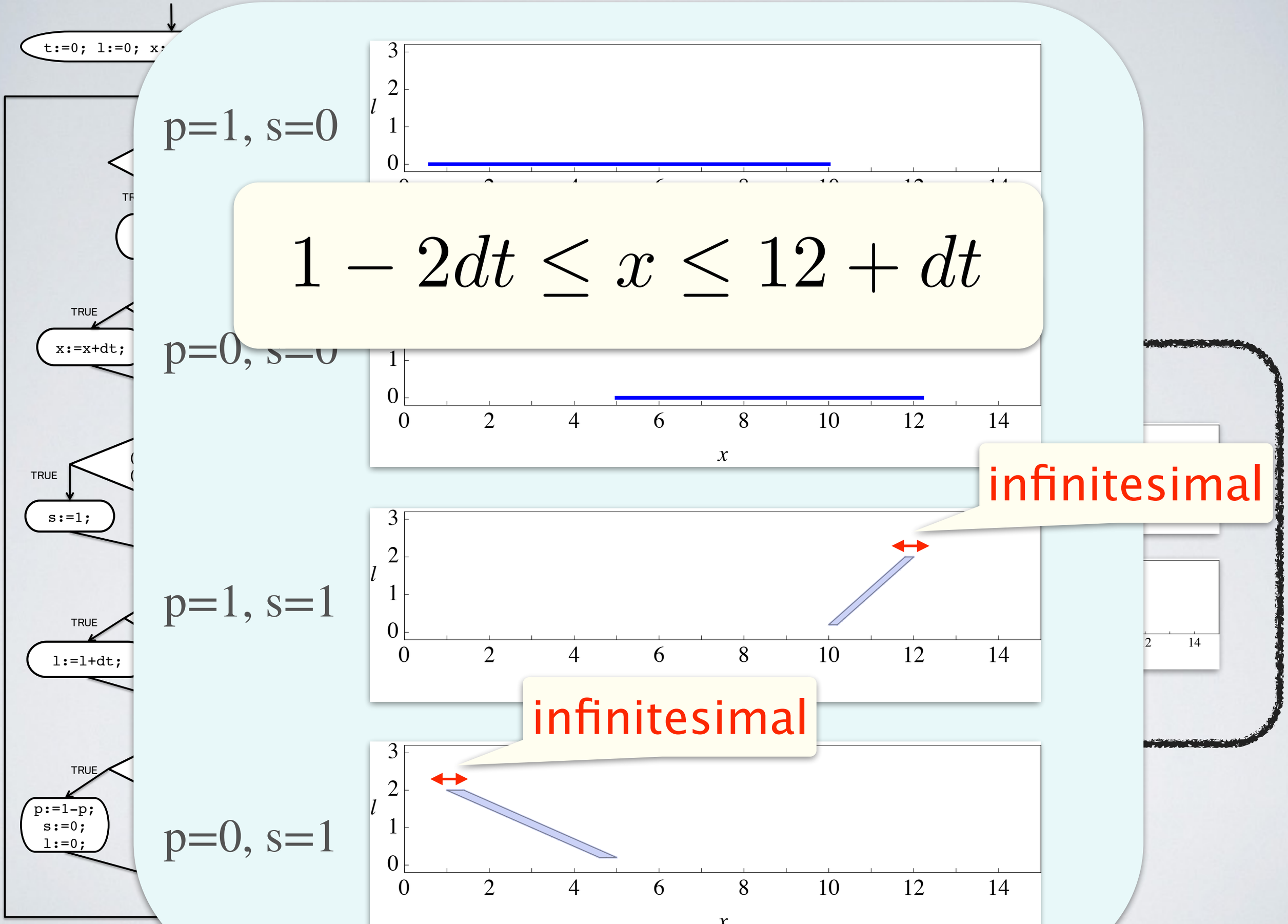
t:=0; l:=0; x:
TRUE
x:=x+dt;
TRUE
s:=1;
TRUE
l:=l+dt;
TRUE
p:=1-p;
s:=0;
l:=0;
p=1, s=0
p=0, s=0
p=1, s=1
p=0, s=1
$1 - 2dt \leq x \leq 12 + dt$
infinitesimal
infinitesimal
l
x
0
2
4
6
8
10
12
14

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where $x \in \mathbf{Var}, r \in \mathbb{R}$ and $\mathtt{aop} \in \{+, -, \cdot, \hat{}\}$

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Nonstandard Analysis

[Robinson 60's]

hyperreals

$$\mathbb{R} \mapsto {}^*\mathbb{R} := \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}}$$

$$\mathbb{N} \mapsto {}^*\mathbb{N} := \mathbb{N}^{\mathbb{N}} / \sim_{\mathcal{F}}$$

