Categorical Büchi and Parity Conditions via Alternating Fixed Points of Functors

Natsuki Urabe & Ichiro Hasuo

University of Tokyo National Institute of Informatics

Goal

- Categorical characterization of Büchi and parity conditions

Goal

- Categorical characterization of Büchi and parity conditions

Goal

- Categorical characterization of Büchi and parity conditions

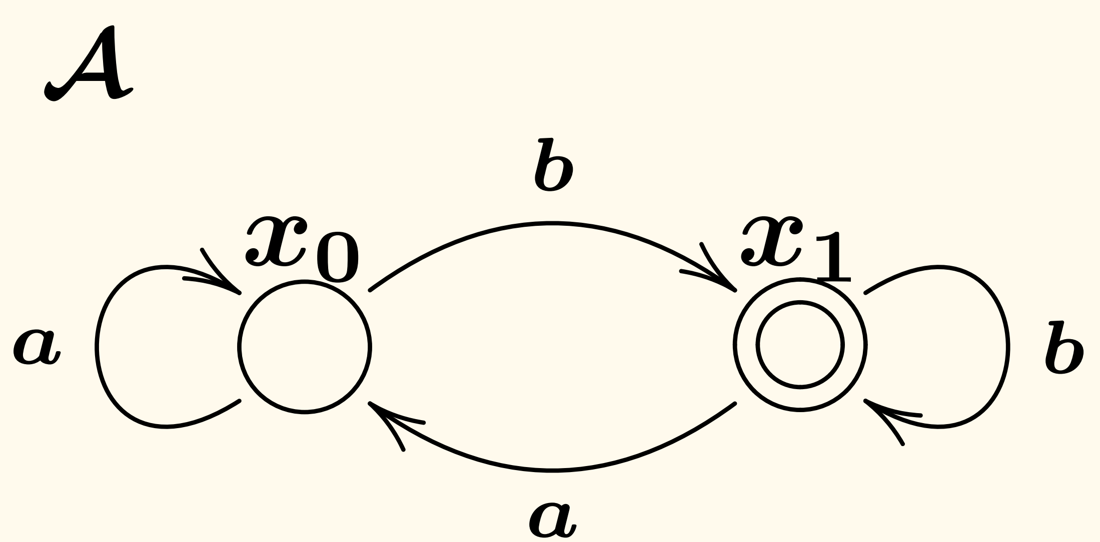
- **Büchi automaton**: an automaton accepting infinite words
- A run is **accepting** if it visits ◎ infinitely many times
- A word is **accepted** if it has an accepting run

Goal

- Categorical characterization of Büchi and parity conditions

- **Büchi automaton**: an automaton accepting infinite words
- A run is **accepting** if it visits ◎ infinitely many times
- A word is **accepted** if it has an accepting run

Example:



$$x_0 \xrightarrow{b} x_1 \xrightarrow{a} x_0 \xrightarrow{b} x_1 \xrightarrow{a} x_0 \dots$$

is accepting

⬇

$baba\dots$ is accepted from x_0

“Behaviors” of Büchi automaton

- We characterized behaviors of Büchi automaton as **decorated trace semantics**

“Behaviors” of Büchi automaton

- We characterized behaviors of Büchi automaton as **decorated trace semantics**

- For a Büchi automaton, we characterize its behavior as:

$$\mathbf{dtr_1}(\mathcal{A}) \;:\; X_1 \longrightarrow \mathcal{P}(\mathsf{A}^+(\mathsf{A}^+)^\omega)$$
$$\mathbf{dtr_2}(\mathcal{A}) \;:\; X_2 \longrightarrow \mathcal{P}((\mathsf{A}^+)^\omega)$$

$$X_1 := \{x : \text{nonaccepting}\}$$
$$X_2 := \{x : \text{accepting}\}$$
$$\mathsf{A} : \text{alphabet}$$

"Behaviors" of Büchi automaton

- We characterized behaviors of Büchi automaton as **decorated trace semantics**

- For a Büchi automaton, we characterize its behavior as:

$$\mathbf{dtr_1}(\mathcal{A}) \,:\, X_1 \longrightarrow \mathcal{P}(\mathsf{A}^{+}(\mathsf{A}^{+})^{\omega})$$
$$\mathbf{dtr_2}(\mathcal{A}) \,:\, X_2 \longrightarrow \mathcal{P}((\mathsf{A}^{+})^{\omega})$$

$$X_1 := \{x : \text{nonaccepting}\}$$
$$X_2 := \{x : \text{accepting}\}$$
$$\mathsf{A} : \text{alphabet}$$

- $(\mathsf{A}^{+})^{\omega}$ and $\mathsf{A}^{+}(\mathsf{A}^{+})^{\omega}$ are regarded as the set of infinite words "decorated" with ◯ and ◎

Decorated Word

$$(a_{00}a_{01}\dots a_{0n_0})(a_{10}a_{11}\dots a_{1n_1})\dots \in (\mathbf{A}^+)^\omega$$

Decorated Word

$$\frac{(a_{00}a_{01}\ldots a_{0n_0})(a_{10}a_{11}\ldots a_{1n_1})\ldots \in (\mathbf{A}^+)^\omega}{}$$

$$a_{00}\ a_{01}\ \ \ldots\ \ a_{0n_0}\ a_{10}\ a_{11}\ \ \ldots\ \ a_{1n_1}\ \ \ldots$$

