Term Evaluation Systems with Refinements

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Two kinds of rewriting in functional programming

evaluation	refinement
run-time rewriting	compile-time rewriting
beta-reduction $E[(\lambda x\,.\,t)\ v] \to E[t\{v/x\}]$	beta-law $C[(\lambda x\,.\,t)\ v] \Rightarrow C[t\{v/x\}]$ eta-law $C[\lambda x\,.\,v\ x] \Rightarrow C[v]$

Two kinds of rewriting in functional programming

- with different roles:

evaluation	refinement
run-time rewriting	compile-time rewriting
beta-reduction $E[(\lambda x\,.\,t)\ v] \to E[t\{v/x\}]$	beta-law $C[(\lambda x\,.\,t)\ v] \Rightarrow C[t\{v/x\}]$ eta-law $C[\lambda x\,.\,v\ x] \Rightarrow C[v]$
specification of *operational semantics*	model of *optimisation* to be **validated** wrt. evaluation

Validation of refinement $\Rightarrow$ wrt. evaluation $\rightarrow$

- by proving *contextual improvement* [Sands '96]
  - "two terms t, u have the same result, and u requires *fewer steps* than t"
  - $t \precsim u \overset{\Delta}{\iff} \forall C, v, v'.$ if $C[t] \rightarrow^k v$ then $C[u] \rightarrow^m v' \wedge v =_{Val} v' \wedge k \geq m$

$\lambda x.t =_{Val} \lambda x'.t'$

$\underline{n} =_{Val} \underline{n}$

- namely, by proving: refinement $t \Rightarrow u$ implies contextual improvement $t \precsim u$
  - our goal: develop a proof methodology

Our goal

- develop a proof methodology
  - to validate refinement $\Rightarrow$ wrt. evaluation $\rightarrow$
  - to establish: refinement $\Rightarrow$ implies contextual improvement $\precsim$
- by a *rewriting-theoretic* approach

contribution 1

  - exploiting critical pair analysis
    - in the hope of (partial) automation
  - by *rewriting-theoretic* modelling of both refinement $\Rightarrow$ and evaluation $\rightarrow$

contribution 2

Our goal

- develop a proof methodology
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  - to establish: refinement $\Rightarrow$ implies contextual improvement $\precsim$
- by a *rewriting-theoretic* approach — contribution 1
  - exploiting critical pair analysis
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  - by *rewriting-theoretic* modelling of both refinement $\Rightarrow$ and evaluation $\rightarrow$ — contribution 2

Modelling evaluation → and refinement ⇒

- being aware of different rewriting-theoretic properties!

evaluation	refinement
run-time rewriting	compile-time rewriting
beta-reduction $E[(\lambda x\,.\,t)\ v] \to E[t\{v/x\}]$	beta-law $C[(\lambda x\,.\,t)\ v] \Rightarrow C[t\{v/x\}]$ eta-law $C[\lambda x\,.\,v\ x] \Rightarrow C[v]$
rule-based rewriting	rule-based rewriting
restricted by *evaluation contexts* [Felleisen '88]	unrestricted

new kind of rewriting

general, standard, rewriting

Modelling evaluation → and refinement ⇒

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$\dfrac{(l \to r) \in \mathcal{E} \quad E \in Ectx}{E[l\theta] \to_{\mathcal{E}} E[r\theta]}$	$\dfrac{(l \Rightarrow r) \in \mathcal{R} \quad C \in Ctx}{C[l\theta] \Rightarrow_{\mathcal{R}} C[r\theta]}$

new kind of rewriting

general, standard, rewriting

Modelling evaluation → and refinement ⇒: an example

- being aware of different rewriting-theoretic properties!

evaluation	refinement
the left-to-right call-by-value λ-calculus $t ::= x \mid \lambda x\,.\,t \mid t @ t$ $v \in Val ::= \lambda x\,.\,t$ $E \in Ectx ::= \square \mid E @ t \mid v @ E$ $\mathscr{E} = \{(\lambda x\,.\,t) @ v \to t\{v/x\}\}$	$C \in Ctx ::= \square \mid \lambda x\,.\,C \mid E @ t \mid t @ E$ $\mathscr{R} = \left\{ \begin{array}{l} (\lambda x\,.\,t) @ v \Rightarrow t\{v/x\}, \\ \lambda x\,.\,v @ x \Rightarrow v \end{array} \right\}$
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terms

Modelling evaluation $\to$ and refinement $\Rightarrow$: an example

- being aware of different rewriting-theoretic properties!

