Memoryful GoI with recursion

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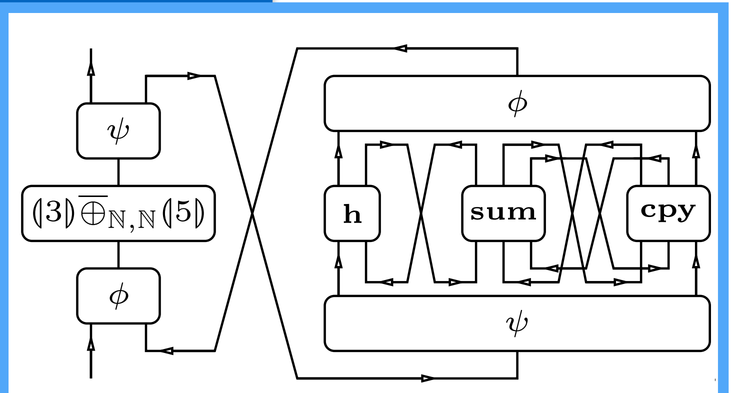
Memoryful GoI [Hoshino, —, Hasuo '14]

effectful terms

$(\lambda \mathtt{x} : \mathtt{nat}.\, \mathtt{x} + \mathtt{x})\,(3 \sqcup 5) \quad : \mathtt{nat}$

translation
- based on Geometry of Interaction
- via coalgebraic component calculus

transducers



Geometry of Interaction (GoI)

- semantics of { linear logic proof [Girard '89], functional programming
- token machine presentation [Mackie '95]

↓

compilation techniques and implementations

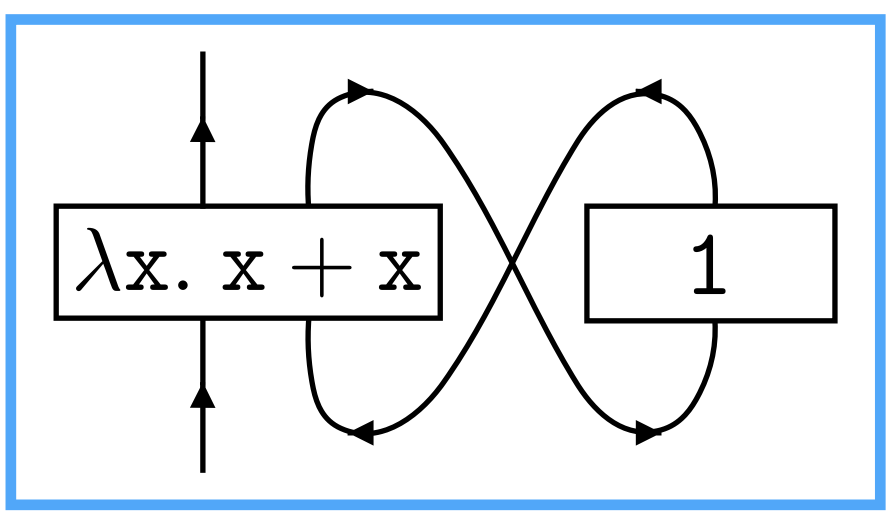
"GoI implementation"

[Mackie '95] [Pinto '01] [Ghica '07]

Geometry of Interaction (GoI)

- token machine semantics

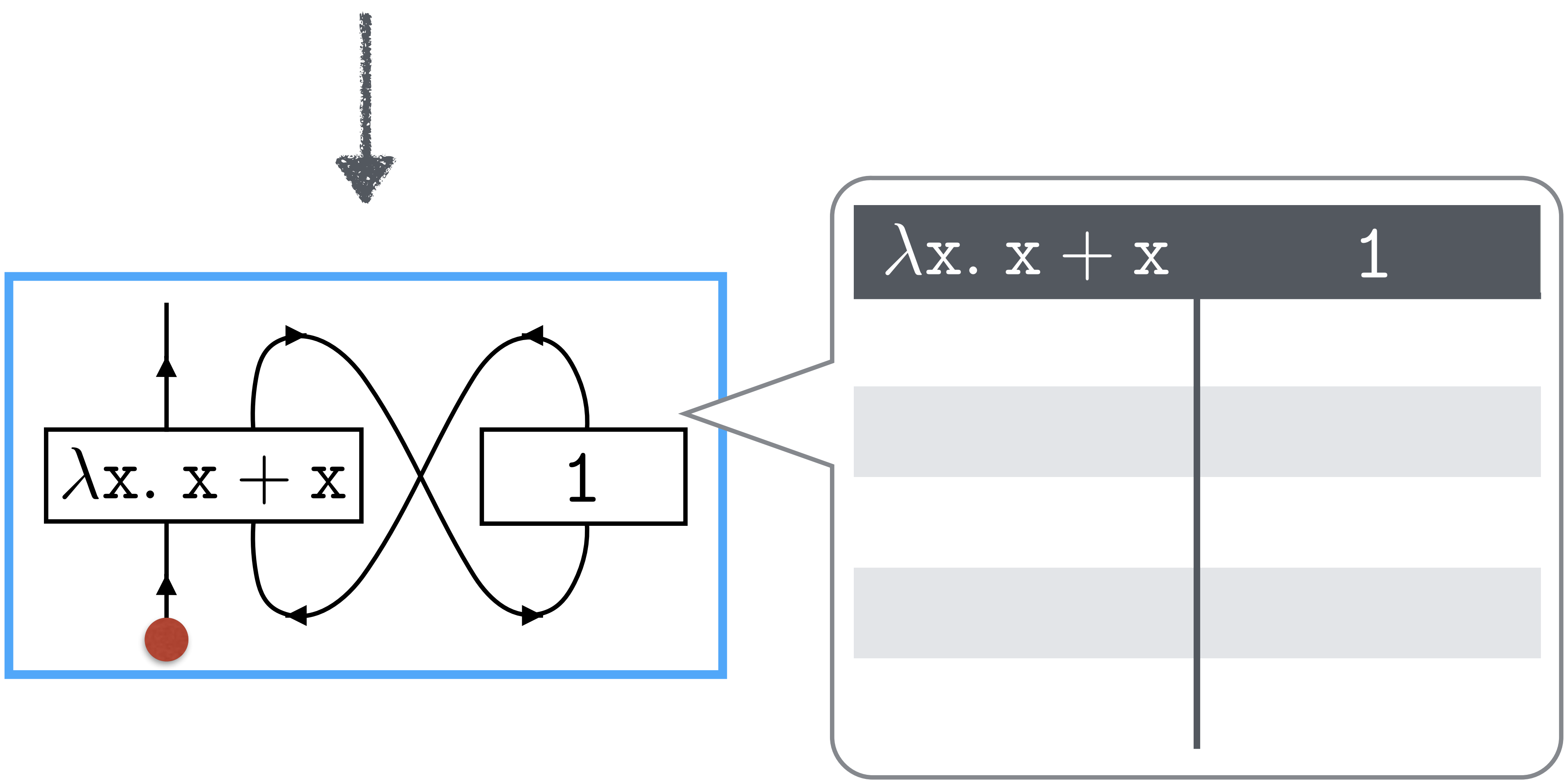
$$(\lambda x.\, x + x)\ 1$$



Geometry of Interaction (GoI)

- token machine semantics

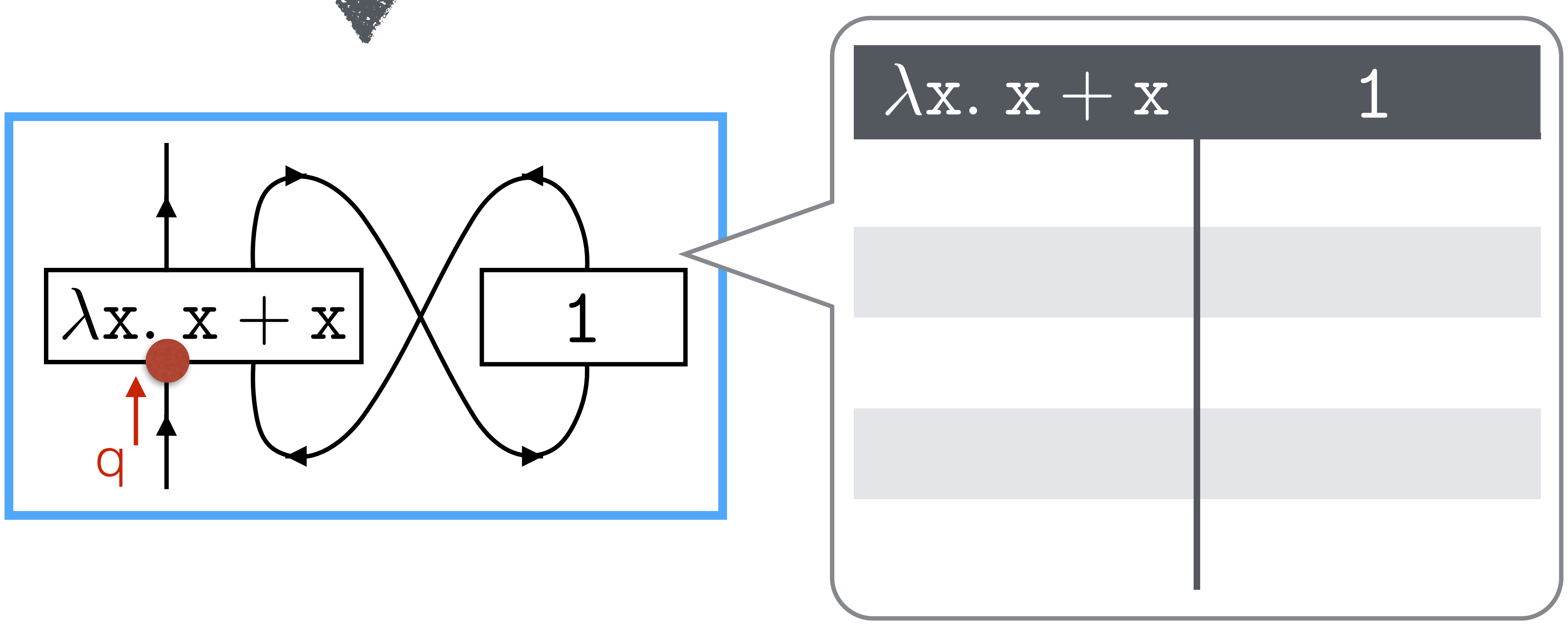
$$(\lambda x.\ x + x)\ 1$$



Geometry of Interaction (GoI)

- token machine semantics

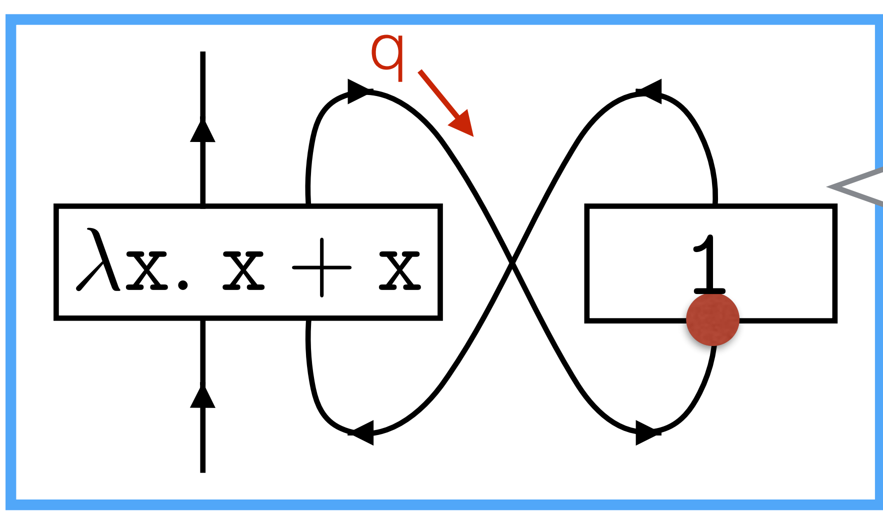
$$(\lambda x.\ x + x)\ 1$$



Geometry of Interaction (GoI)

- token machine semantics

$$(\lambda x.\ x + x)\ 1$$

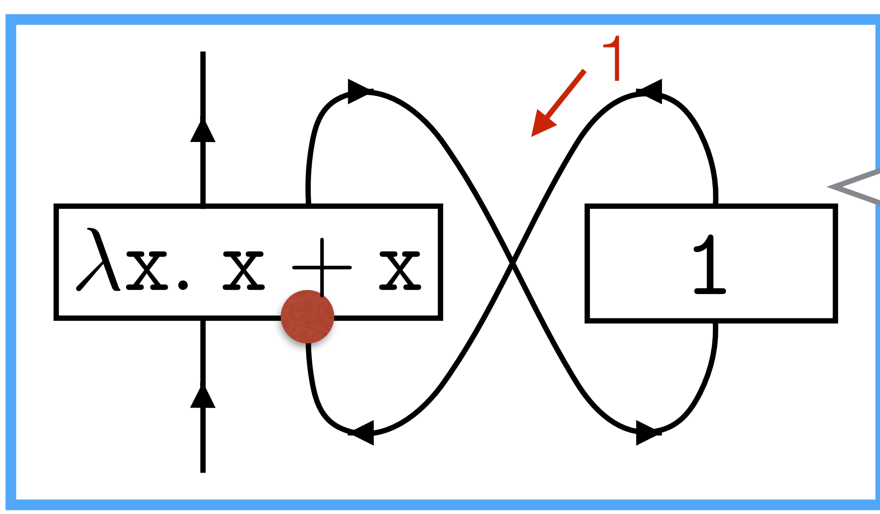


$\lambda x.\ x + x$	1
ask (left) x	

Geometry of Interaction (GoI)

- token machine semantics

$$(\lambda x.\ x + x)\ 1$$

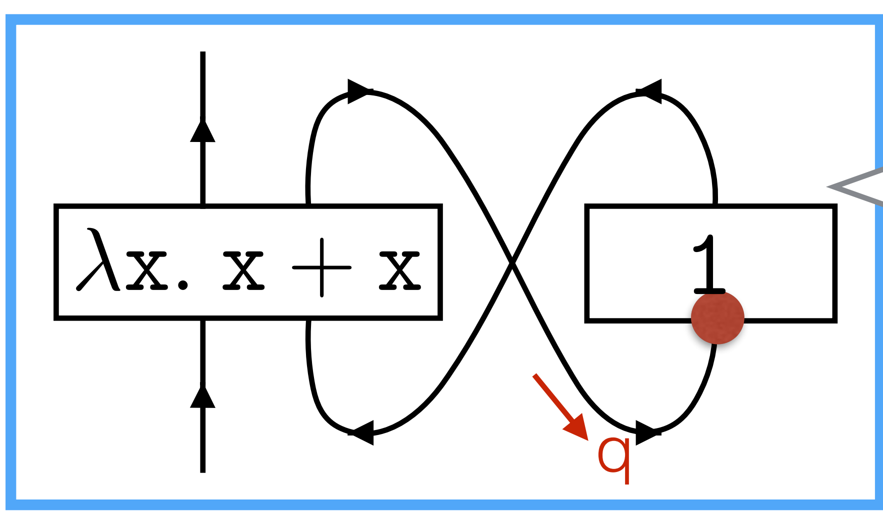


$\lambda x.\ x + x$	1
ask (left) x	
	answer 1

Geometry of Interaction (GoI)

