Local Coherence and Program Refinement
(work in progress)

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58th TRS Meeting (Niigata), 20 February 2023

Term *Evaluation* and Term *Rewriting* (work in progress)

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Is program semantics a TRS?

- Yes?
  - Arithmetics is a TRS.

$$
\begin{aligned}
(1+2)+(3+4) &\to 3+(3+4) \to 3+7 \to 10 \\
&\to (1+2)+7 \to 3+7 \to
\end{aligned}
$$

Is program semantics a TRS?

- No!
  - Left-to-right arithmetics is not a TRS.

$$(1+2)+(3+4) \to 3+(3+4) \to 3+7 \to 10$$

$$\not\to (1+2)+7 \to 3+7 \to$$

  - The evaluation order is specified by *evaluation contexts* [Felleisen, LFP ’88 & POPL ‘88], e.g. $E ::= \square \mid E + t \mid n + E$
    - $\square + (3+4)$ is
    - $(1+2) + \square$ is

1. evaluate the first argument to a number
2. evaluate the second argument

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  - The evaluation order is specified by *evaluation contexts* [Felleisen, LFP ’88 & POPL ‘88], e.g. $E ::= \square \mid E + t \mid n + E$
    - $\square + (3+4)$ is an evaluation context.
    - $(1+2) + \square$ is not an evaluation context.

Is program semantics a TRS?

- No!
  - The evaluation *order* is specified by *evaluation contexts* [Felleisen, LFP ’88 & POPL ’88], e.g. $E ::= \Box \mid E + t \mid n + E$
    - Context-sensitive rewriting [Lucas, ’00] is not enough.
- **Question** How can then we transfer TRS techniques to program semantics?
  - e.g. critical pair analysis
- **Answer** Use Term *Evaluation* Systems, a variant of TRS!

Term Evaluation Systems (TES)

- *evaluation*

$$\frac{(l \to r) \in R \quad \theta : \text{subst.} \quad E \in \mathscr{E}}{E[l\theta] \to_R E[r\theta]}$$

closed under evaluation contexts $\mathscr{E}$ only

- cf. ordinary rewriting

$$\frac{(l \to r) \in R \quad \theta : \text{subst.} \quad C : \text{context}}{C[l\theta] \to_R C[r\theta]}$$

closed under any contexts

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study interaction between evaluation and rewriting

Interaction between evaluation and rewriting

- From program semantics perspective:

terms $t ::= \underline{n} \mid t + t \mid t \times t$

Interaction between evaluation and rewriting (refinement)

- From program semantics perspective:

evaluation contexts	$E ::= \Box \mid E + t \mid \underline{n} + E \mid E \times t \mid \underline{n} \times E$
evaluation rules	$\underline{m} + \underline{n} \to \underline{m + n}, \quad \underline{m} \times \underline{n} \to \underline{m \times n}$
evaluation relation	$\dfrac{l \to r \qquad E}{E[l] \to E[r]}$

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evaluation relation	$\dfrac{l \to r \qquad E}{E[l] \to E[r]}$
refinement rules	$\underline{l} \times (\underline{m} + \underline{n}) \Rightarrow \underline{l} \times \underline{m} + \underline{l} \times \underline{n}, \quad \underline{m} + \underline{n} \Rightarrow \underline{m} \times \underline{n}$

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- **Question** Is refinement *correct* wrt. evaluation?
- **Goal** to prove that $t \Rightarrow u$ implies, for any context C,
  - $C[t]$: →-normalising $\Longrightarrow$ $C[u]$: →-normalising

Interaction between evaluation and rewriting (refinement)

- From program semantics perspective:

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evaluation rules $\underline{m} + \underline{n} \to \underline{m+n}, \quad \underline{m} \times \underline{n} \to \underline{m \times n}$

evaluation relation $\dfrac{l \to r \qquad E}{E[l] \to E[r]}$

correct *incorrect*

refinement rules $\underline{l} \times (\underline{m} + \underline{n}) \Rightarrow \underline{l} \times \underline{m} + \underline{l} \times \underline{n}, \quad \underline{m} + \underline{n} \Rightarrow \underline{m} \times \underline{n}$