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$t ::= x \mid \lambda x\,.\,t \mid t @ t$ $v \in Val ::= \lambda x\,.\,t$ $E \in Ectx ::= \Box \mid E @ t \mid v @ E$ $\mathscr{E} = \{(\lambda x\,.\,t) @ v \to t\{v/x\}\}$	$C \in Ctx ::= \Box \mid \lambda x\,.\,C \mid E @ t \mid t @ E$ $\mathscr{R} = \left\{ \begin{array}{l} (\lambda x\,.\,t) @ v \Rightarrow t\{v/x\}, \\ \lambda x\,.\,v @ x \Rightarrow v \end{array} \right\}$
$Val \subseteq \mathrm{NF}(\to_{\mathscr{E}})$	
$\dfrac{(l \to r) \in \mathcal{E} \quad E \in Ectx}{E[l\theta] \to_{\mathcal{E}} E[r\theta]}$	$\dfrac{(l \Rightarrow r) \in \mathcal{R} \quad C \in Ctx}{C[l\theta] \Rightarrow_{\mathcal{R}} C[r\theta]}$

values

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evaluation contexts

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(arbitrary) contexts

new kind of rewriting

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refinement rules

new kind of rewriting

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Term Evaluation and Refinement Systems (TERS)

Def. A TERS $(\Sigma, \mathscr{E}, \mathscr{R}, Ectx, SClass)$ is given by:

- a signature Σ
- a set $\mathscr{E}$ of evaluation rules $l \to r$
- a set $\mathscr{R}$ of refinement rules $l \Rightarrow r$
- a set $Ectx \subseteq Ctx$ of evaluation contexts
- a set $SClass$ of syntax classes including $Val \subseteq \mathrm{NF}(\to_{\mathscr{E}})$

$$\frac{(l \to r) \in \mathcal{E} \quad E \in Ectx}{E[l\theta] \to_{\mathcal{E}} E[r\theta]} \qquad \frac{(l \Rightarrow r) \in \mathcal{R} \quad C \in Ctx}{C[l\theta] \Rightarrow_{\mathcal{R}} C[r\theta]}$$

Term Evaluation and Refinement Systems (TERS)

Ex. A TERS $\mathbf{CBV}\lambda$ for the left-to-right call-by-value λ-calculus

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- a set $\mathscr{E} = \{(\lambda x\,.\,t)\,@\,v \rightarrow t\{v/x\}\}$ of an evaluation rule
- a set $\mathscr{R} = \{(\lambda x\,.\,t)\,@\,v \Rightarrow t\{v/x\},\ \ \lambda x\,.\,v\,@\,x \Rightarrow v\}$ of refinement rules
- evaluation contexts $E \in Ectx ::= \square \mid E\,@\,t \mid v\,@\,E$
- a set $SClass$ of syntax classes including $Val = \{\lambda x\,.\,t\} \subseteq \mathrm{NF}(\rightarrow_{\mathscr{E}})$

$$\frac{(l \to r) \in \mathcal{E} \quad E \in Ectx}{E[l\theta] \to_{\mathcal{E}} E[r\theta]} \qquad \frac{(l \Rightarrow r) \in \mathcal{R} \quad C \in Ctx}{C[l\theta] \Rightarrow_{\mathcal{R}} C[r\theta]}$$

Term Evaluation and Refinement Systems (TERS)

<u>Ex.</u> A TERS $\mathbf{Hndl}$ for a λ-calculus with effect handlers [Pretnar '15]

Syntax class *Sclass*

functions $F ::= x \mid \mathtt{fun}(x.P)$

values $V ::= \mathtt{true} \mid \mathtt{false} \mid F \mid H$

handlers $H ::= \mathtt{handler}_1(x.P, x.k.P_1) \mid \mathtt{handler}_0(x.P)$

computations $P, P_1, P_2 ::= \mathtt{return}(V) \mid \mathtt{op}(V, y.P) \mid \mathtt{do}(P_1, x.P_2)$
$\mid \mathtt{if}(V, P_1, P_2) \mid F\ V \mid \mathtt{with\_handle}(H, P)$