- token machine semantics

$(\lambda x.\ x + x)\ 1$

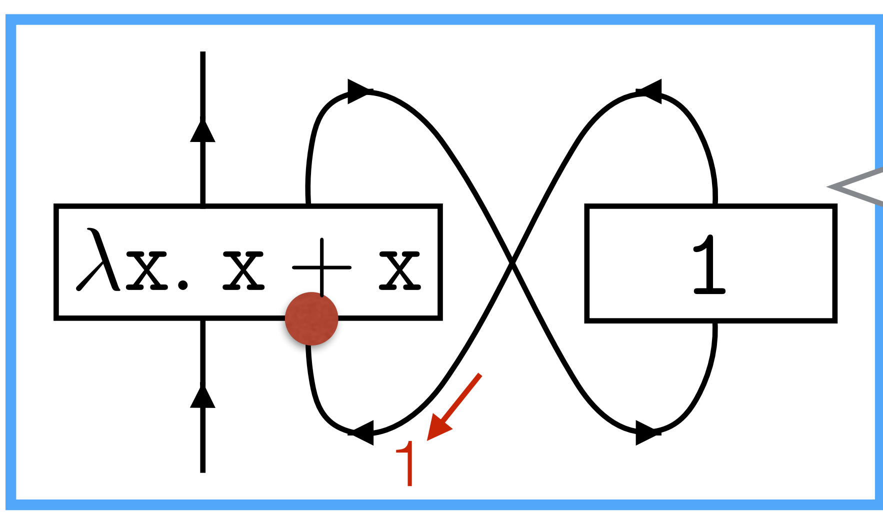


$\lambda x.\ x + x$	1
ask (left) x	
	answer 1
ask (right) x	

Geometry of Interaction (GoI)

- token machine semantics

$$(\lambda x.\ x + x)\ 1$$

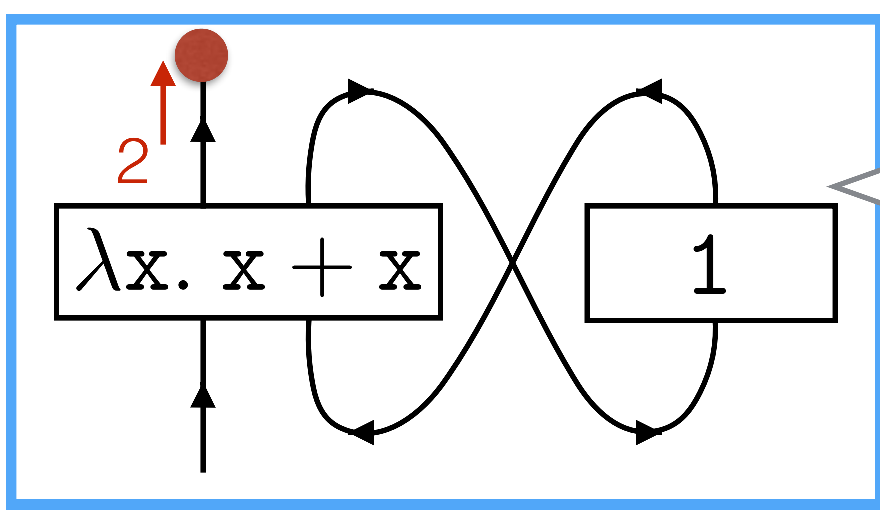


$\lambda x.\ x + x$	1
ask (left) x	
	answer 1
ask (right) x	
	answer 1

Geometry of Interaction (GoI)

- token machine semantics

$$(\lambda x.\ x + x)\ 1$$



$\lambda x.\ x + x$	1
ask (left) x	
	answer 1
ask (right) x	
	answer 1
answer 2	

Memoryful GoI — Input

effectful terms

↓

transducers

CBV λ-terms with <u>algebraic effects</u>

algebraic operations [Plotkin, Power '01]

- nondeterministic choice $M \sqcup N$
- probabilistic choice $M \sqcup_p N$
- actions on global state

$$\mathrm{lookup}_l(v\colon \mathtt{Val}.\ M) \qquad \mathrm{update}_{l,v}(M)$$

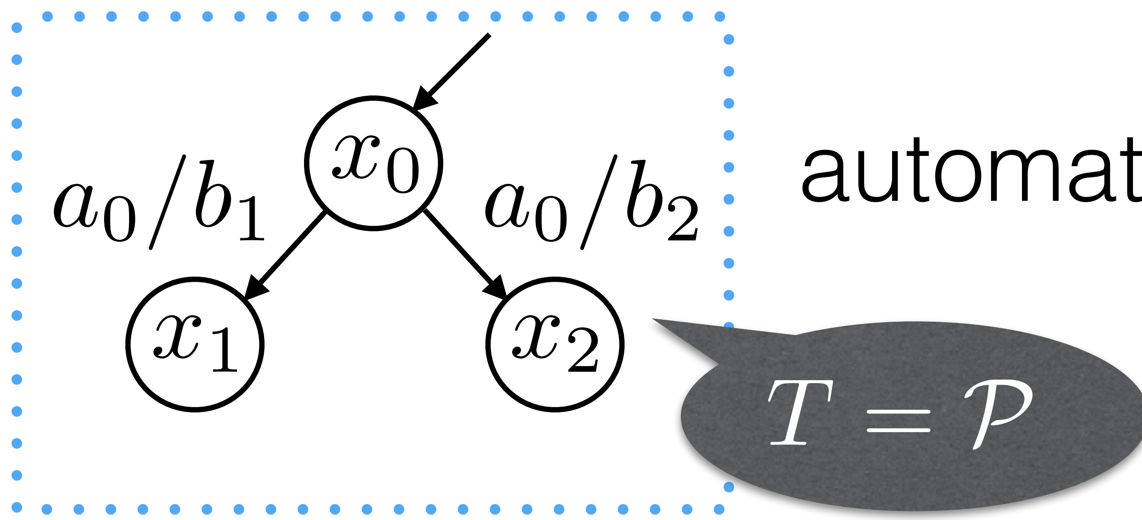
Memoryful GoI — Output

effectful terms

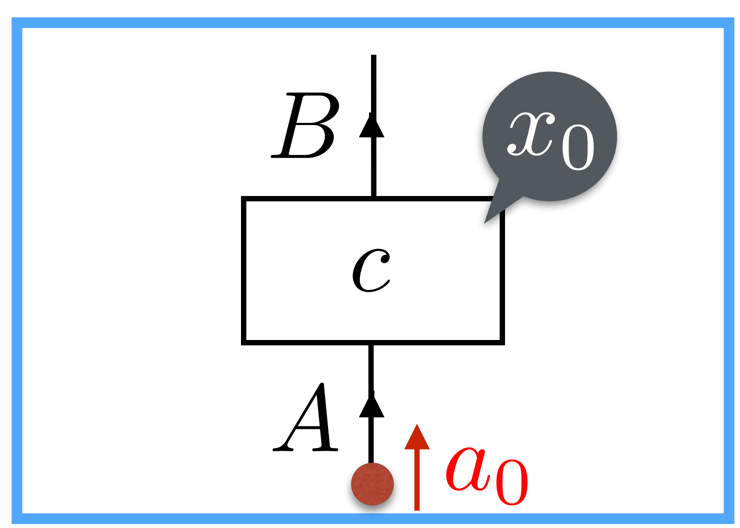
transducers

stream transducers (Mealy machines)

$$(X,\ c\colon X \times A \to T(X \times B),\ x_0 \in X)\colon A \rightarrowtriangle B$$



automaton style



string diagram style

Memoryful GoI — Output

effectful terms

↓

transducers

stream transducers (Mealy machines)

$$(X,\ c\colon X \times A \to T(X \times B),\ x_0 \in X)\colon A \rightharpoonup B$$

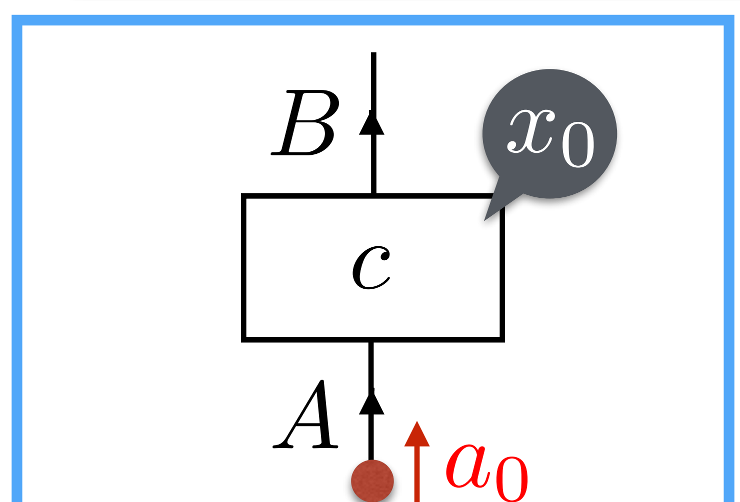
automaton style

$\mathbf{Res}(T)$

objects: sets

arrows: transducers modulo behavioral equivalence

$$[(X,\ c\colon X \times A \to T(X \times B),\ x_0 \in X)]_{\simeq}\colon A \rightharpoonup B$$



string diagram style

Memoryful GoI — Output

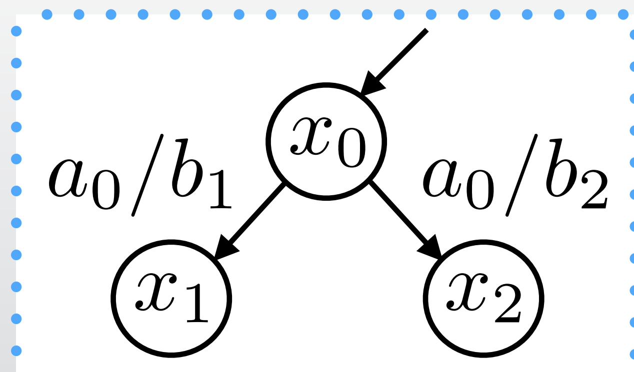
effectful terms

transducers

stream transducers (Mealy machines)

$$(X,\ c\colon X \times A \to T(X \times B),\ x_0 \in X)\colon A \rightharpoonup B$$

$T = \mathcal{P}$ $\quad (x_0, a_0) \longmapsto \{(x_1, b_1), (x_2, b_2)\}$

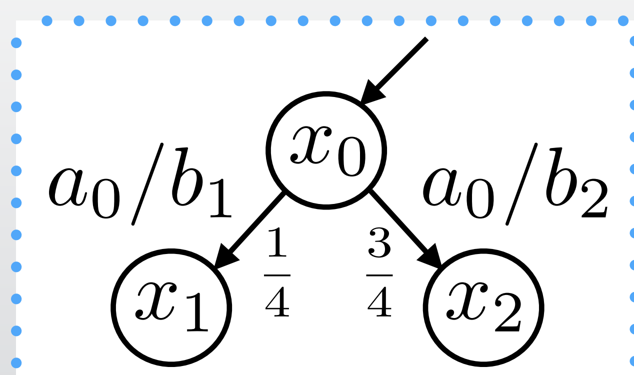


nondeterministic computation

$T = \mathcal{S} = (1 + (-) \times S)^S$

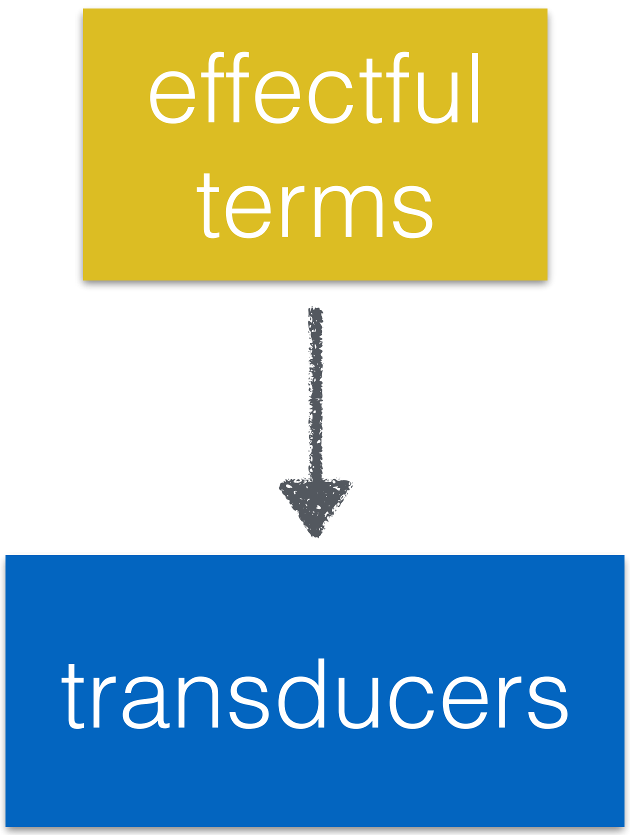
computation with global state

$T = \mathcal{D}$ $\quad (x_0, a_0) \longmapsto \begin{bmatrix} (x_1, b_1) \mapsto 1/4, \\ (x_2, b_2) \mapsto 3/4 \end{bmatrix}$



probabilistic computation

Memoryful GoI — Translation



- idea: resumptions + categorical GoI

[Abramsky, Haghverdi, Scott '02]

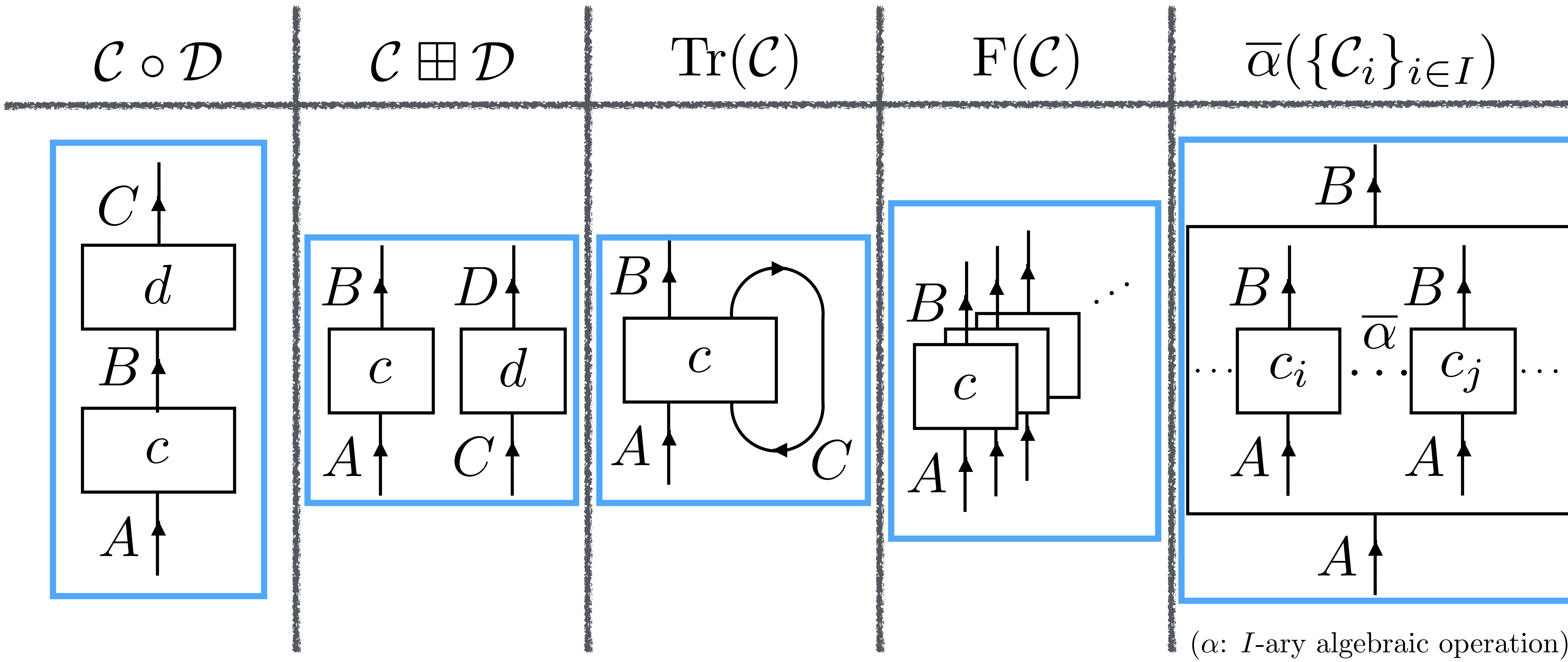
- use **coalgebraic component calculus**

[Barbosa '03] [Hasuo, Jacobs '11]

- **composition operations for software components**
- **(many-sorted) process calculus**

Memoryful GoI — Translation

Def. (component calculus)