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hypernaturals

$${}^*r := [(r, r, r, \cdots)]$$

$$\omega := [(1, 2, 3, \cdots)]$$

$$\omega^{-1} := [(1, \frac{1}{2}, \frac{1}{3}, \cdots)]$$

Nonstandard Analysis

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∋ infinitesimals, infinites

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∋ infinites

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$$\omega := [(1, 2, 3, \cdots)]$$

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Collecting Semantics for WHILE^{dt}

$$[\![x]\!]\boldsymbol{\sigma} := \boldsymbol{\sigma}(x) \text{ for each } x \in \mathbf{Var}$$
$$[\![r]\!]\boldsymbol{\sigma} := r \text{ for each } r \in \mathbb{R}$$
$$[\![a_1 \ \mathtt{aop} \ a_2]\!]\boldsymbol{\sigma} := [\![a_1]\!] \ \mathtt{aop} \ [\![a_2]\!]$$
$$[\![\mathtt{dt}]\!]\boldsymbol{\sigma} := [(1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots)]$$

$$[\![\mathtt{true}]\!]\boldsymbol{\sigma} := \mathrm{tt}$$
$$[\![\mathtt{false}]\!]\boldsymbol{\sigma} := \mathrm{ff}$$
$$[\![b_1 \wedge b_2]\!]\boldsymbol{\sigma} := [\![b_1]\!] \wedge [\![b_2]\!]$$
$$[\![\neg b]\!]\boldsymbol{\sigma} := \neg([\![b]\!]\boldsymbol{\sigma})$$

$$[\![\mathtt{skip}]\!]\mathbf{S} := \mathbf{S}$$
$$[\![x := a]\!]\mathbf{S} := \{\boldsymbol{\sigma}[[\![a]\!]\boldsymbol{\sigma}/x] \mid \boldsymbol{\sigma} \in \mathbf{S}\}$$
$$[\![c_1; c_2]\!]\mathbf{S} := [\![c_2]\!]([\![c_1]\!]\mathbf{S})$$
$$[\![\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2]\!]\mathbf{S} := \begin{array}{l}\{[\![c_1]\!]\boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in \mathbf{S},\ [\![b]\!]\boldsymbol{\sigma} = \mathrm{tt}\} \\ \cup \{[\![c_2]\!]\boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in \mathbf{S},\ [\![b]\!]\boldsymbol{\sigma} = \mathrm{ff}\}\end{array}$$
$$[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!] := \mathrm{lfp}\left({}^{*}\Phi\left([\![b]\!]\right)\left([\![c]\!]\right)\right)$$

where $\Phi : (\mathbf{St} \to \mathbb{B} \cup \{\bot\}) \to \left(\mathcal{P}(\mathbf{Var} \to \mathbb{R}) \to \mathcal{P}(\mathbf{Var} \to \mathbb{R})\right) \to$

$$\left(\left(\mathcal{P}(\mathbf{Var} \to \mathbb{R}) \to \mathcal{P}(\mathbf{Var} \to \mathbb{R})\right) \to \left(\mathcal{P}(\mathbf{Var} \to \mathbb{R}) \to \mathcal{P}(\mathbf{Var} \to \mathbb{R})\right)\right)$$

is defined by $\Phi(f)(g) = \lambda\psi.\ \lambda S.\ S \cup \psi\{(g(\sigma)) \mid \sigma \in S,\ f(\sigma) = \mathrm{tt}\} \cup \{\sigma \mid \sigma \in S,\ f(\sigma) = \mathrm{ff}\}$

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Soundness

$$\mathcal{P}(\mathbb{R}^n) \xleftarrow{\gamma} \mathbb{CP}_n$$

Thm.

The least fixed point in L is
overapproximated by a prefixed point in $\overline{L}$

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Thm.

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Termination

Definition (Widening operator) Let $(L, \sqsubseteq)$ be a poset. An operator $\nabla : L \times L \to L$ is said to be a *widening operator* if the following two conditions hold:

- (Covering) for any $x, y \in L$, $x \sqsubseteq x \nabla y$ and $y \sqsubseteq x \nabla y$;
- (Termination) for any ascending chain $\langle x_i \rangle \in L^\omega$, the chain $\langle y_i \rangle \in L^\omega$ defined by $\begin{cases} y_0 = x_0 \\ y_{i+1} = y_i \nabla x_{i+1} \quad \text{(for all } i \in \mathbb{N}\text{)} \end{cases}$ is ultimately stationary.

Thm.