Evaluation contexts *Ectx* $E ::= \square \mid \mathtt{do}(E, x.P) \mid \mathtt{with\_handle}(H, E)$

Evaluation rules $\mathcal{E}$ where $i \in [2]$

$\mathtt{do}(\mathtt{return}(V), x.P[x]) \to P[V]$ (1)

$\mathtt{do}(\mathtt{op}_i(V, y.P_1[y]), x.P_2[x]) \to \mathtt{op}_i(V, y.\mathtt{do}(P_1[y], x.P_2[x]))$ (2)

$\mathtt{if}(\mathtt{true}, P_1, P_2) \to P_1$ (3)

$\mathtt{if}(\mathtt{false}, P_1, P_2) \to P_2$ (4)

$\mathtt{fun}(x.P[x])\ V \to P[V]$ (5)

In the following three rules, $h_1 \equiv \mathtt{handler}_1(x.P[x], x.k.P_1[x, k])$.

$\mathtt{with\_handle}(h_1, \mathtt{return}(V)) \to P[V]$ (6)

$\mathtt{with\_handle}(h_1, \mathtt{op}_1(V, y.P'[y])) \to P_1[V, \mathtt{fun}(y.P'[y])]$ (7)

$\mathtt{with\_handle}(h_1, \mathtt{op}_1(V, y.P'[y])) \to P_1[V, \mathtt{fun}(y.\mathtt{with\_handle}(h_1, P'[y]))]$ (7')

$\mathtt{with\_handle}(h_1, \mathtt{op}_2(V, y.P'[y])) \to \mathtt{op}_2(V, y.\mathtt{with\_handle}(h_1, P'[y]))$ (8)

In the following two rules, $h_0 \equiv \mathtt{handler}_0(x.P[x])$.

$\mathtt{with\_handle}(h_0, \mathtt{return}(V)) \to P[V]$ (9)

$\mathtt{with\_handle}(h_0, \mathtt{op}_i(V, y.P'[y])) \to \mathtt{op}_i(V, y.\mathtt{with\_handle}(h_0, P'[y]))$ (10)

Refinement rules $\mathcal{R}$

$\mathtt{do}(P, x.\mathtt{return}(x)) \Rightarrow P$ (r3)

$\mathtt{do}(\mathtt{do}(P_1, x_1.P_2[x_1]), x_2.P_3[x_2]) \Rightarrow \mathtt{do}(P_1, x_1.\mathtt{do}(P_2[x_1], x_2.P_3[x_2]))$ (r4)

$\mathtt{if}(V, P[\mathtt{true}], P[\mathtt{false}]) \Rightarrow P[V]$ (r7)

$\mathtt{fun}(x.F\ x) \Rightarrow F$ (r9)

$\mathtt{with\_handle}(\mathtt{handler}_0(x.P[x]), P') \Rightarrow \mathtt{do}(P', x.P[x])$ (r13)

Our goal

- develop a proof methodology
  - to validate refinement $\Rightarrow$ wrt. evaluation $\rightarrow$
  - to establish refinement $\Rightarrow$ implies contextual improvement $\preceq$

- by a *rewriting-theoretic* approach
  contribution 1
  - exploiting critical pair analysis
  - in the hope of (partial) automation
  - by *rewriting-theoretic* modelling of both refinement $\Rightarrow$ and evaluation $\rightarrow$
    as Term Evaluation and Refinement Systems (TERS)
    contribution 2

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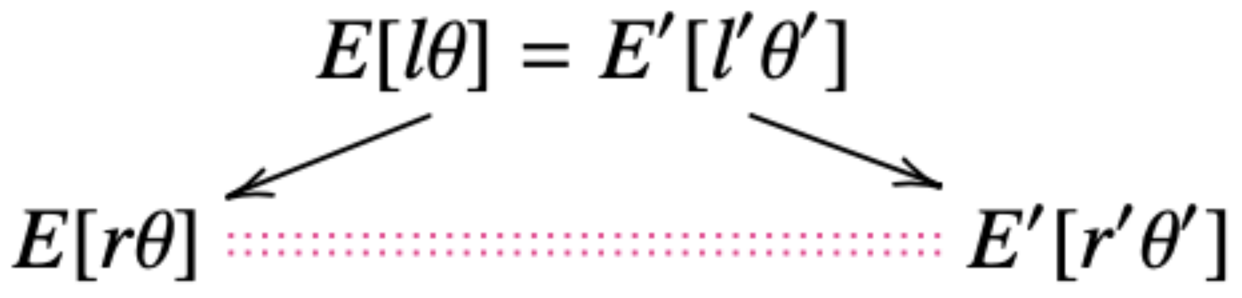
contribution 2