Memoryful GoI — Translation

Def. (component calculus)

sequential composition

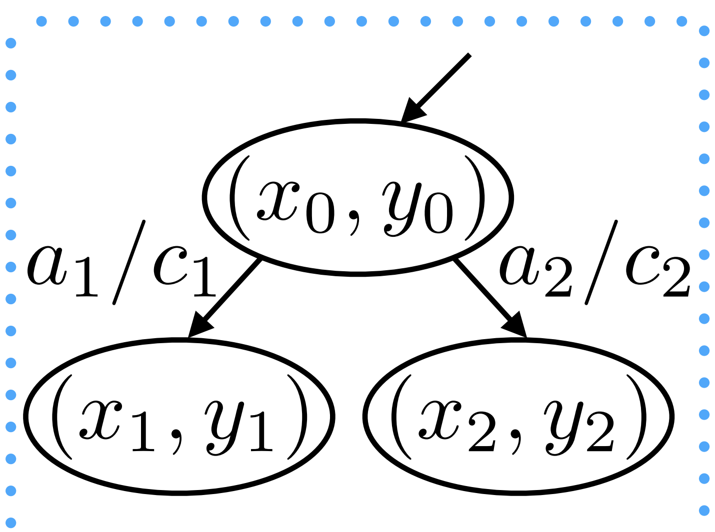
$\mathcal{C} \circ \mathcal{D}$

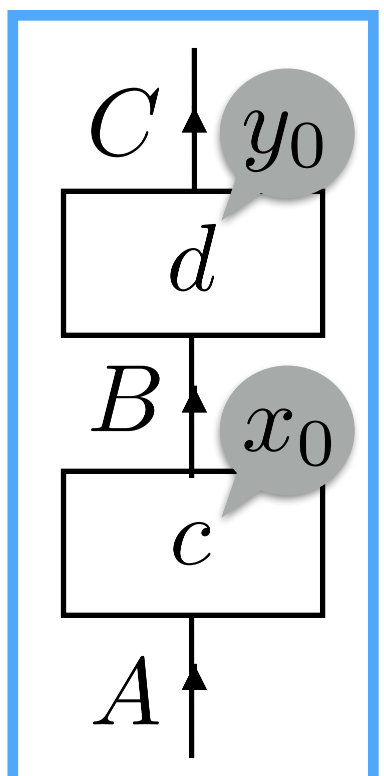
parallel composition

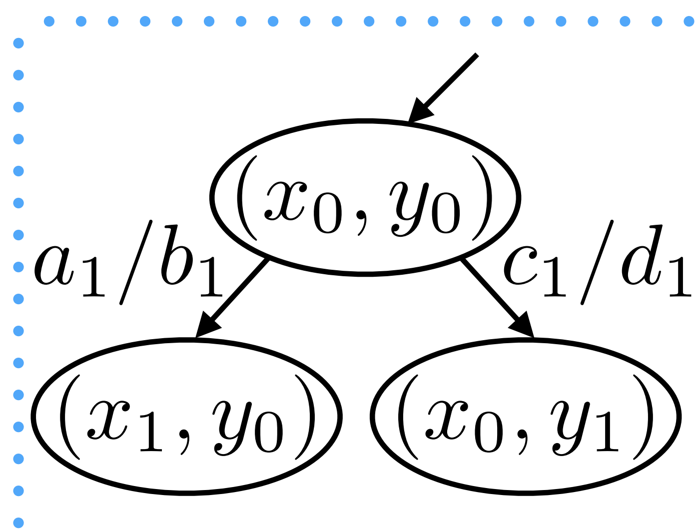
$\mathcal{C} \boxplus \mathcal{D}$

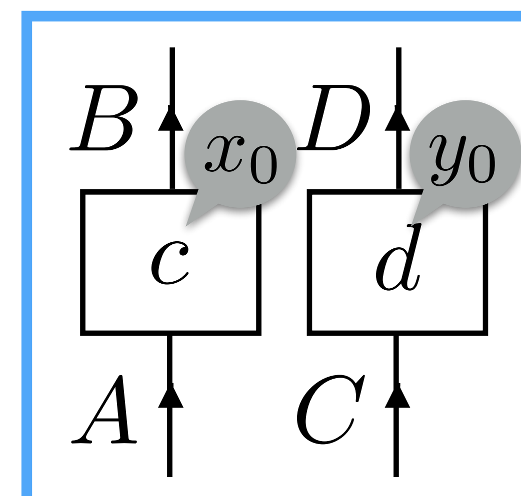
$$\begin{pmatrix} Y, \\ Y \times B \overset{d}{\to} T(Y \times C), \\ y_0 \in Y \end{pmatrix} \circ \begin{pmatrix} X, \\ X \times A \overset{c}{\to} T(X \times B), \\ x_0 \in X \end{pmatrix} = \begin{pmatrix} X \times Y, \\ \cdots \\ (x_0, y_0) \in X \times Y \end{pmatrix}$$

$$\begin{pmatrix} X, \\ X \times A \overset{c}{\to} T(X \times B), \\ x_0 \in X \end{pmatrix} \boxplus \begin{pmatrix} Y, \\ Y \times C \overset{d}{\to} T(Y \times D), \\ y_0 \in Y \end{pmatrix} = \begin{pmatrix} X \times Y, \\ \cdots \\ (x_0, y_0) \in X \times Y \end{pmatrix}$$







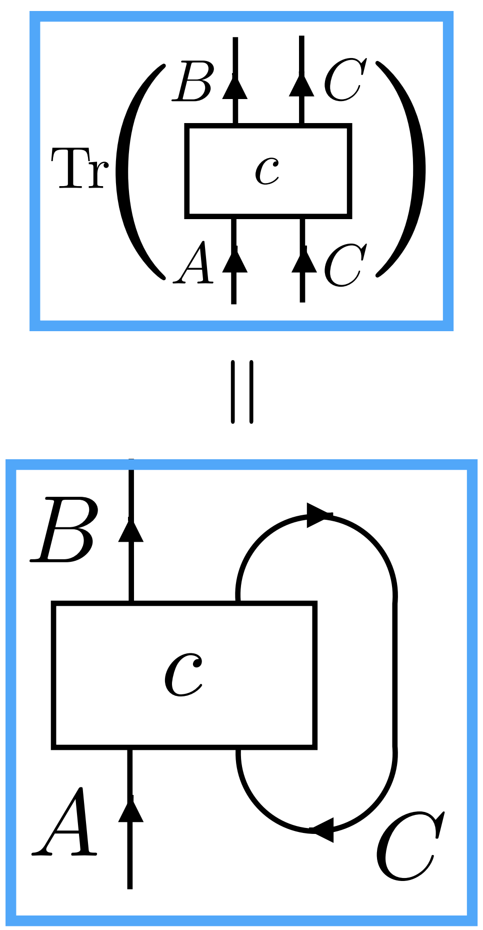


Memoryful GoI — Translation

Def. (component calculus)

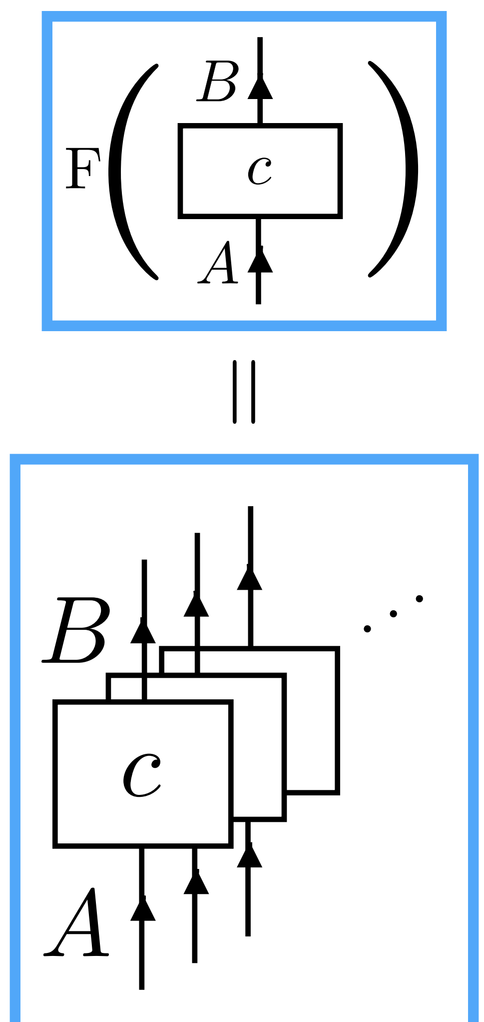
application

$\mathrm{Tr}(\mathcal{C})$



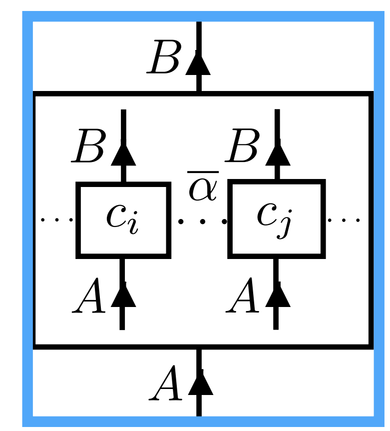
!-modality

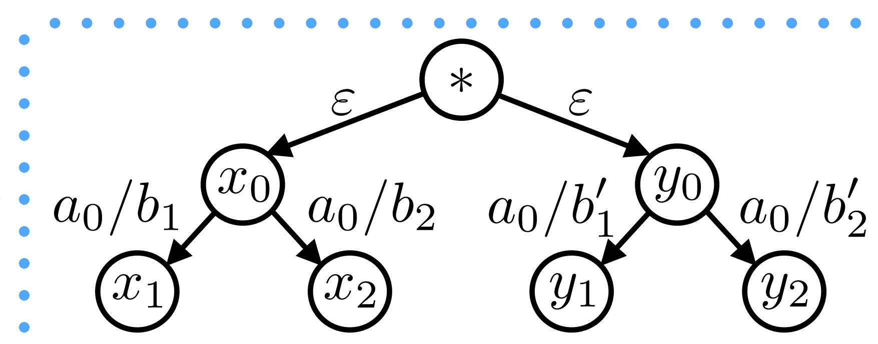
$\mathrm{F}(\mathcal{C})$



algebraic effect

$\overline{\alpha}(\{\mathcal{C}_i\}_{i\in I})$

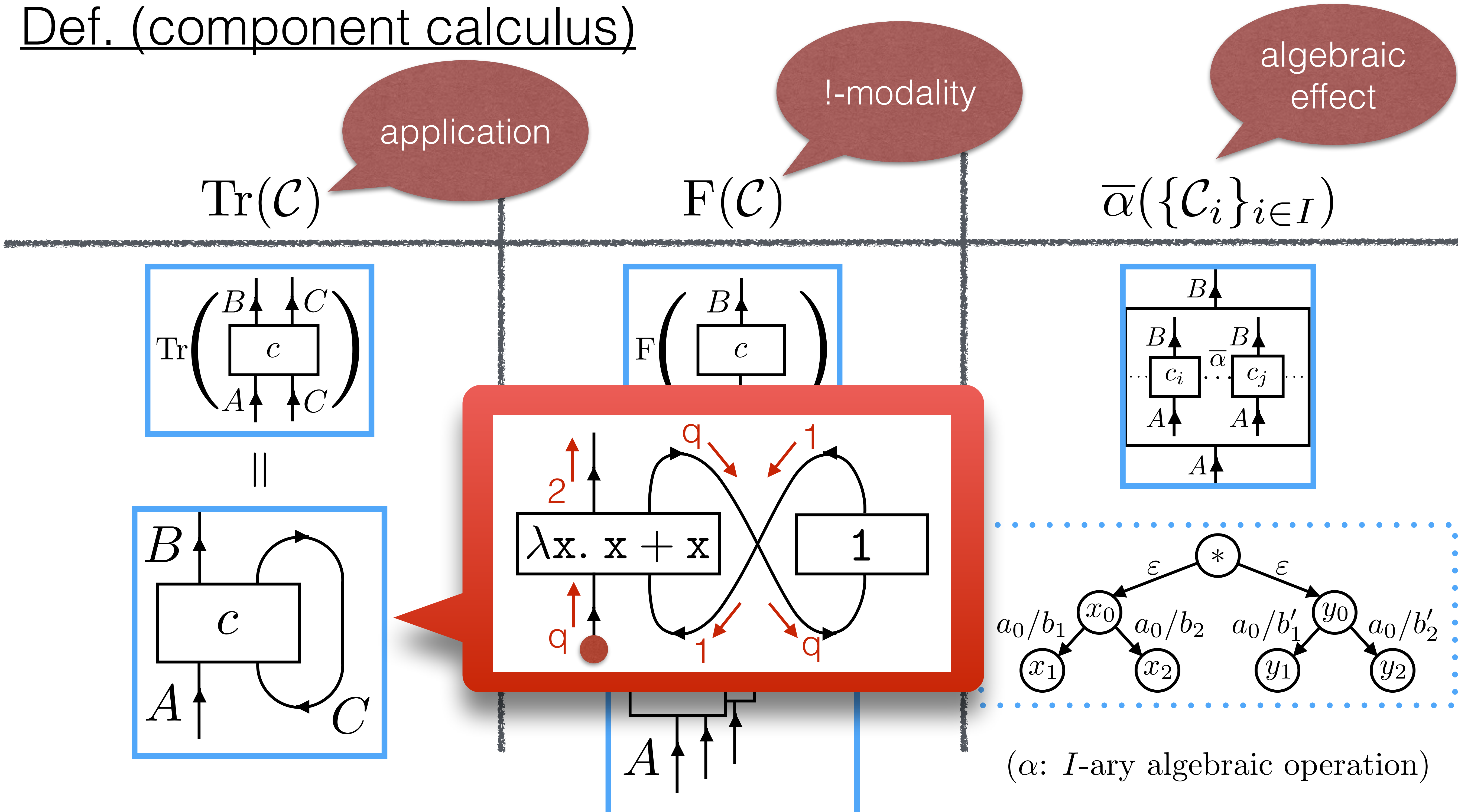




(α: I-ary algebraic operation)

Memoryful GoI — Translation

Def. (component calculus)