Prefixed point is computed within $n \in \mathbb{N}$ steps using ∇.

transfer principle

(ϕ: 1st-order $\mathcal{L}_{\mathbb{U}}$-sentence)

$$\mathbb{R} \models \phi \Leftrightarrow {}^*\mathbb{R} \models {}^*\phi$$

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$$\mathbb{R} \models \phi \Leftrightarrow {}^*\mathbb{R} \models {}^*\phi$$

$$\forall x \in \mathbb{R}.\ \forall y \in \mathbb{R}.\ (x \leq y \vee y \leq x)$$

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$$\mathbb{R} \models \phi \Leftrightarrow {}^*\mathbb{R} \models {}^*\phi$$

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$$\forall x \in {}^*\mathbb{R}.\ \forall y \in {}^*\mathbb{R}.\ (x \leq y \vee y \leq x)$$

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Standard

Nonstandard

$$\mathcal{P}(\mathbb{R}^n) \xleftarrow{\gamma} \mathbb{CP}_n$$

Thm.
The least fixed point in $\mathcal{P}(\mathbb{R}^n)$ is overapproximated by a prefixed point in $\mathbb{CP}_n$.

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Widening operator: ∇

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Prefixed point is computed within $n \in \mathbb{N}$ steps using ∇.

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Hyperwidening operator: $^*\nabla$

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Lemma (Convergence of iteration sequences in $\mathcal{L}_{\mathbb{U}}$) *Let $L \in \mathbb{U}$ be a set, $\sqsubseteq \in \mathcal{P}(L \times L)$ be a binary relation on L and $\nabla : L \times L \to L$ be a function. Then, the following $\mathcal{L}_{\mathbb{U}}$-sentence holds:*

$$\forall F \in L \to L.\ \forall \bot \in L.\ \forall X \in \mathbb{N} \to L.$$
$$\mathsf{Poset}_{L,\sqsubseteq} \wedge \mathsf{Monotone}_{L,\sqsubseteq,L,\sqsubseteq}(F) \wedge \mathsf{Basis}_{L,\sqsubseteq}(\bot, F) \wedge \mathsf{Widen}_{L,\sqsubseteq,\nabla}$$
$$\wedge \mathsf{WidenSeq}_{L,\sqsubseteq,\nabla}(X, \bot, F)$$
$$\Rightarrow \exists i \in \mathbb{N}.\ \forall j \in \mathbb{N}.\ i \leq j \Rightarrow X(i) = X(j)$$
$$\wedge \forall k \in \mathbb{N}.\ \Big(\big(\forall l \in \mathbb{N}.\ k \leq l \Rightarrow X(k) = X(l) \big) \Rightarrow F\big(X(k)\big) \sqsubseteq X(k) \Big).$$

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Theorem

Let $L \in \mathbb{U}$ be a set, $\sqsubseteq \in \mathcal{P}(L \times L)$ be a binary relation on L and $\nabla : L \times L \to L$ be a function. Then, the following $\mathcal{L}_{\mathbb{U}}$-sentence holds:

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include infinites

Uniformity of Widening Operators

– (Termination) for any ascending chain $\langle x_i \rangle \in L^\omega$, the chain $\langle y_i \rangle \in L^\omega$ defined by $\begin{cases} y_0 = x_0 \\ y_{i+1} = y_i \nabla x_{i+1} \quad \text{(for all } i \in \mathbb{N}\text{)} \end{cases}$ is ultimately stationary.

– (Uniform termination) for any $x_0 \in L$, there exists a constant $i \in \mathbb{N}$ such that for any ascending chain $\langle x_i \rangle \in L^\omega$ starting from x_0, there exists $j \in \mathbb{N}$ such that $j \leq i$ and the chain $\langle y_i \rangle \in L^\omega$ defined by

$$\begin{cases} y_0 = x_0 \\ y_{i+1} = y_i \nabla x_{i+1} \quad \text{(for all } i \in \mathbb{N}\text{)} \end{cases}$$

satisfies $y_j = y_{j+1}$.

$$\mathsf{Term}_{L,\sqsubseteq,\nabla} :\equiv \forall x \in \mathbb{N} \to L.\ \mathsf{AscCn}(x) \Rightarrow \left(\forall y \in \mathbb{N} \to L.\ \left((y(0) = x(0) \wedge \forall n \in \mathbb{N}.\ y(n+1) = y(n) \nabla x(n+1)\right) \Rightarrow \exists k \in \mathbb{N}.\ y(k) = y(k+1)\right)\right)$$

$$\mathsf{UnifTerm}_{L,\sqsubseteq,\nabla} :\equiv \forall x_0 \in L.\ \exists i \in \mathbb{N}.\ \forall x \in \mathbb{N} \to L.\ (\mathsf{AscCn}(x) \wedge x(0) = x_0) \Rightarrow \left(\forall y \in \mathbb{N} \to L.\ \left((y(0) = x(0) \wedge \forall n \in \mathbb{N}.\ y(n+1) = y(n) \nabla x(n+1)\right) \Rightarrow \exists j \in \mathbb{N}.\ (j \leq i \wedge y(j) = y(j+1))\right)\right)$$