Sufficient conditions for contextual improvement $\preceq$

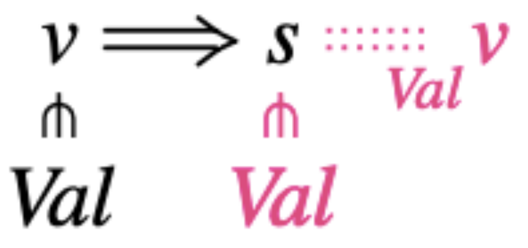
Thm. (sufficient conditions for improvement)

If a TERS is

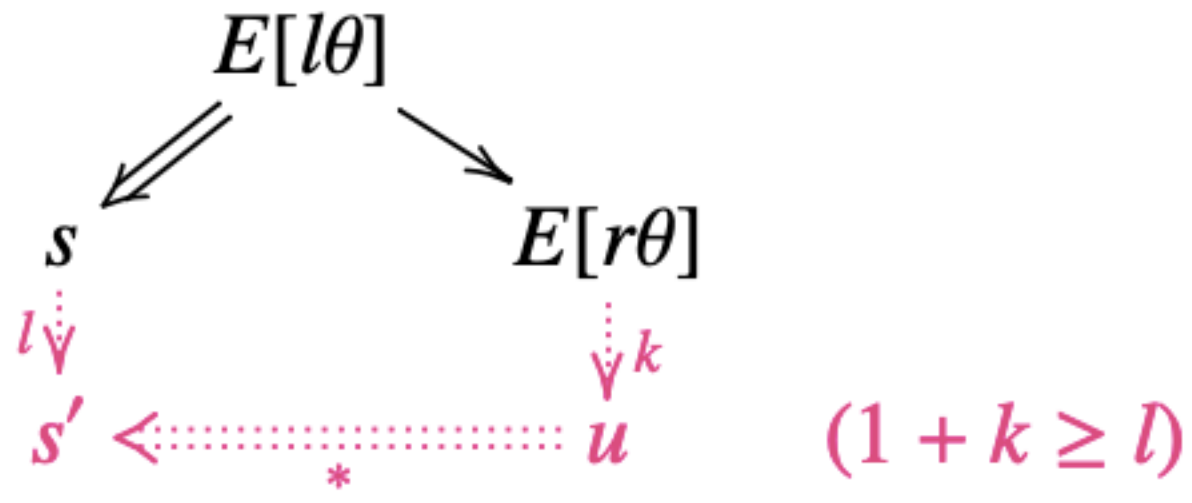
- *deterministic*



- *value-invariant*



- *locally coherent*,



akin to *local coherence* for equational rewriting [Huet '80] [Aoto & Toyama '12]

then refinement $\mathscr{R}$ is *contextual improvement* w.r.t. evaluation $\mathscr{E}$

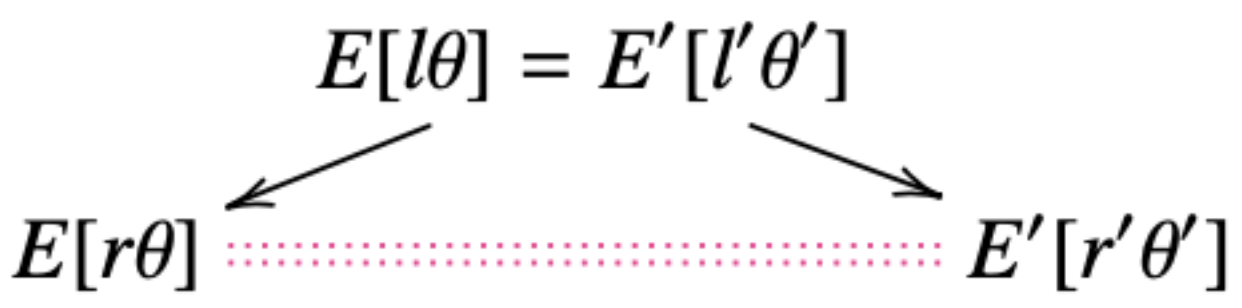
(i.e. $t \preceq u$ for each $t \Rightarrow_{\mathscr{R}} u$)