Memoryful GoI — Translation

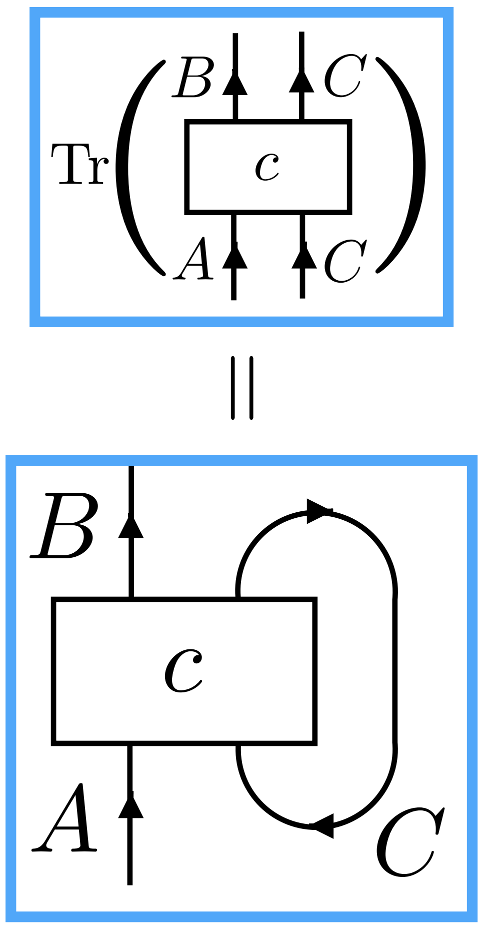
<u>Def. (component calculus)</u>

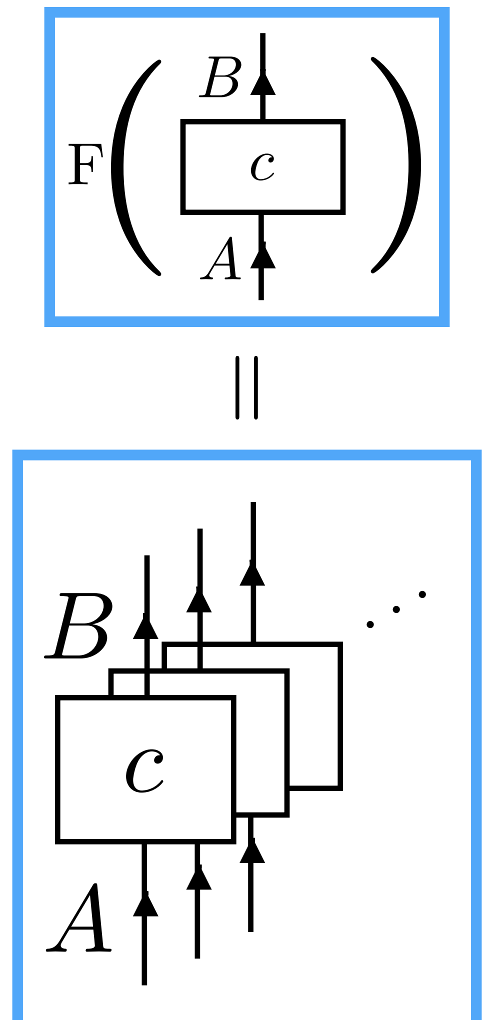
application

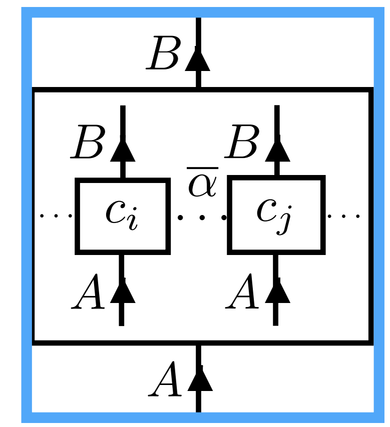
!-modality

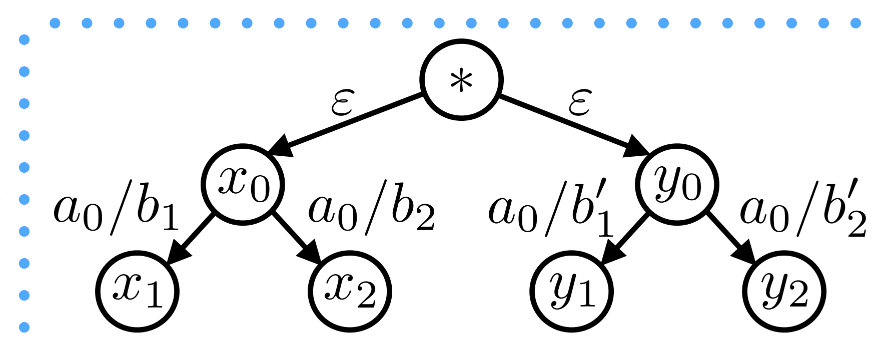
algebraic effect

$\mathrm{Tr}(\mathcal{C})$	$\mathrm{F}(\mathcal{C})$	$\overline{\alpha}(\{\mathcal{C}_i\}_{i\in I})$









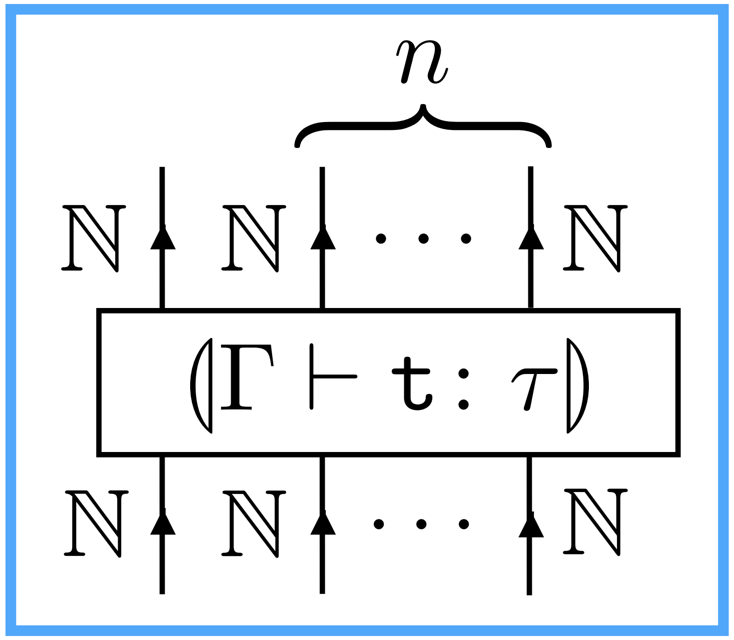
(α: I-ary algebraic operation)

Memoryful GoI — Translation

<u>Def. (interpretation $(\!|\Gamma \vdash \mathtt{t} \colon \tau|\!)$)</u>

For a type judgement $(\!|\Gamma \vdash \mathtt{t} \colon \tau|\!)$ $(\Gamma = x_1 \colon \tau_1, \ldots, x_n \colon \tau_n)$,

we inductively define

$(\!|\Gamma \vdash \mathtt{t} \colon \tau|\!) =$ 

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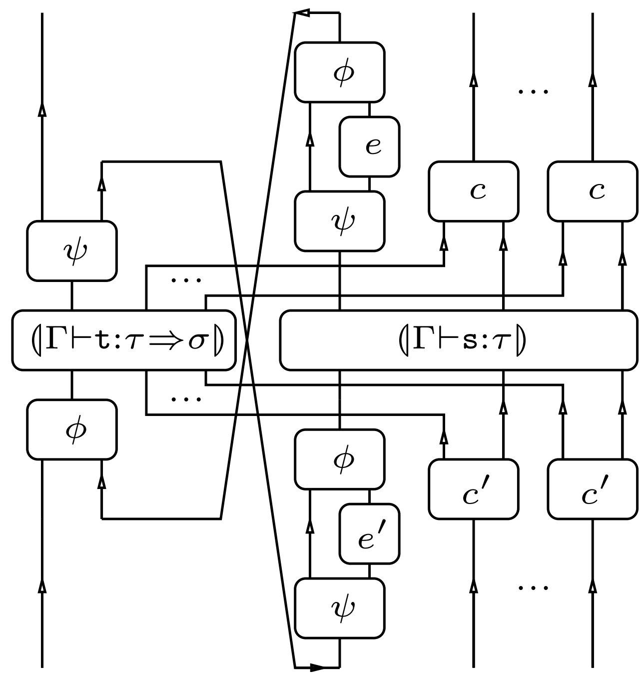
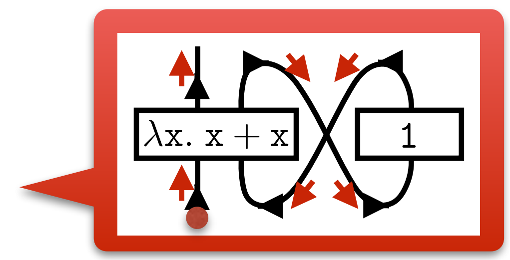
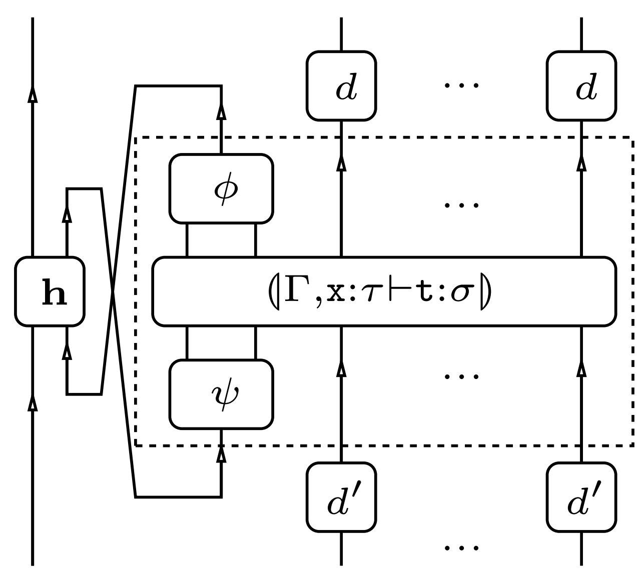
Memoryful GoI — Translation

Def. (interpretation $(\!|\Gamma \vdash \mathtt{t} : \tau|\!)$)

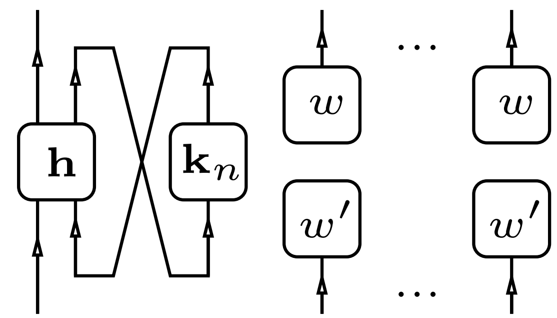
$(\!|\Gamma \vdash \mathtt{t\ s} : \sigma|\!) =$ 

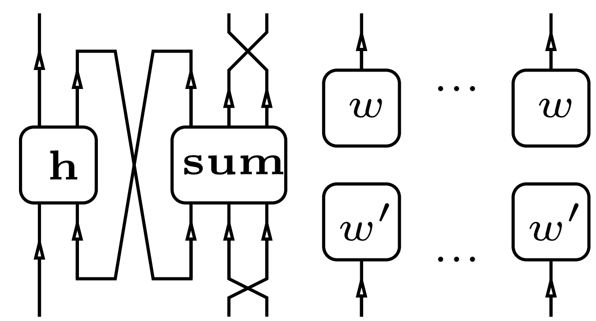


$(\!|\Gamma \vdash \lambda\mathtt{x} : \tau.\,\mathtt{t} : \tau \Rightarrow \sigma|\!) =$ 

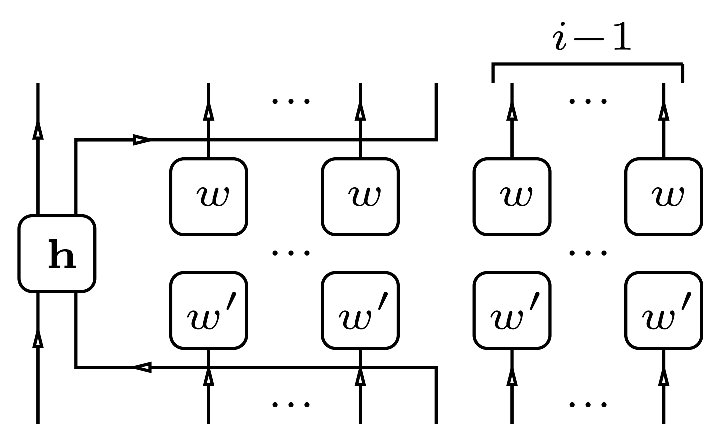
Memoryful GoI — Translation

Def. (interpretation $(\!|\Gamma \vdash \mathtt{t} : \tau|\!)$)

$(\!|\Gamma \vdash n : \mathtt{nat}|\!) =$ 

$(\!|\Gamma, \mathtt{x} : \mathtt{nat}, \mathtt{y} : \mathtt{nat} \vdash \mathtt{x} + \mathtt{y} : \mathtt{nat}|\!) =$ 

$$(\!|\Gamma \vdash \mathtt{t} + \mathtt{s} : \mathtt{nat}|\!) = (\!|\Gamma \vdash (\lambda \mathtt{xy} : \mathtt{nat}.\, \mathtt{x} + \mathtt{y})\, \mathtt{t}\, \mathtt{s} : \mathtt{nat}|\!)$$

$(\!|\mathtt{x}_1 : \tau_1, \cdots, \mathtt{x}_n : \tau_n \vdash \mathtt{x}_i : \tau_i|\!) =$ 

Memoryful GoI — Translation

Thm. (soundness)

For closed terms M and N of type τ,

- $\vdash M = N : \tau$ implies $([(\!|M|\!)]_{\simeq}, [(\!|N|\!)]_{\simeq}) \in \Phi[\![\tau]\!]$
- $\vdash M = N : \mathtt{nat}$ implies $(\!|M|\!) \simeq (\!|N|\!)$.

- Moggi's equations for computational lambda-calculus
- equations for algebraic operations

$$M \sqcup M = M$$

$$E[\mathrm{opr}(M_1, \ldots, M_n)] = \mathrm{opr}(E[M_1], \ldots, E[M_n])$$

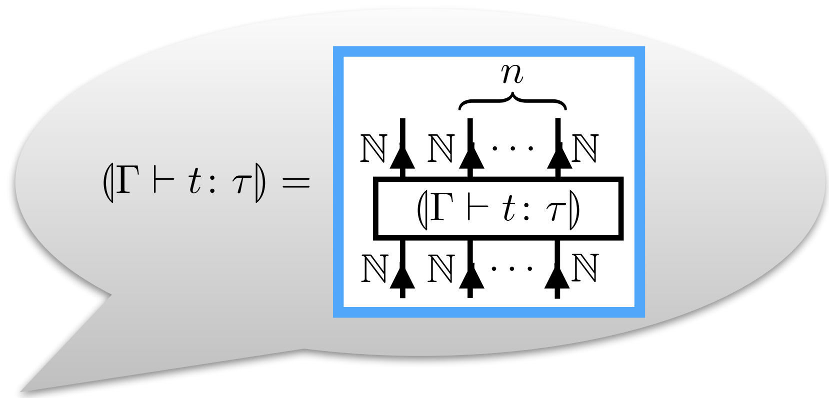
$$(\lambda x.\ M)\ (N_1 \sqcup N_2) = (\lambda x.\ M)\ N_1\ \sqcup\ (\lambda x.\ M)\ N_2$$

behavioral equivalence

Memoryful GoI [Hoshino, —, Hasuo '14]

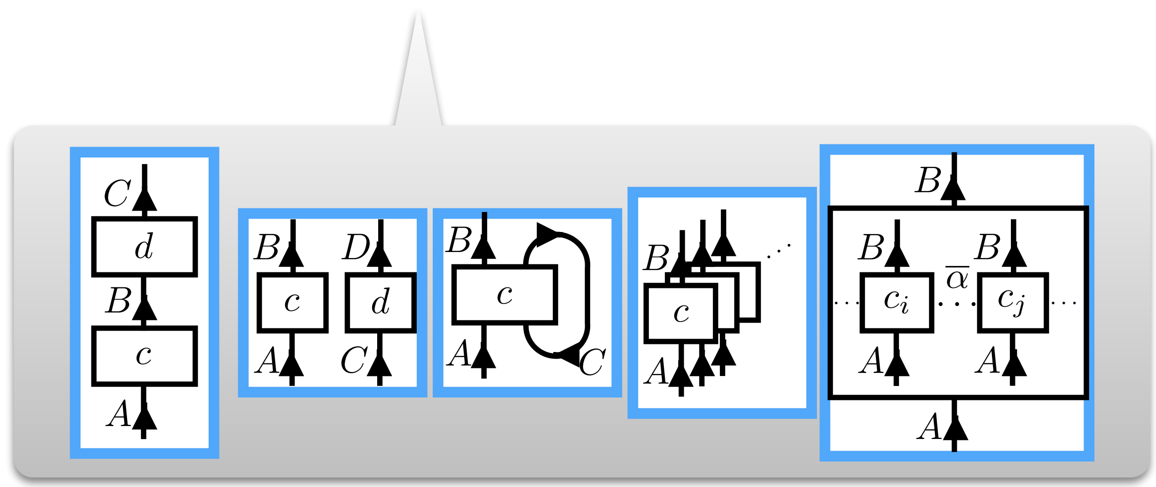
effectful terms

$(\lambda \mathtt{x} : \mathtt{nat}.\, \mathtt{x} + \mathtt{x})\ (3 \sqcup 5) \quad : \mathtt{nat}$

