

Local Coherence and Program Refinement (work in progress)

Makoto Hamana
(Gunma University)

Koko Muroya
(RIMS, Kyoto University)

Term *Evaluation* and Term *Rewriting* (work in progress)

Makoto Hamana
(Gunma University)

Koko Muroya
(RIMS, Kyoto University)

Is program semantics a TRS?

- Yes?
 - Arithmetics is a TRS.

$$(1 + 2) + (3 + 4) \rightarrow 3 + (3 + 4) \rightarrow 3 + 7 \rightarrow 10$$
$$\rightarrow (1 + 2) + 7 \rightarrow 3 + 7 \rightarrow$$

Is program semantics a TRS?

- No!
 - **Left-to-right** arithmetics is not a TRS.

$$(1 + 2) + (3 + 4) \rightarrow 3 + (3 + 4) \rightarrow 3 + 7 \rightarrow 10$$
$$\not\rightarrow (1 + 2) + 7 \rightarrow 3 + 7 \rightarrow$$

- The evaluation order is specified by *evaluation contexts*

[Felleisen, LFP '88 & POPL '88], e.g. $E ::= \square \mid E + t \mid n + E$

- $\square + (3 + 4)$ is

- $(1 + 2) + \square$ is

1. evaluate the first argument to a number
2. evaluate the second argument

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- The evaluation order is specified by *evaluation contexts*

[Felleisen, LFP '88 & POPL '88], e.g. $E ::= \square \mid E + t \mid n + E$

- $\square + (3 + 4)$ is an evaluation context.
- $(1 + 2) + \square$ is not an evaluation context.

Is program semantics a TRS?

- No!
 - The evaluation *order* is specified by *evaluation contexts*
[Felleisen, LFP '88 & POPL '88], e.g. $E ::= \square \mid E + t \mid n + E$
 - Context-sensitive rewriting [Lucas, '00] is not enough.
- **Question** How can then we transfer TRS techniques to program semantics?
 - e.g. critical pair analysis
- **Answer** Use Term *Evaluation* Systems, a variant of TRS!

Term Evaluation Systems (TES)

- *evaluation*

$$\frac{(l \rightarrow r) \in R \quad \theta: \text{subst.} \quad E \in \mathcal{E}}{E[l\theta] \rightarrow_R E[r\theta]}$$

closed under evaluation contexts \mathcal{E} only

- cf. ordinary rewriting

$$\frac{(l \rightarrow r) \in R \quad \theta: \text{subst.} \quad C: \text{context}}{C[l\theta] \rightarrow_R C[r\theta]}$$

closed under any contexts

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closed under any contexts

study interaction between
evaluation and rewriting

Interaction between evaluation and rewriting

- From program semantics perspective:

terms

$$t ::= \underline{n} \mid t + t \mid t \times t$$

Interaction between evaluation and rewriting (refinement)

- From program semantics perspective:

evaluation contexts $E ::= \square \mid E + t \mid \underline{n} + E \mid E \times t \mid \underline{n} \times E$

evaluation rules $\underline{m} + \underline{n} \rightarrow \underline{m + n}, \quad \underline{m} \times \underline{n} \rightarrow \underline{m \times n}$

evaluation relation
$$\frac{l \rightarrow r \quad E}{E[l] \rightarrow E[r]}$$

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refinement rules $\underline{l} \times (\underline{m} + \underline{n}) \Rightarrow \underline{l} \times \underline{m} + \underline{l} \times \underline{n}, \quad \underline{m} + \underline{n} \Rightarrow \underline{m} \times \underline{n}$

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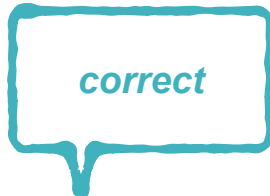

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- **Question** Is refinement *correct* wrt. evaluation?
- **Goal** to prove that $t \Rightarrow u$ implies, for any context C ,
 - $C[t] : \rightarrow$ -normalising $\implies C[u] : \rightarrow$ -normalising

Interaction between evaluation and rewriting (refinement)

- From program semantics perspective:

| | | | |
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| evaluation contexts | $E ::= \square \mid E + t \mid \underline{n} + E \mid E \times t \mid \underline{n} \times E$ | | |
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Interaction between evaluation and rewriting (refinement)

- From TES/TRS perspective:

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| refinement relation | $\frac{(l \Rightarrow r) \in A \quad \theta: \text{subst.} \quad C: \text{context}}{C[l\theta] \Rightarrow_A C[r\theta]}$ |

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- **Question** Is refinement *correct* wrt. evaluation?
- **Goal** to prove that $t \Rightarrow_A u$ implies, for any context C ,
 - $C[t] : \rightarrow_R\text{-normalising} \implies C[u] : \rightarrow_R\text{-normalising}$
- Sufficient to prove that $t \Rightarrow_A u$ implies
 - $t : \rightarrow_R\text{-normalising} \implies u : \rightarrow_R\text{-normalising}$

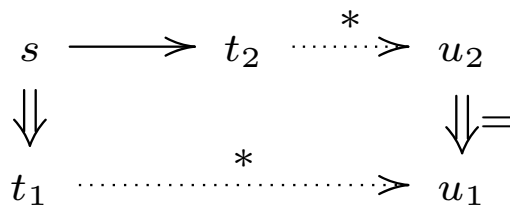
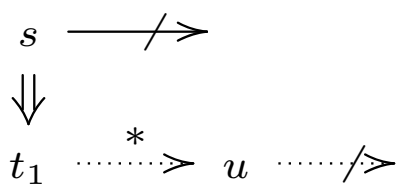
Interaction between evaluation and rewriting (refinement)