Sufficient conditions for contextual improvement $\precsim$

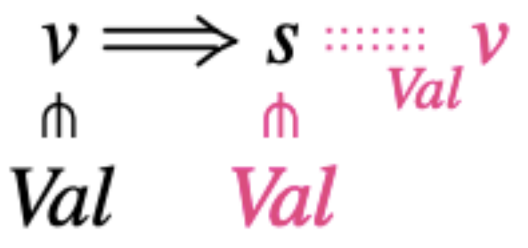
Thm. (sufficient conditions for improvement)

If a TERS is

- *deterministic*



- *value-invariant*



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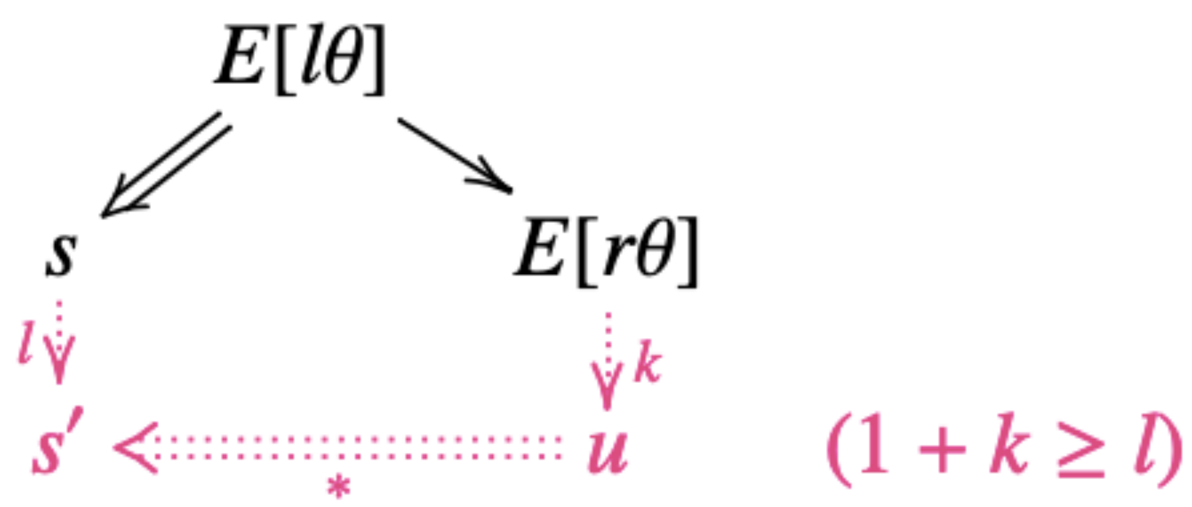

feasible
pen-and-paper
proof

tedious, error-prone,
case analysis

where *critical pair analysis* can help!

then refinement $\mathscr{R}$ is *contextual improvement* w.r.t. evaluation $\mathscr{E}$

(i.e. $t \precsim u$ for each $t \Rightarrow_{\mathscr{R}} u$)

Sufficient conditions for contextual improvement $\preceq$

Thm. (critical pair theorem)

A *well-behaved* TERS

- refinement respecting evaluation contexts

$$\begin{array}{ccc} E & \Longrightarrow & C \\ \pitchfork & & \pitchfork \\ \mathit{Ectx} & & \mathit{Ectx} \end{array}$$

- linearity for $(l \Rightarrow r) \in \mathscr{R}$
- left-linearity for $(l \rightarrow r) \in \mathscr{E}$
- lhs l being Miller's HO pattern for $(l \multimap r) \in \mathscr{R} \cup \mathscr{E}$
- …

is *locally coherent* iff every critical pair is *joinable*.

feasible
pen-and-paper
proof

automated
enumeration &
joinability check

Term Evaluation and Refinement Systems (TERS)

Ex. A TERS $\mathbf{CBV}\lambda$ for the left-to-right call-by-value λ-calculus

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- a set $\mathscr{E} = \{(\lambda x\,.\,t)\,@\,v \to t\{v/x\}\}$ of an evaluation rule
- a set $\mathscr{R} = \{(\lambda x\,.\,t)\,@\,v \Rightarrow t\{v/x\},\ \ \lambda x\,.\,v\,@\,x \Rightarrow v\}$ of refinement rules
- evaluation contexts $E \in Ectx ::= \square \mid E\,@\,t \mid v\,@\,E$
- a set $SClass$ of syntax classes including $Val = \{\lambda x\,.\,t\} \subseteq \mathrm{NF}(\to_{\mathscr{E}})$

Prop. For the TERS $\mathbf{CBV}\lambda$, refinement $\mathscr{R}$ is contextual improvement $\preceq$.