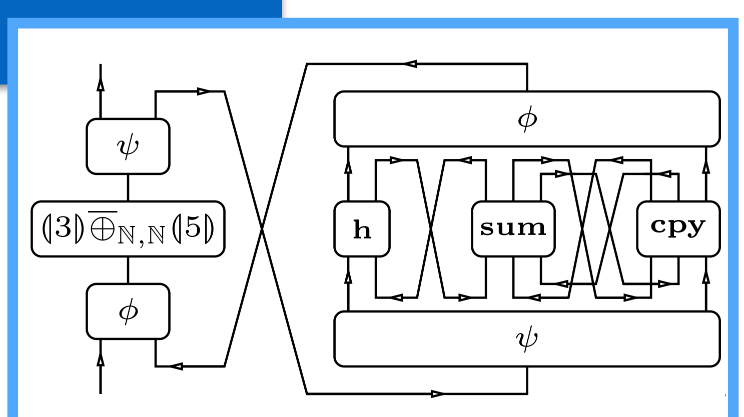


translation

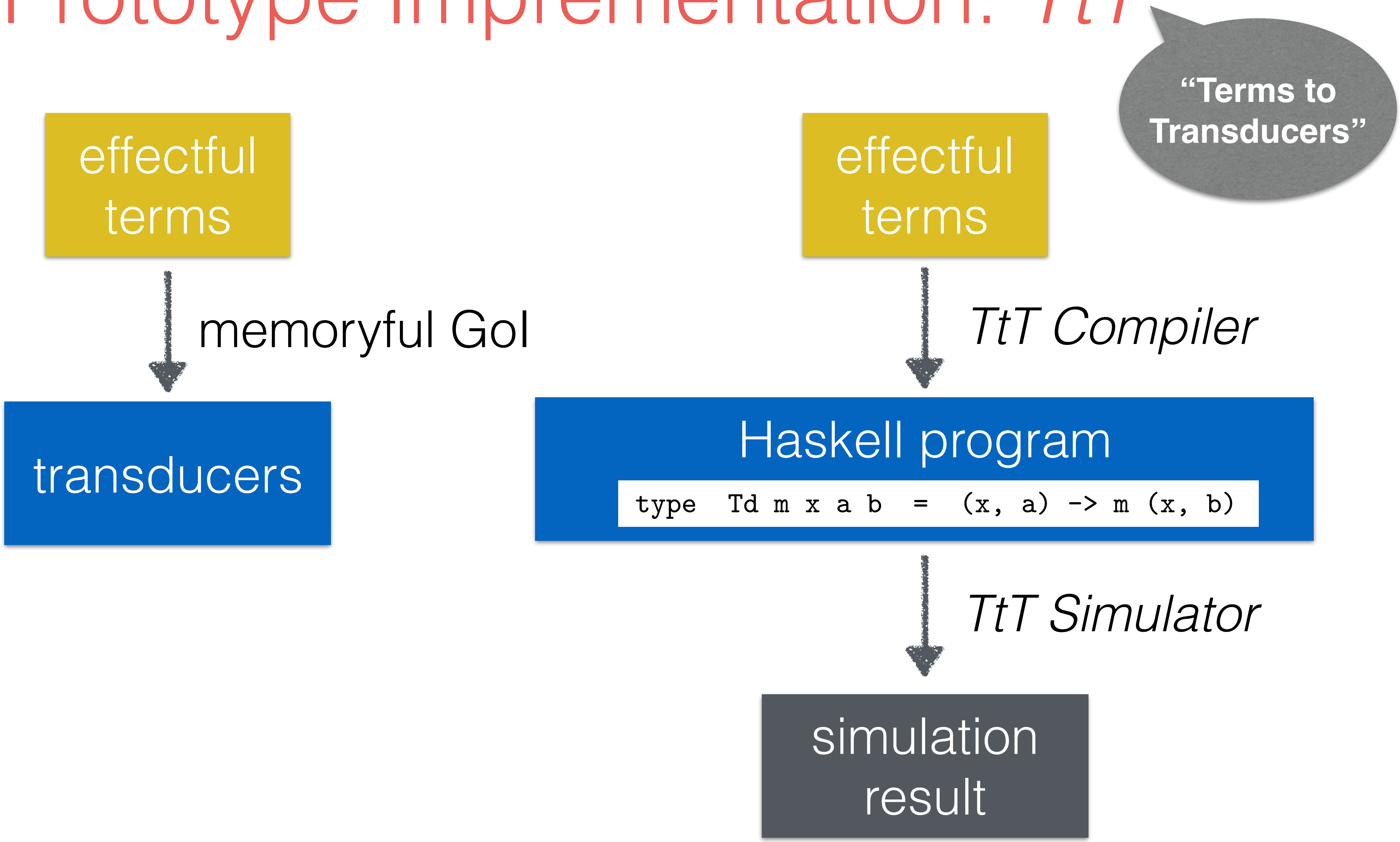
- based on Geometry of Interaction
- via coalgebraic component calculus



transducers



Prototype Imprementation: *TtT*



Prototype Imprementation: *TtT*

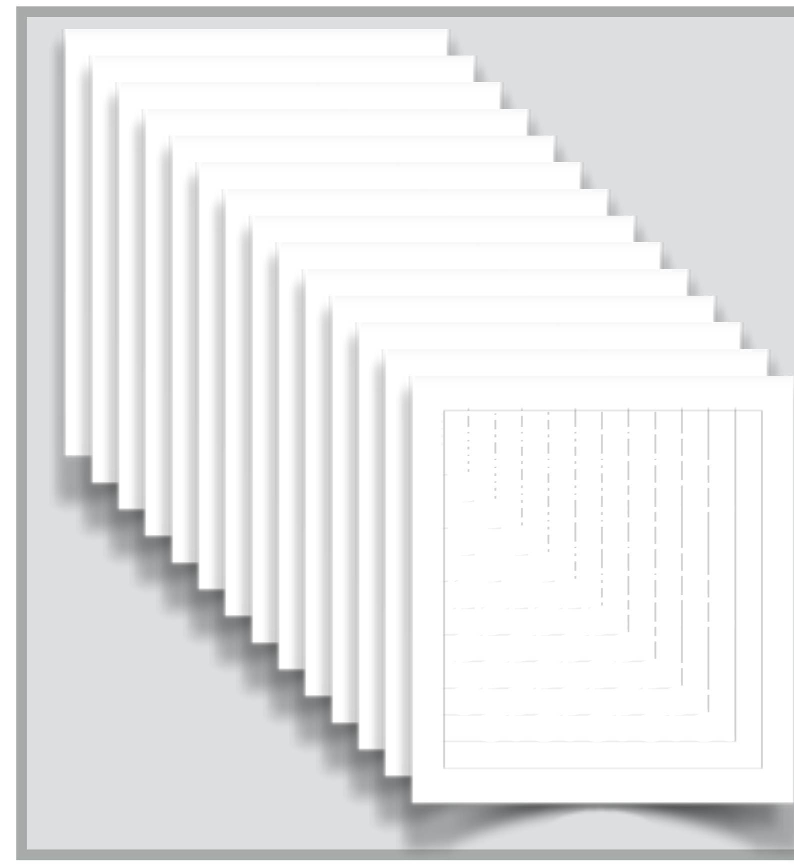
$3 \sqcup 5$

```
.    ---- dd<42,137> --->
|    Query:[ [3|_|5] @ Nothing ]
+-.    ---- dd<42,137> --->
| |    Query:[ [3]|_|5 @ * ]
| |      h; k_3; h
| |    [ [3]|_|5 @ * ]:Answer
| |    ---- dd<42,3> --->
| |  [ [3|_|5] @ Just (Left (*)) ]:Answer
| |  ---- dd<42,3> --->
| Result: 3 / State: Just (Left (*))
`-.    ---- dd<42,137> --->
  |    Query:[ 3|_|[5] @ * ]
  |      h; k_5; h
  |    [ 3|_|[5] @ * ]:Answer
  |    ---- dd<42,5> --->
  |  [ [3|_|5] @ Just (Right (*)) ]:Answer
  |  ---- dd<42,5> --->
  Result: 5 / State: Just (Right (*))
```

$(\lambda x.\, x)\ 1$

```
---- dd<42,137> --->
Query:[ [(\x.x) 1] @ ({_: *}, *) ]
  phi
  ---- gdd<42,137> --->
  Query:[ [\x.x] 1 @ {_: *} ]
    h
  [ [\x.x] 1 @ {_: *} ]:Answer
  ---- dgdd<42,137> --->
  psi; psi; phi
  ---- gdd<42,137> --->
  Query:[ (\x.x) [1] @ * ]
    h
  [ (\x.x) [1] @ * ]:Answer
  ---- dgdd<42,137> --->
  psi; e; phi; phi
  ---- dd<0,gdd<42,137>> --->
  Query:[ [\x.x] 1 @ {_: *} ]
    h; v
    0 {
      psi
      ---- dd<42,137> --->
      Query:[ (\x.[x]) 1 @ * ]
        h
      [ (\x.[x]) 1 @ * ]:Query x
      ---- <42,137> --->
      phi
    } 0
    u; h
  [ [\x.x] 1 @ {_: *} ]:Answer
  ---- dd<0,d<42,137>> --->
  psi; psi; e'; phi
  ---- dd<42,137> --->
  Query:[ (\x.x) [1] @ * ]
    h; k_1; h
  [ (\x.x) [1] @ * ]:Answer
  ---- dd<42,1> --->
  psi; e; phi; phi
  ---- dd<0,d<42,1>> --->
  Query:[ [\x.x] 1 @ {_: *} ]
    h; v
    0 {
      psi
      ---- <42,1> --->
      Answer x:[ (\x.[x]) 1 @ * ]
        h
      [ (\x.[x]) 1 @ * ]:Answer
      ---- dd<42,1> --->
      phi
    } 0
    u; h
  [ [\x.x] 1 @ {_: *} ]:Answer
  ---- dd<0,gdd<42,1>> --->
  psi; psi; e'; phi
  ---- dgdd<42,1> --->
  Query:[ (\x.x) [1] @ * ]
    h
  [ (\x.x) [1] @ * ]:Answer
  ---- gdd<42,1> --->
  psi; phi; phi
  ---- dgdd<42,1> --->
  Query:[ [\x.x] 1 @ {_: *} ]
    h
  [ [\x.x] 1 @ {_: *} ]:Answer
  ---- gdd<42,1> --->
  psi
[ [(\x.x) 1] @ ({_: *}, *) ]:Answer
---- dd<42,1> --->
Result: 1 / State: ({_: *}, *)
```

$(\lambda f.\, f\ 0 + f\ 1)\ (\lambda x.\, 3 \sqcup 5)$

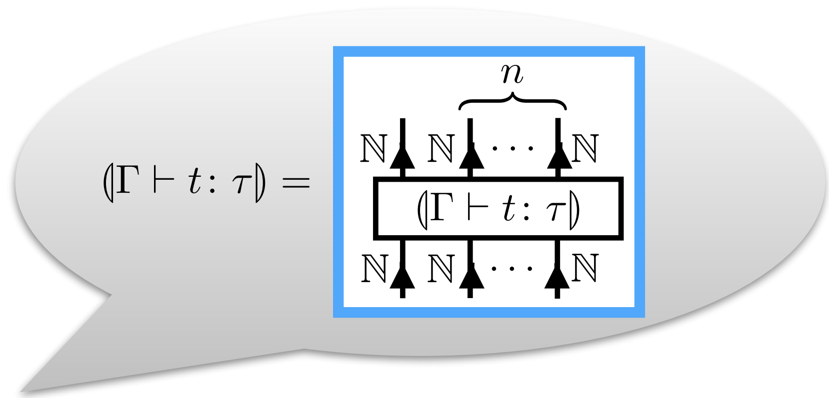


(4,526 lines)

Memoryful GoI [Hoshino, —, Hasuo ’14]

effectful terms

$(\lambda \mathtt{x} : \mathtt{nat}.\, \mathtt{x} + \mathtt{x})\ (3 \sqcup 5) \quad : \mathtt{nat}$



translation

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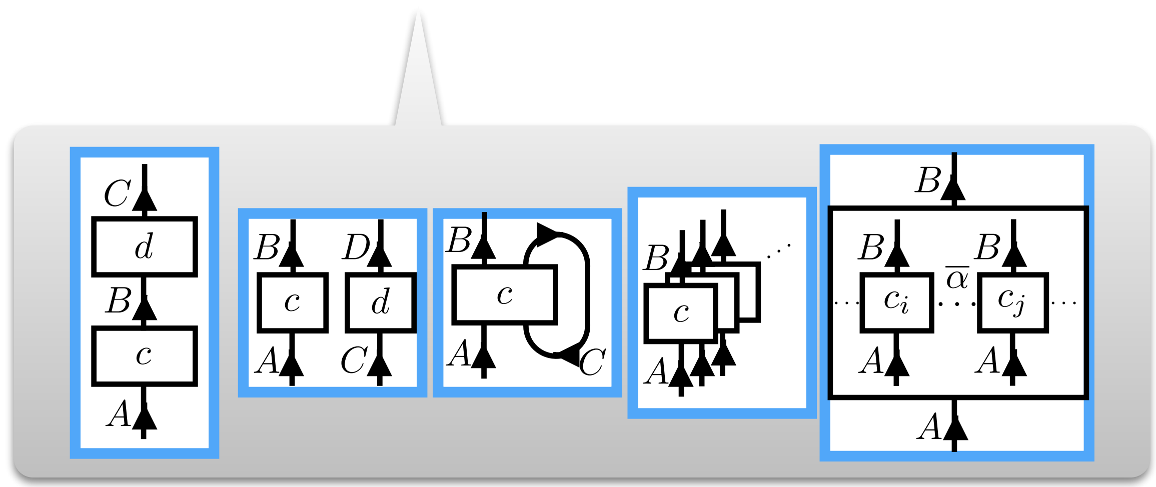
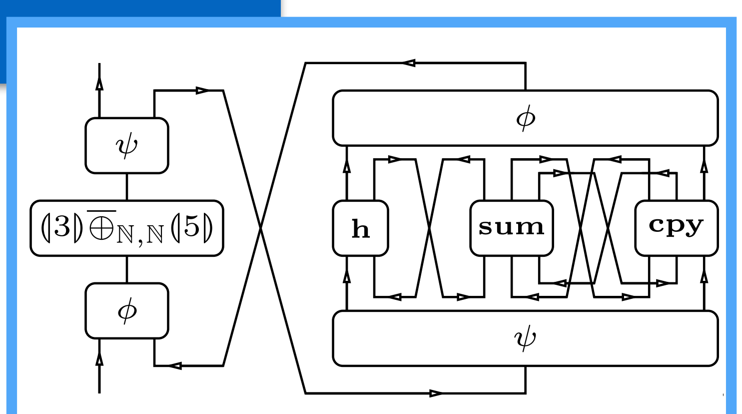


transducers



Memoryful GoI with recursion

effectful terms

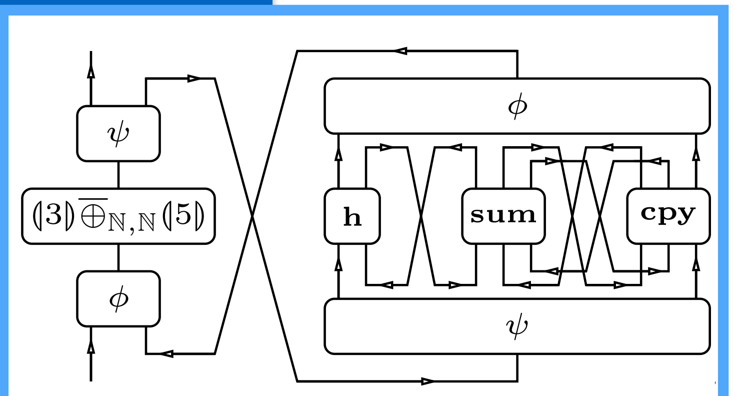
recursion

$(\lambda \mathtt{x} : \mathtt{nat}.\, \mathtt{x} + \mathtt{x})\,(3 \sqcup 5) \quad : \mathtt{nat}$

translation

- based on Geometry of Interaction
- via coalgebraic component calculus

transducers



Memoryful GoI with recursion

effectful terms

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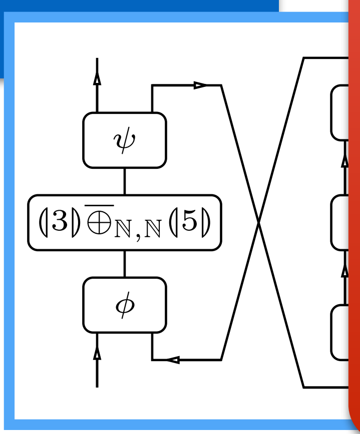