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Interaction between evaluation and rewriting (refinement)

- From TES/TRS perspective:

evaluation relation

$$\frac{(l \to r) \in R \quad \theta : \text{subst.} \quad E \in \mathscr{E}}{E[l\theta] \to_R E[r\theta]}$$

refinement relation

$$\frac{(l \Rightarrow r) \in A \quad \theta : \text{subst.} \quad C : \text{context}}{C[l\theta] \Rightarrow_A C[r\theta]}$$

- **Question** Is refinement *correct* wrt. evaluation?
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Interaction between evaluation and rewriting (refinement)

evaluation relation $$\frac{(l \to r) \in R \quad \theta : \text{subst.} \quad E \in \mathscr{E}}{E[l\theta] \to_R E[r\theta]}$$

refinement relation $$\frac{(l \Rightarrow r) \in A \quad \theta : \text{subst.} \quad C : \text{context}}{C[l\theta] \Rightarrow_A C[r\theta]}$$

- **Question** Is refinement *correct* wrt. evaluation?
- **Goal** to prove that $t \Rightarrow_A u$ implies, for any context C,
  - $C[t] : \to_R$-normalising $\Longrightarrow C[u] : \to_R$-normalising
- Sufficient to prove that $t \Rightarrow_A u$ implies
  - $t : \to_R$-normalising $\Longrightarrow u : \to_R$-normalising

Interaction between evaluation and rewriting (refinement)

evaluation relation $$\frac{(l \to r) \in R \quad \theta:\text{ subst.} \quad E \in \mathscr{E}}{E[l\theta] \to_R E[r\theta]}$$

refinement relation $$\frac{(l \Rightarrow r) \in A \quad \theta:\text{ subst.} \quad C:\text{ context}}{C[l\theta] \Rightarrow_A C[r\theta]}$$

- Sufficient to prove that $t \Rightarrow_A u$ implies
  - $t : \to_R$-normalising $\Longrightarrow$ $u : \to_R$-normalising

[M., PhD thesis '20]

- For deterministic $\to_R$, sufficient to prove that $t \Rightarrow_A u$ implies

$$\begin{tikzcd}
s \arrow[r, "/" marking] \arrow[d, Rightarrow] & {} & \\
t_1 \arrow[r, dotted, "*"] & u \arrow[r, dotted, "/" marking] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
s \arrow[r] \arrow[d, Rightarrow] & t_2 \arrow[r, dotted, "*"] & u_2 \arrow[d, Rightarrow, "="] \\
t_1 \arrow[rr, dotted, "*"] & & u_1
\end{tikzcd}$$

local coherence
[Aoto & Toyama, LMCS '12]

Critical pair analysis for *local coherence*

Definition 2.8 (R-peaks, R-joinability, (A, R)-peaks, (A, R)-joinability)**.** Let $\mathcal{E}(\Sigma, R)$ be a TES with a template A.

- An R-*peak* is given by a triple (t_1, s, t_2) such that $s \to_R t_1$ and $s \to_R t_2$.
- An R-peak (t_1, s, t_2) is R-*joinable* if there exists a term u such that $t_1 \overset{*}{\to}_R u$ and $t_2 \overset{*}{\to}_R u$.
- An (A, R)-*peak* is given by a triple (t_1, s, t_2) such that $s \Rightarrow_A t_1$ and $s \to_R t_2$.
- An (A, R)-peak (t_1, s, t_2) is (A, R)-*joinable* if there exist terms u_1, u_2 such that $t_1 \overset{*}{\to} u_1$, $t_2 \overset{*}{\to} u_2$ and $u_2 \overset{=}{\Rightarrow}_A u_1$.