- ✓ determinism
- ✓ value-invariance
- ✓ local coherence
  - ✓ well-behavedness
  - ✓ 2 critical pairs joinable

automatically
enumerated & checked

Term Evaluation and Refinement Systems (TERS)

Ex. A TERS **Hndl** for a λ-calculus with effect handlers [Pretnar ’15]

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$\mathtt{if}(\mathtt{true}, P_1, P_2) \to P_1$ (3)

$\mathtt{if}(\mathtt{false}, P_1, P_2) \to P_2$ (4)

$\mathtt{fun}(x.P[x])\ V \to P[V]$ (5)

In the following three rules, $h_1 \equiv \mathtt{handler}_1(x.P[x], x.k.P_1[x,k])$.

$\mathtt{with\_handle}(h_1, \mathtt{return}(V)) \to P[V]$ (6)

$\mathtt{with\_handle}(h_1, \mathtt{op}_1(V, y.P'[y])) \to P_1[V, \mathtt{fun}(y.P'[y])]$ (7)

$\mathtt{with\_handle}(h_1, \mathtt{op}_1(V, y.P'[y])) \to P_1[V, \mathtt{fun}(y.\mathtt{with\_handle}(h_1, P'[y]))]$ (7’)

$\mathtt{with\_handle}(h_1, \mathtt{op}_2(V, y.P'[y])) \to \mathtt{op}_2(V, y.\mathtt{with\_handle}(h_1, P'[y]))$ (8)

In the following two rules, $h_0 \equiv \mathtt{handler}_0(x.P[x])$.

$\mathtt{with\_handle}(h_0, \mathtt{return}(V)) \to P[V]$ (9)

$\mathtt{with\_handle}(h_0, \mathtt{op}_i(V, y.P'[y])) \to \mathtt{op}_i(V, y.\mathtt{with\_handle}(h_0, P'[y]))$ (10)

Refinement rules $\mathcal{R}$

$\mathtt{do}(P, x.\mathtt{return}(x)) \Rightarrow P$ (r3)

$\mathtt{do}(\mathtt{do}(P_1, x_1.P_2[x_1]), x_2.P_3[x_2]) \Rightarrow \mathtt{do}(P_1, x_1.\mathtt{do}(P_2[x_1], x_2.P_3[x_2]))$ (r4)

$\mathtt{if}(V, P[\mathtt{true}], P[\mathtt{false}]) \Rightarrow P[V]$ (r7)

$\mathtt{fun}(x.F\ x) \Rightarrow F$ (r9)

$\mathtt{with\_handle}(\mathtt{handler}_0(x.P[x]), P') \Rightarrow \mathtt{do}(P', x.P[x])$ (r13)

Prop. For the TERS **Hndl**,

refinement $\mathscr{R}$ is

contextual improvement $\preceq$.

- ✓ determinism
- ✓ value-invariance
- ✓ local coherence
  - ✓ well-behavedness
  - ✓ 10 critical pairs joinable

automatically enumerated & checked

Overview

evaluation	refinement
run-time rewriting	compile-time rewriting
specification of operational semantics	model of optimisation to be **validated** wrt. evaluation
$\dfrac{(l \to r) \in \mathcal{E} \quad E \in Ectx}{E[l\theta] \to_{\mathcal{E}} E[r\theta]}$	$\dfrac{(l \Rightarrow r) \in \mathcal{R} \quad C \in Ctx}{C[l\theta] \Rightarrow_{\mathcal{R}} C[r\theta]}$

formalise **Term Evaluation and Refinement Systems (TERS)**

$$(\Sigma, \mathscr{E}, \mathscr{R}, Ectx, SClass)$$

Overview

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- formally, *contextual improvement*
- develop a proof methodology centred around critical pair analysis