Memoryful GoI with recursion

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translation

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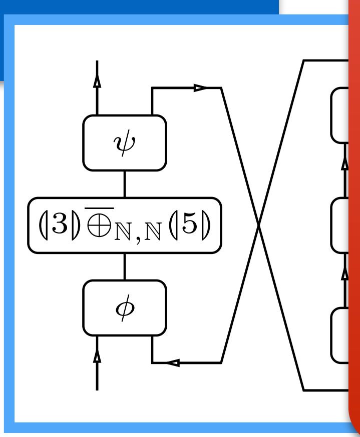
$$\mathrm{fix}(F) = F(F(F(\cdots)))$$

transducers



Memoryful GoI with recursion

effectful terms

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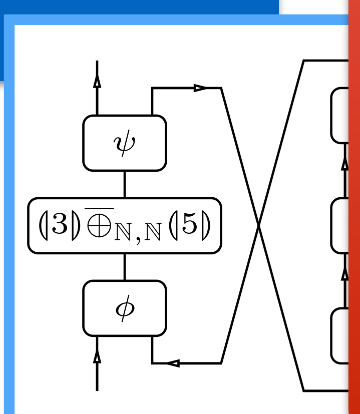

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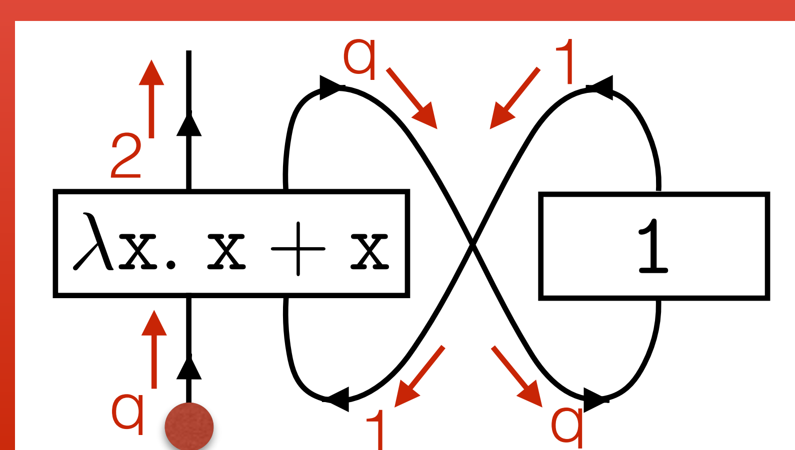

Memoryful GoI with recursion

effectful terms

recursion

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translation
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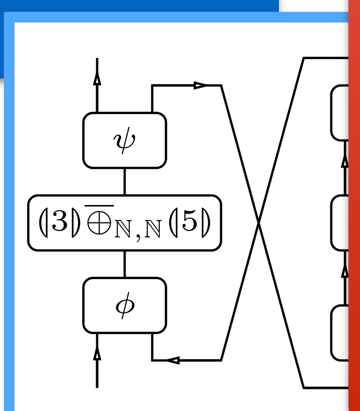
transducers



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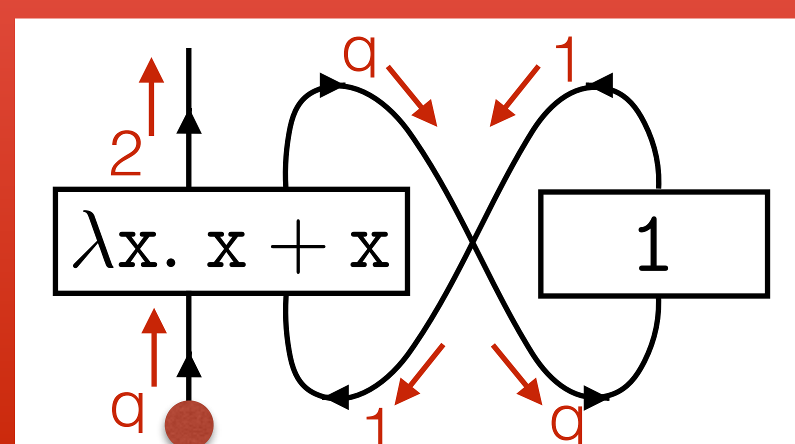
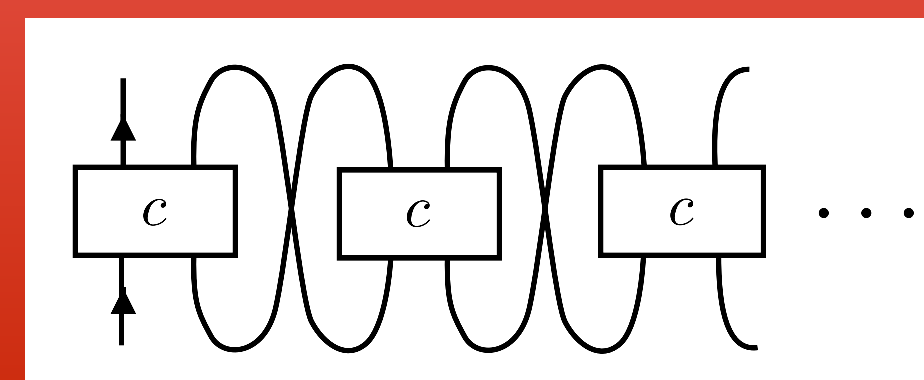




Memoryful GoI with recursion

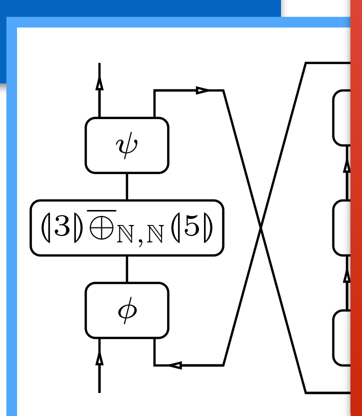
effectful terms

recursion

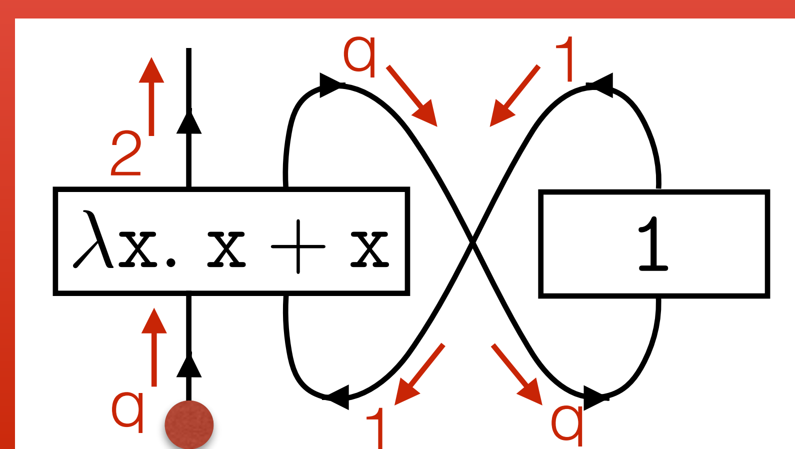
$(\lambda \mathtt{x} : \mathtt{nat}.\, \mathtt{x} + \mathtt{x})\, (3 \sqcup 5) \quad : \mathtt{nat}$

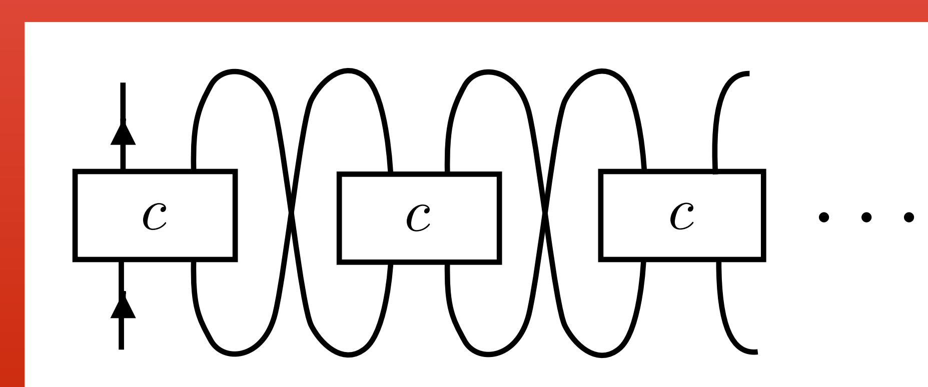
translation
- based on Geometry of Interaction
- via **extended** coalgebraic component calculus

transducers



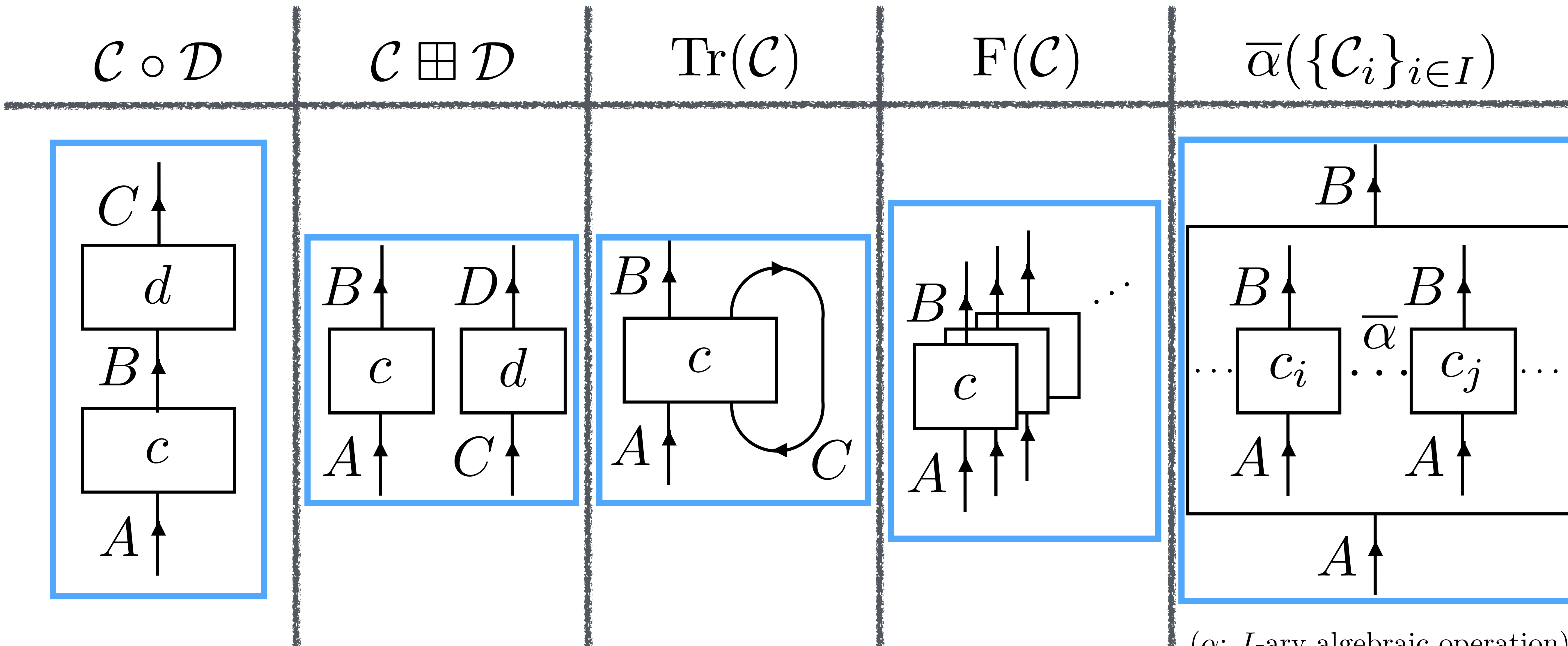
$$\mathrm{fix}(F) = F(F(F(\cdots)))$$





Memoryful GoI with recursion

Def. (component calculus)



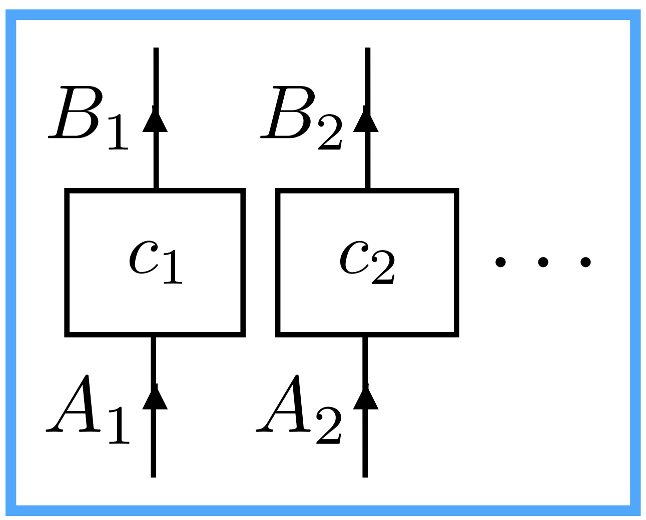
(α: I-ary algebraic operation)

Memoryful GoI with recursion

Def. (**extended** component calculus)