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- Sufficient to prove that $t \Rightarrow_A u$ implies
 - $t : \rightarrow_R$ -normalising $\implies u : \rightarrow_R$ -normalising

[M., PhD thesis '20]

- For deterministic \rightarrow_R , sufficient to prove that $t \Rightarrow_A u$ implies



local coherence
[Aoto & Toyama, LMCS '12]

Critical pair analysis for *local coherence*

Definition 2.8 (*R*-peaks, *R*-joinability, (*A*, *R*)-peaks, (*A*, *R*)-joinability). Let $\mathcal{E}(\Sigma, R)$ be a TES with a template *A*.

- An *R*-peak is given by a triple (t_1, s, t_2) such that $s \rightarrow_R t_1$ and $s \rightarrow_R t_2$.
- An *R*-peak (t_1, s, t_2) is *R*-joinable if there exists a term *u* such that $t_1 \xrightarrow{*}_R u$ and $t_2 \xrightarrow{*}_R u$.
- An (*A*, *R*)-peak is given by a triple (t_1, s, t_2) such that $s \Rightarrow_A t_1$ and $s \rightarrow_R t_2$.
- An (*A*, *R*)-peak (t_1, s, t_2) is (*A*, *R*)-joinable if there exist terms u_1, u_2 such that $t_1 \xrightarrow{*} u_1$, $t_2 \xrightarrow{*} u_2$ and $u_2 \xRightarrow{=}_A u_1$.

$$\begin{array}{ccc}
 s \longrightarrow t_2 & & s \longrightarrow t_2 \xrightarrow{*} u_2 \\
 \downarrow & & \Downarrow \\
 t_1 \xrightarrow{*} u & & t_1 \xrightarrow{*} u_1 \\
 & & \Downarrow =
 \end{array}$$

Definition 2.11 (local coherence). A TES $\mathcal{E}(\Sigma, R)$ with a template *A* is *locally coherent* if any (*A*, *R*)-peak is (*A*, *R*)-joinable.

Critical pair analysis for *local coherence*

Definition 5.2 (overlaps). Let $(l_1 \Rightarrow r_1) \in A$ and $(l_2 \rightarrow r_2) \in R$.

- A *shrinking overlap* between $(l_1 \Rightarrow r_1)$ and $(l_2 \rightarrow r_2)$ is given by data $(l_1 \Rightarrow r_1, l_2 \rightarrow r_2, p, \theta)$, such that p is a non-variable position of l_1 and θ is a most general unifier between $l_1|_p$ and l_2 .
- An *expanding overlap* between $(l_1 \Rightarrow r_1)$ and $(l_2 \rightarrow r_2)$ is given by data $(l_1 \Rightarrow r_1, l_2 \rightarrow r_2, p, \theta)$, such that p is a non-variable position of l_2 and θ is a most general unifier between $l_2|_p$ and l_1 .

Definition 5.3 (critical pairs).

- The (*shrinking*) *critical pair* generated by a shrinking overlap $(l_1 \Rightarrow r_1, l_2 \rightarrow r_2, p, \theta)$ is given by a triple $(r_1\theta, l_1\theta, l_1\theta[r_2\theta]_p)$.
- The (*expanding*) *critical pair* generated by an expanding overlap $(l_1 \Rightarrow r_1, l_2 \rightarrow r_2, p, \theta)$ is given by a triple $(l_2\theta[r_1\theta]_p, l_2\theta, r_2\theta)$.

3. Correctness of refinement $t \Rightarrow_A u$ wrt. evaluation $t \rightarrow_R u$

Theorem 5.5 (Critical Pair Theorem). *Let $\mathcal{E}_{S,V}(\Sigma, R)$ be a well-behaved TES with a template A . If R is linear, and A is linear, right-ready and compatible with V , the TES $\mathcal{E}_{S,V}(\Sigma, R)$ with the template A is locally coherent if and only if every critical pair is joinable.*

- well-behaved evaluation contexts, with a notion of *values*
 - \mathcal{E} is defined by a certain BNF, e.g.: $E ::= \square \mid \dots \mid f(v, E) \mid f(E, t)$
- linear rules
 - l, r are linear terms in $(l \rightarrow r) \in R, (l \Rightarrow r) \in A$
- right-ready rules
 - in r of $(l \Rightarrow r) \in A$, if p is a variable position, $r[\square]_p \in \mathcal{E}$
- compatible rules wrt. values V
 - A template A is said to be *compatible with* a set $V \in T(\Sigma, X)$ if, for any $v \in V$ and any $(l \Rightarrow r) \in A$, if there exist a position p and a substitution θ such that $v[l\theta]_p \in V$ and $v[r\theta]_p \notin V$, then $v[r\theta]_p \xrightarrow{*}_R v[l\theta]_p$.

Overview

- *Term Evaluation Systems* with refinement (ordinary rewriting)

| | |
|---------------------|---|
| evaluation relation | $\frac{(l \rightarrow r) \in R \quad \theta: \text{subst.} \quad E \in \mathcal{E}}{E[l\theta] \rightarrow_R E[r\theta]}$ |
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