Theorem 4.9. *Let $(L, \sqsubseteq)$ be a preorder and $\nabla \in L \times L \to L$ be a uniform widening operator on L. Let $F : {}^*L \to {}^*L$ be a monotone and internal function; and $\bot \in L$ be such that ${}^*\bot \; {}^*\sqsubseteq F({}^*\bot)$. The iteration sequence $\langle X_i \rangle_{i \in \mathbb{N}}$ defined by*

$$X_0 = {}^*\bot, \quad X_{i+1} = \begin{cases} X_i & (if\ F(X_i)\ {}^*\sqsubseteq X_i) \\ X_i\ {}^*\nabla\ F(X_i) & (otherwise) \end{cases} \quad \text{for all } i \in \mathbb{N}$$

reaches its limit within some finite number of steps; and the limit $\bigsqcup_{i \in \mathbb{N}} X_i$ is a prefixed point of F such that ${}^*\bot \; {}^*\sqsubseteq \bigsqcup_{i \in \mathbb{N}} X_i$. □

Uniformity of Widening Operators on ℂℍℙn

- Standard widening
  [Halbwachs Ph.D. Thesis 79]

- Widening up to
  [Halbwachs CAV 93]

- Precise widening
  [Bagnara, Hill, Ricci and Zaffanella SCP 05]

Uniformity of Widening Operators on $\mathbb{CH}_n$

- Standard widening
  [Halbwachs Ph.D. Thesis 79] ✔

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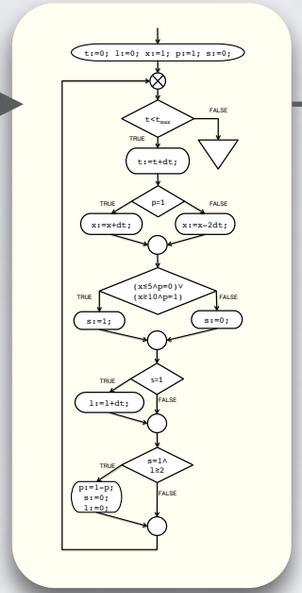
- Example of analysis
- Semantics of WHILE^{dt}
- Abstract interpretation with infinitesimals
    - Soundness
    - Termination
- Implementation

Tool Overview

WHILE^{dt} program

$t := 0; l := 0; x := 1; p := 1; s := 0;$
while $t < t_{max}$ **do**
  $t := t +$ dt;
  if $p = 1$ **then** $x := x +$ dt; **else** $x := x - 2$dt;
  if $(x \leq 5 \wedge p = 0) \vee (x \geq 10 \wedge p = 1)$ **then** $s := 1$ **else** $s := 0$;
  if $s = 1$ **then** $l := l +$ dt;
  if $s = 1 \wedge l \geq 2$ **then** $p := 1 - p; s := 0; l := 0$

CFG



- operations on $\mathbb{CP}_n$
- uniform widenings for While loops

Use CAS to deal with dt

Over-approximation of reachable set (with dt)

$$1 - 2dt \leq x \leq 12 + dt$$

1. Arithmetic Operations

e.g. $(1 + \mathtt{dt}) + \mathtt{dt} = 1 + 2\mathtt{dt}$

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2. Validity of Formulas with dt's

e.g. $y \geq x - \mathtt{dt} \land y \geq -x - \mathtt{dt} \Rightarrow y \geq -1$

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$\exists r > 0.\ \forall a \in (0, r).\ (y \geq x - a \land y \geq -x - a \Rightarrow y \geq -1)$

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Proposition 3.14. *Let A be an $\mathscr{L}_{\mathbb{R}}$-formula with a unique free variable x; to emphasize it we write $A(x)$ for A. Then the validity of the formula*

$$\exists r \in \mathbb{R}.\ (0 < r \wedge \forall x \in \mathbb{R}.\ (0 < x < r \Rightarrow A(x)))$$

(in $V(\mathbb{R})$) implies the validity of ${}^*A(\mathtt{dt})$ in $V({}^*\mathbb{R})$. □

Abstract interpretation is extended with infinitesimals

- Soundness
- Termination with uniform widenings
- Prototype implementation

Future Work

- Numerical implementation
- Transferring other abstract domains
- Hyperwidening operators other than transferred uniform widening operators