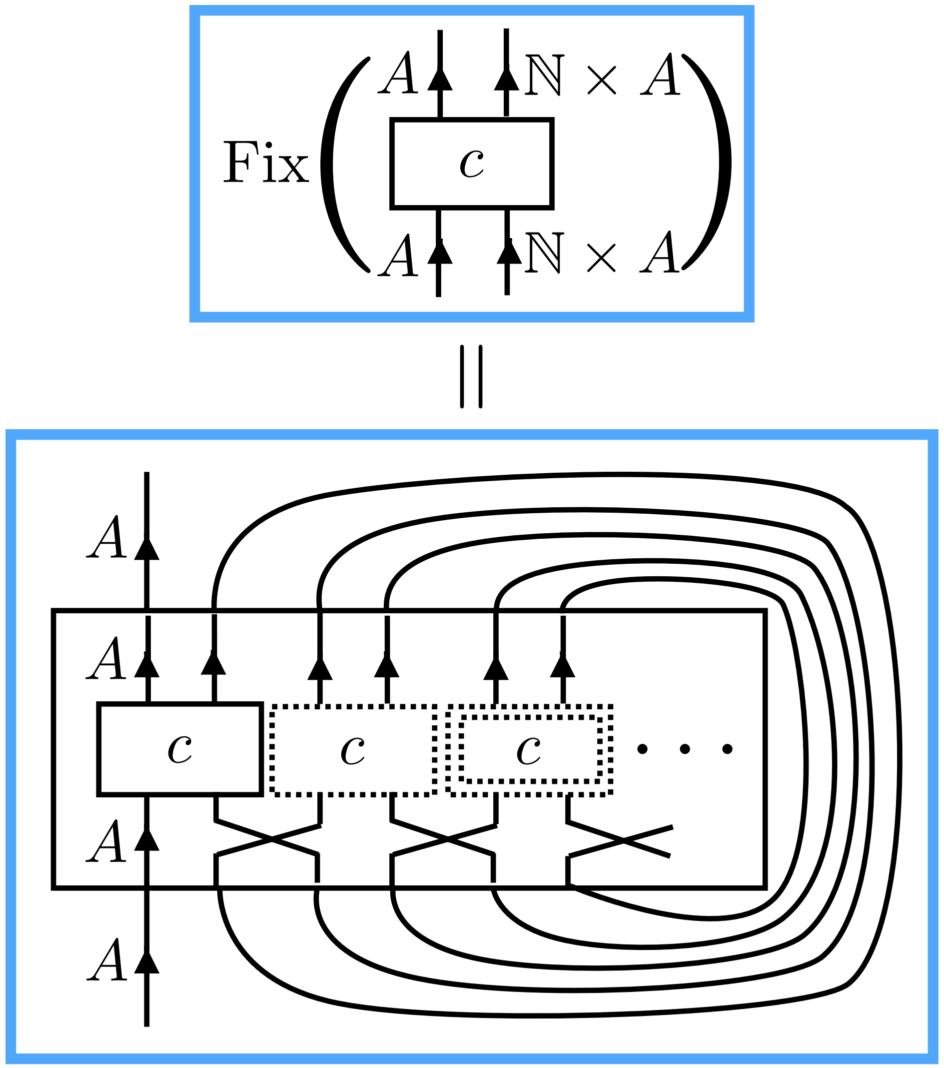
$\bigboxplus_{i \in I} \mathcal{C}_i$

countable parallel composition



$\mathrm{Fix}(\mathcal{C})$

fixpoint



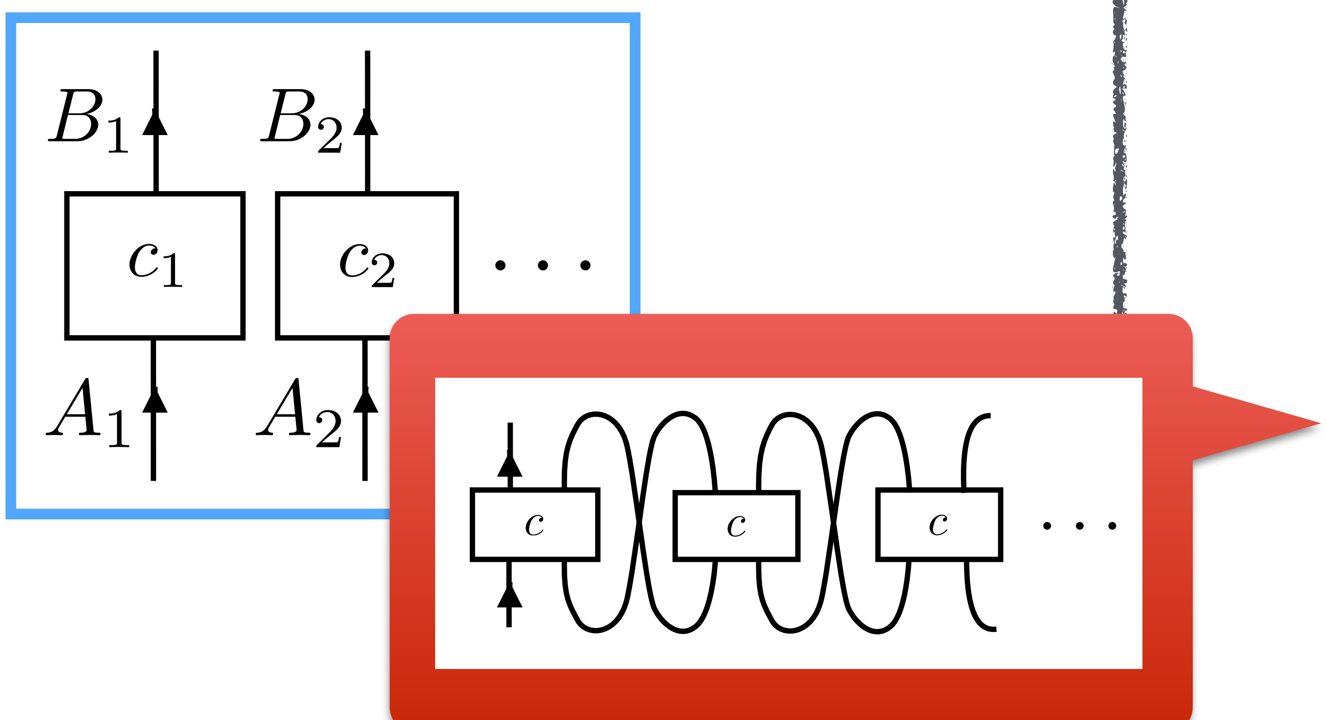
Muroya (U. Tokyo)

Memoryful GoI with recursion

Def. (**extended** component calculus)

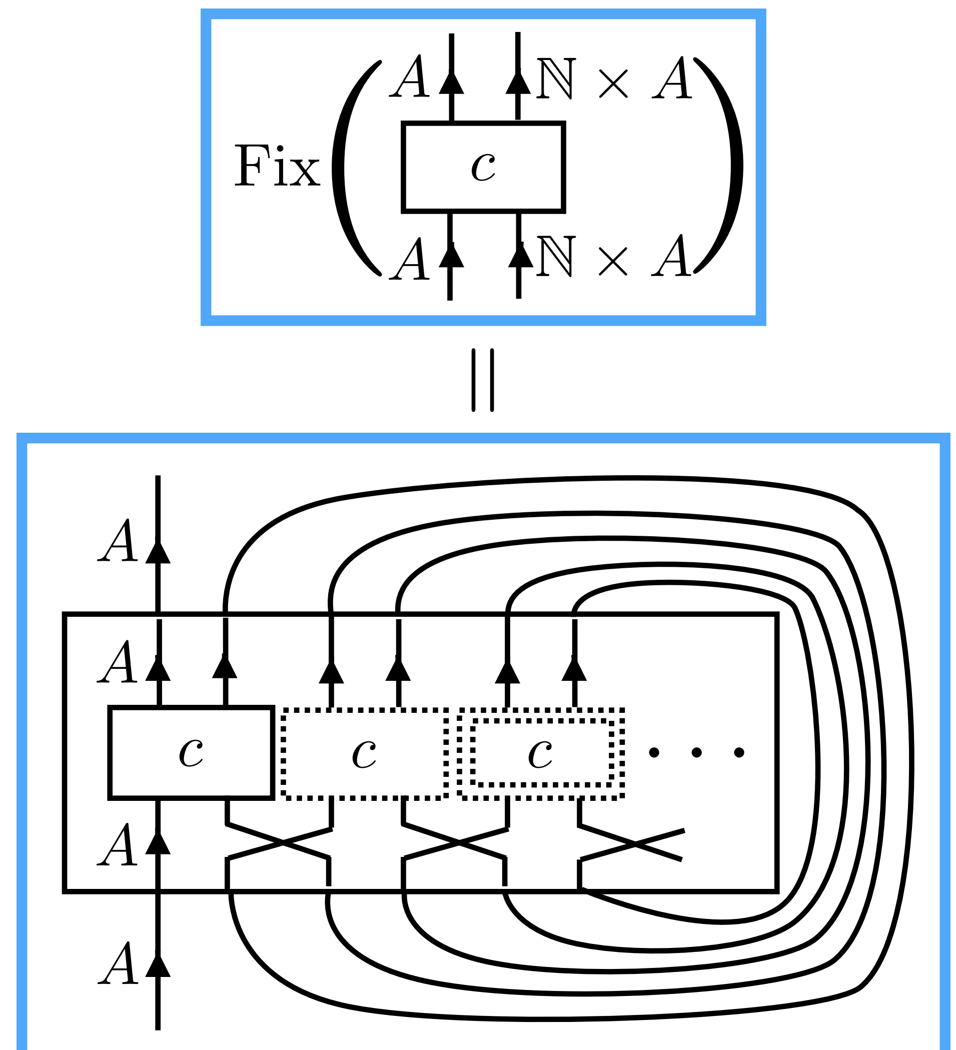
$\boxplus_{i \in I} \mathcal{C}_i$

countable parallel composition



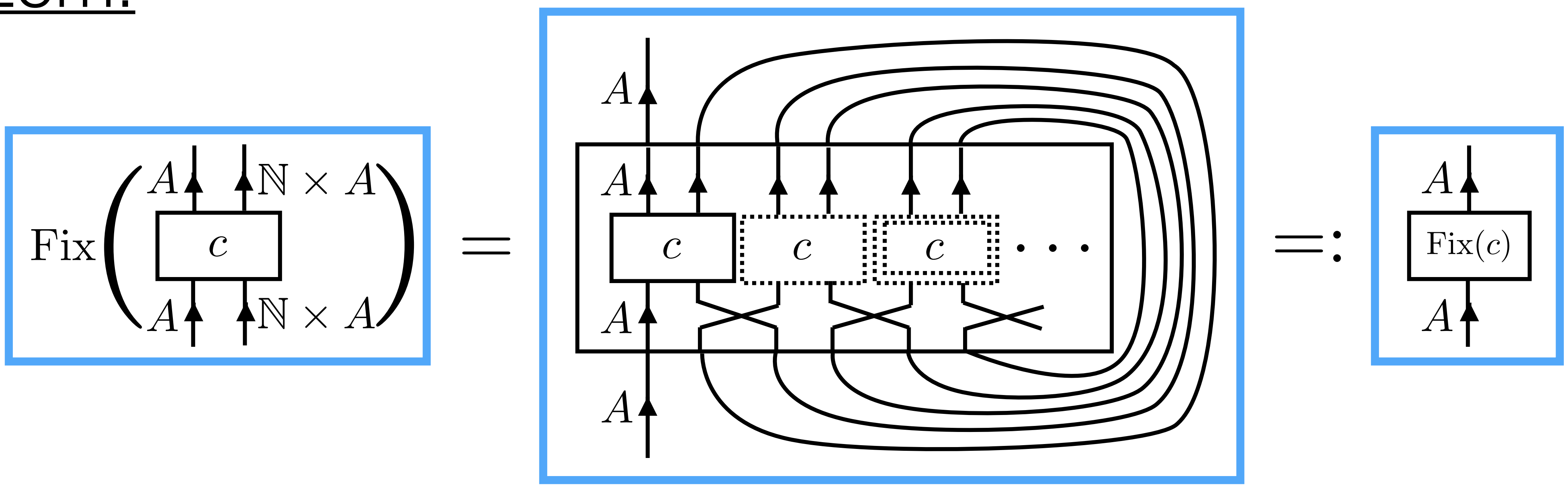
$\mathrm{Fix}(\mathcal{C})$

fixpoint

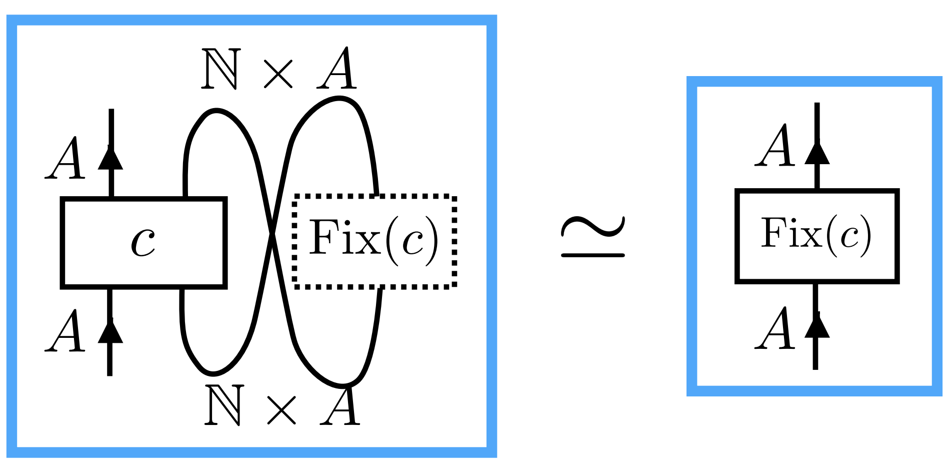


Memoryful GoI with recursion

Lem.



satisfies



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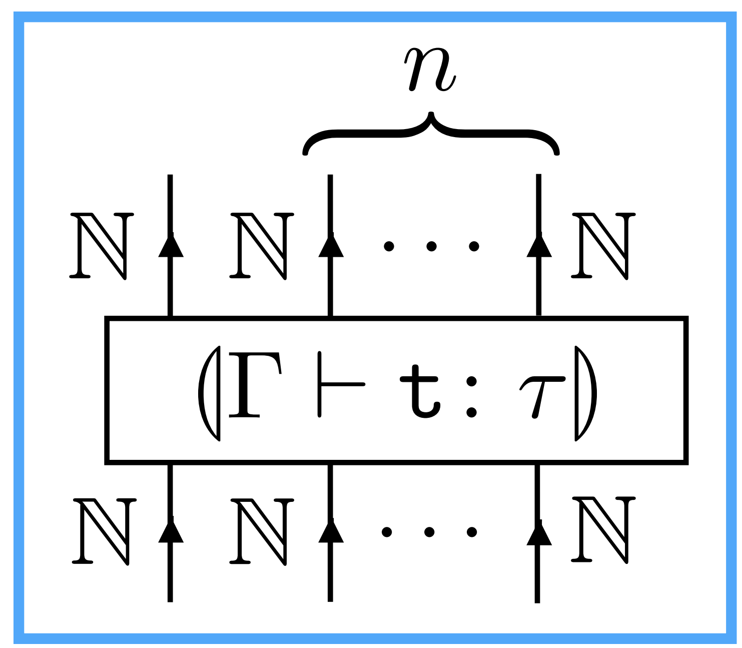
Memoryful GoI with recursion

<u>Def. (interpretation $(\!|\Gamma \vdash \mathtt{t} \colon \tau|\!)$)</u>

For a type judgement $(\!|\Gamma \vdash \mathtt{t} \colon \tau|\!)$ $(\Gamma = x_1 \colon \tau_1, \ldots, x_n \colon \tau_n)$,

we inductively define

$$(\!|\Gamma \vdash \mathtt{t} \colon \tau|\!) =$$



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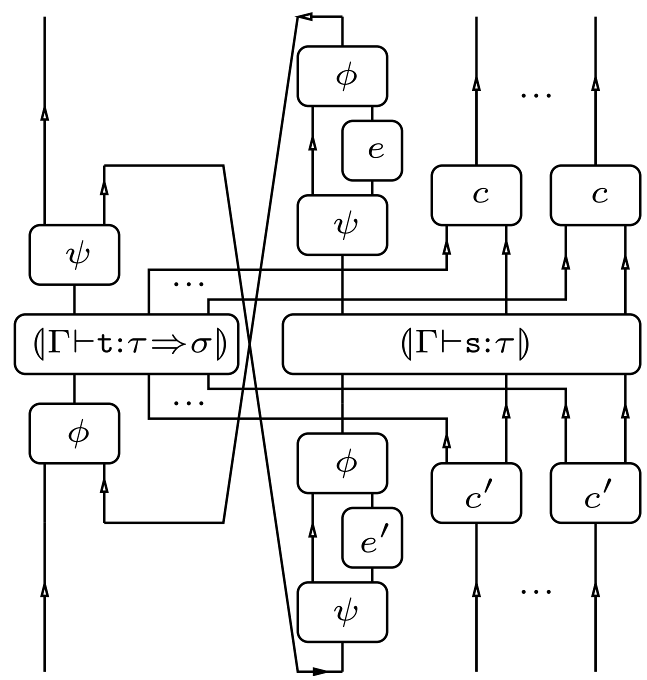
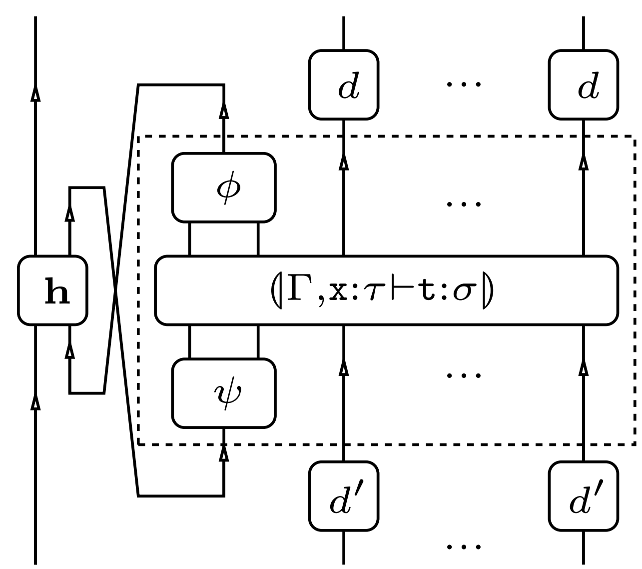
Memoryful GoI with recursion