$$\begin{tikzcd}
s \arrow[r] \arrow[d] & t_2 \arrow[d, dotted, "*"] \\
t_1 \arrow[r, dotted, "*"] & u
\end{tikzcd}$$

$$\begin{tikzcd}
s \arrow[r] \arrow[d, Rightarrow] & t_2 \arrow[r, dotted, "*"] & u_2 \arrow[d, Rightarrow, "="] \\
t_1 \arrow[rr, dotted, "*"] & & u_1
\end{tikzcd}$$

Definition 2.11 (local coherence)**.** A TES $\mathcal{E}(\Sigma, R)$ with a template A is *locally coherent* if any (A, R)-peak is (A, R)-joinable.

Critical pair analysis for *local coherence*

Definition 5.2 (overlaps)**.** Let $(l_1 \Rightarrow r_1) \in A$ and $(l_2 \to r_2) \in R$.

- A *shrinking overlap* between $(l_1 \Rightarrow r_1)$ and $(l_2 \to r_2)$ is given by data $(l_1 \Rightarrow r_1, l_2 \to r_2, p, \theta)$, such that p is a non-variable position of l_1 and θ is a most general unifier between $l_1|_p$ and l_2.
- An *expanding overlap* between $(l_1 \Rightarrow r_1)$ and $(l_2 \to r_2)$ is given by data $(l_1 \Rightarrow r_1, l_2 \to r_2, p, \theta)$, such that p is a non-variable position of l_2 and θ is a most general unifier between $l_2|_p$ and l_1.

Definition 5.3 (critical pairs)**.**

- The *(shrinking) critical pair* generated by a shrinking overlap $(l_1 \Rightarrow r_1, l_2 \to r_2, p, \theta)$ is given by a triple $(r_1\theta, l_1\theta, l_1\theta[r_2\theta]_p)$.
- The *(expanding) critical pair* generated by an expanding overlap $(l_1 \Rightarrow r_1, l_2 \to r_2, p, \theta)$ is given by a triple $(l_2\theta[r_1\theta]_p, l_2\theta, r_2\theta)$.

3. Correctness of refinement $t \Rightarrow_A u$ wrt. evaluation $t \to_R u$

Theorem 5.5 (Critical Pair Theorem)**.** *Let* $\mathcal{E}_{S,V}(\Sigma, R)$ *be a well-behaved TES with a template* A. *If* R *is linear, and* A *is linear, right-ready and compatible with* V*, the TES* $\mathcal{E}_{S,V}(\Sigma, R)$ *with the template* A *is locally coherent if and only if every critical pair is joinable.*

- well-behaved evaluation contexts, with a notion of *values*
  - $\mathscr{E}$ is defined by a certain BNF, e.g.: $E ::= \square \mid \cdots \mid f(v, E) \mid f(E, t)$
- linear rules
  - l, r are linear terms in $(l \to r) \in R, (l \Rightarrow r) \in A$
- right-ready rules
  - in r of $(l \Rightarrow r) \in A$, if p is a variable position, $r[\,\square\,]_p \in \mathscr{E}$
- compatible rules wrt. values V
  - A template A is said to be *compatible with* a set $V \in T(\Sigma, X)$ if, for any $v \in V$ and any $(l \Rightarrow r) \in A$, if there exist a position p and a substitution θ such that $v[l\theta]_p \in V$ and $v[r\theta]_p \notin V$, then $v[r\theta]_p \overset{*}{\to}_R v[l\theta]_p$.

Overview

- *Term Evaluation Systems* with refinement (ordinary rewriting)

evaluation relation
$$\frac{(l \to r) \in R \quad \theta\text{: subst.} \quad E \in \mathscr{E}}{E[l\theta] \to_R E[r\theta]}$$

refinement relation
$$\frac{(l \Rightarrow r) \in A \quad \theta\text{: subst.} \quad C\text{: context}}{C[l\theta] \Rightarrow_A C[r\theta]}$$

- **Question** Is refinement *correct* wrt. evaluation?
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