<u>Def. (interpretation $(\!|\Gamma \vdash \mathtt{t} : \tau|\!)$)</u>

$(\!|\Gamma \vdash \mathtt{t\,s} : \sigma|\!) =$

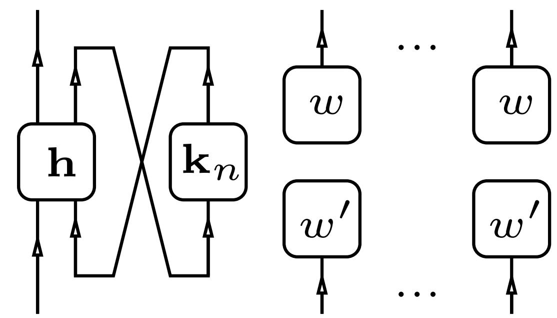


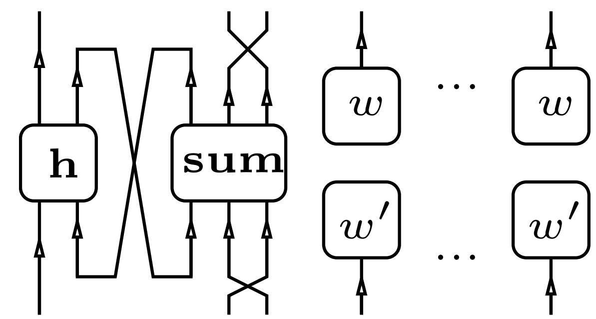
$(\!|\Gamma \vdash \lambda\mathtt{x} : \tau.\,\mathtt{t} : \tau \Rightarrow \sigma|\!) =$



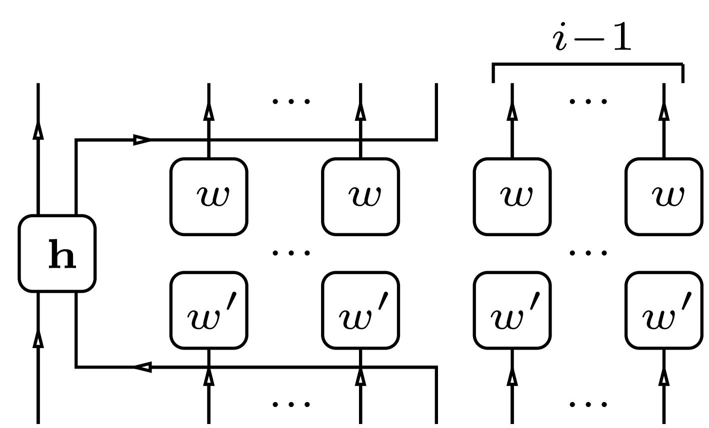
Memoryful GoI with recursion

Def. (interpretation $(\!|\Gamma \vdash \mathtt{t} : \tau|\!)$)

$(\!|\Gamma \vdash n : \mathtt{nat}|\!) =$ 

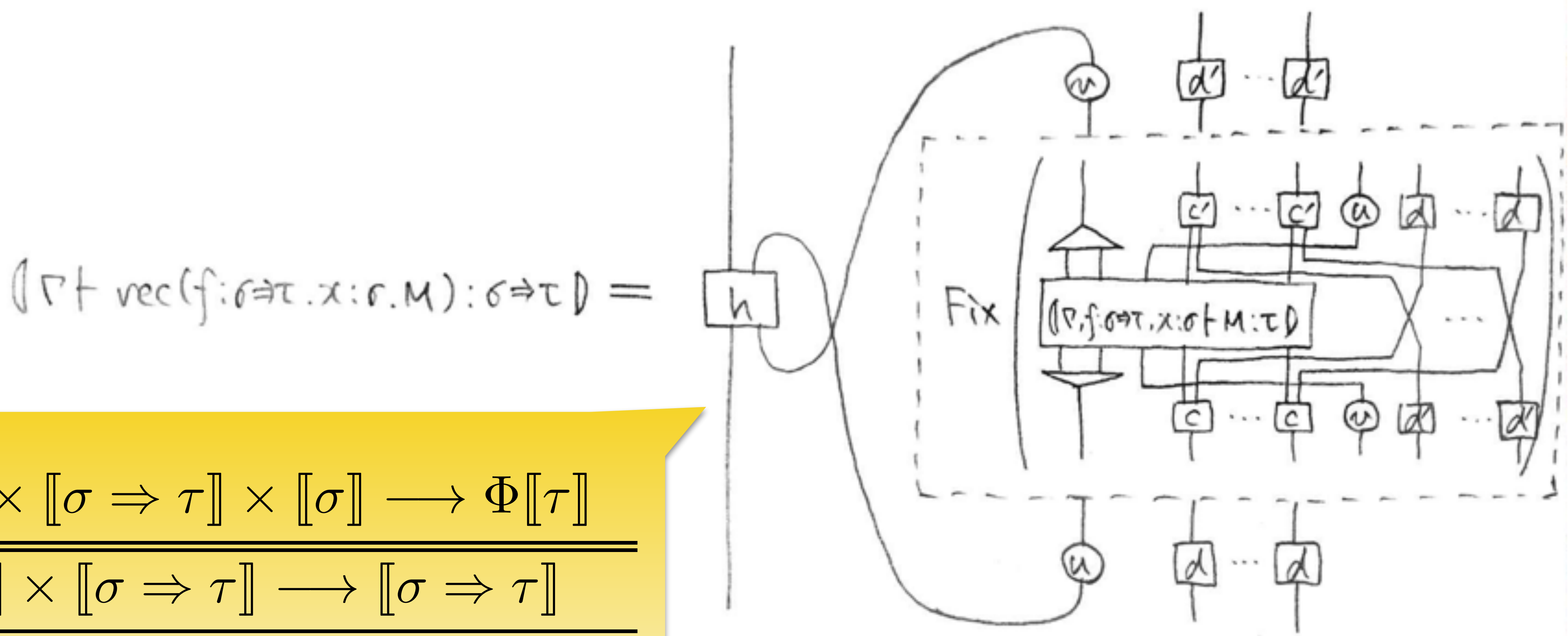
$(\!|\Gamma, \mathtt{x} : \mathtt{nat}, \mathtt{y} : \mathtt{nat} \vdash \mathtt{x} + \mathtt{y} : \mathtt{nat}|\!) =$ 

$$(\!|\Gamma \vdash \mathtt{t} + \mathtt{s} : \mathtt{nat}|\!) = (\!|\Gamma \vdash (\lambda \mathtt{xy} : \mathtt{nat}.\, \mathtt{x} + \mathtt{y})\, \mathtt{t}\, \mathtt{s} : \mathtt{nat}|\!)$$

$(\!|\mathtt{x}_1 : \tau_1, \cdots, \mathtt{x}_n : \tau_n \vdash \mathtt{x}_i : \tau_i|\!) =$ 

Memoryful GoI with recursion

Def. (interpretation $(\!|\Gamma \vdash \mathtt{t} : \tau|\!)$)



$$\llbracket\Gamma\rrbracket \times \llbracket\sigma \Rightarrow \tau\rrbracket \times \llbracket\sigma\rrbracket \longrightarrow \Phi\llbracket\tau\rrbracket$$

$$\llbracket\Gamma\rrbracket \times \llbracket\sigma \Rightarrow \tau\rrbracket \longrightarrow \llbracket\sigma \Rightarrow \tau\rrbracket$$

$$\llbracket\Gamma\rrbracket \Rightarrow \llbracket\sigma \Rightarrow \tau\rrbracket \longrightarrow \llbracket\Gamma\rrbracket \Rightarrow \llbracket\sigma \Rightarrow \tau\rrbracket$$

$$\llbracket\Gamma\rrbracket \longrightarrow \llbracket\sigma \Rightarrow \tau\rrbracket$$

Memoryful GoI with recursion

Thm. (soundness)

For closed terms M and N of type τ,

- $\vdash M = N : \tau$ implies $([\llparenthesis M \rrparenthesis]_{\simeq}, [\llparenthesis N \rrparenthesis]_{\simeq}) \in \Phi[\![\tau]\!]$
- $\vdash M = N : \mathtt{nat}$ implies $\llparenthesis M \rrparenthesis \simeq \llparenthesis N \rrparenthesis$.

behavioral equivalence

- Moggi's equations for computational lambda-calculus
- equations for algebraic operations

$$M \sqcup M = M$$

$$E[\mathrm{opr}(M_1, \ldots, M_n)] = \mathrm{opr}(E[M_1], \ldots, E[M_n])$$

$$(\lambda x.\ M)\ (N_1 \sqcup N_2) = (\lambda x.\ M)\ N_1\ \sqcup\ (\lambda x.\ M)\ N_2$$

$$rec(f : \sigma \Rightarrow \tau,\ x : \sigma.\ M) = \lambda x.\ M[rec(f : \sigma \Rightarrow \tau,\ x : \sigma.\ M)/f]$$

Memoryful GoI with recursion

<u>Thm. (domain-theoretic characterization of Fix)</u>

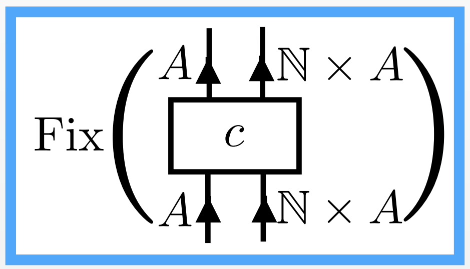
Under the assumption that

- $\mathbf{Set}_T$ is a **Cppo**-enriched category with **Cppo**-enriched (countable) cotuplings
- compositions $\circ_T$ of $\mathbf{Set}_T$ is strict in the restricted form: $f \circ_T \bot = \bot$ and $\bot \circ_T (\eta_Y \circ g) = \bot$ hold for any $f \colon X \to TY$ and $g \colon X \to Y$ in $\mathbf{Set}$
- premonoidal structures $X \otimes -, - \otimes X$ of $\mathbf{Set}_T$ is locally continuous and strict for any X in $\mathbf{Set}$

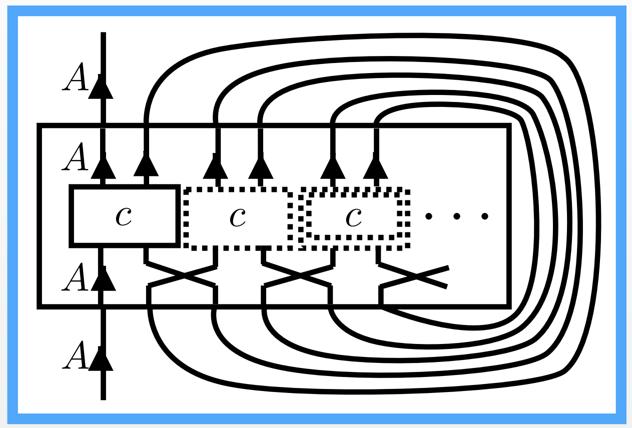
it holds that:

Memoryful GoI with recursion

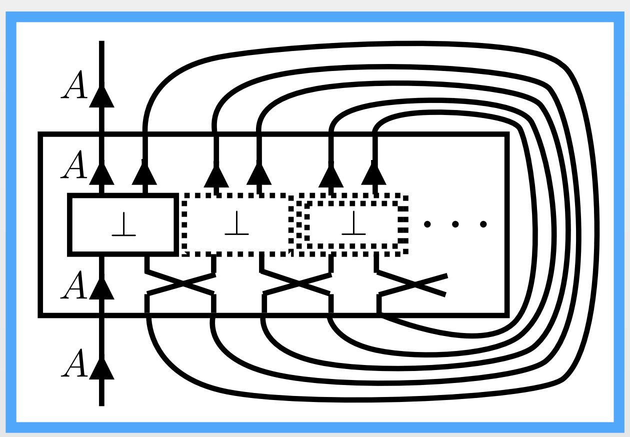
Thm. (domain-theoretic characterization of Fix)



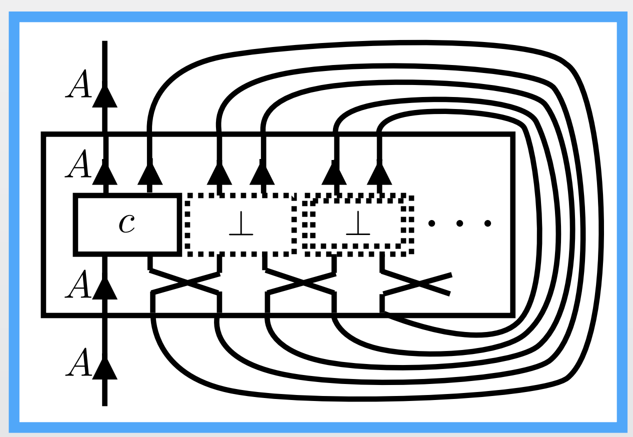
$=$



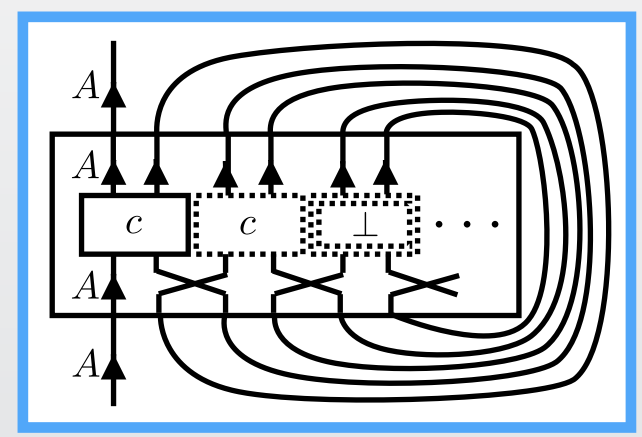
is a supremum of an ω-chain



$\trianglelefteq$



$\trianglelefteq$



$\trianglelefteq \quad \cdots$

where $(X,\ c\colon X \times A \to T(X \times B),\ x_0 \in X) \trianglelefteq (Y,\ c\colon Y \times A \to T(Y \times B),\ y_0 \in Y)$

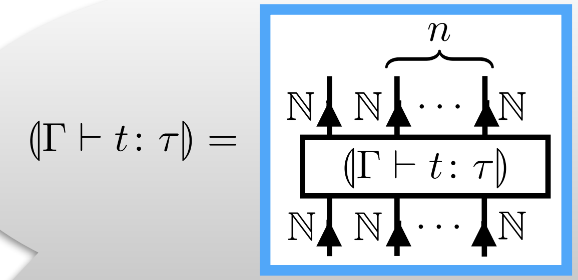
$\overset{\text{def.}}{\iff} X = Y \ \wedge\ x = y \ \wedge\ c \sqsubseteq d$ in $\mathbf{Set}_T(X \times A, X \times B)$

Memoryful GoI with recursion

effectful terms

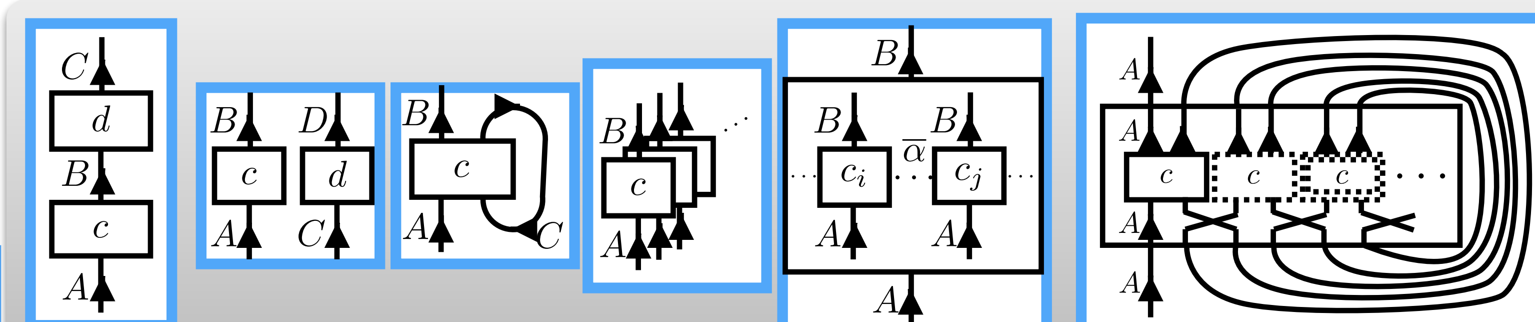
recursion

$(\lambda \mathtt{x} : \mathtt{nat}.\, \mathtt{x} + \mathtt{x})\,(3 \sqcup 5) \quad : \mathtt{nat}$



translation

- based on Geometry of Interaction
- via **extended** coalgebraic component calculus



transducers