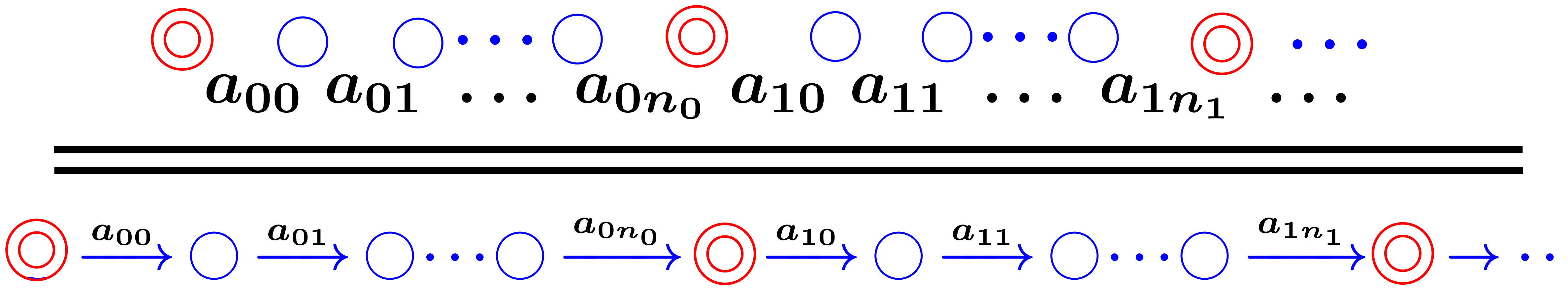
Decorated Word

$$\frac{(a_{00}a_{01}\ldots a_{0n_0})(a_{10}a_{11}\ldots a_{1n_1})\ldots \in (\mathbf{A}^+)^\omega}{}$$

$$a_{00}\ a_{01}\ \ldots\ a_{0n_0}\ a_{10}\ a_{11}\ \ldots\ a_{1n_1}\ \ldots$$

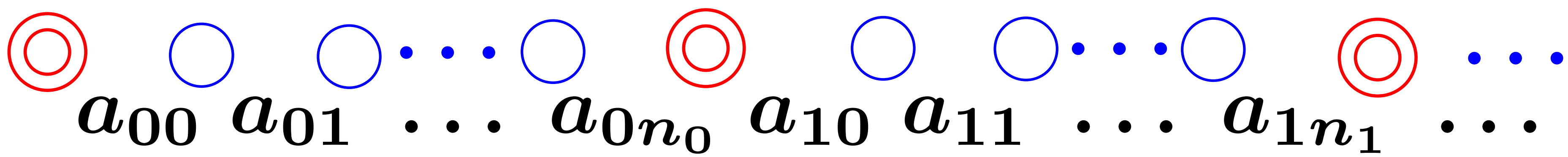
Decorated Word

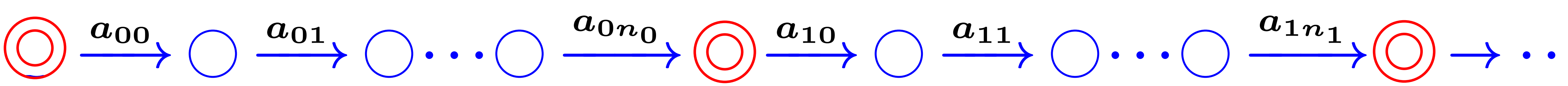
$$(a_{00}a_{01}\ldots a_{0n_0})(a_{10}a_{11}\ldots a_{1n_1})\ldots \in (\mathbf{A}^+)^\omega$$



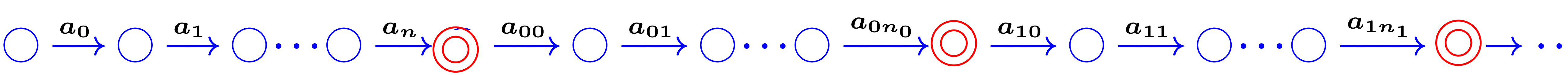
Decorated Word

$$(a_{00}a_{01}\dots a_{0n_0})(a_{10}a_{11}\dots a_{1n_1})\dots \in (\mathbf{A}^+)^\omega$$





$$a_0a_1\dots a_n(a_{00}a_{01}\dots a_{0n_0})(a_{10}a_{11}\dots a_{1n_1})\dots \in \mathbf{A}^+(\mathbf{A}^+)^\omega$$

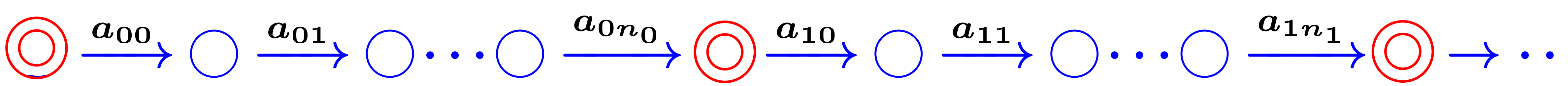


Decorated Word

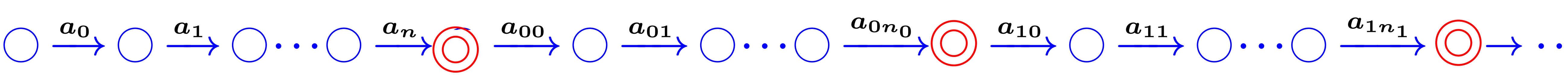
Note:
Büchi condition satisfied

$$(a_{00}a_{01}\ldots a_{0n_0})(a_{10}a_{11}\ldots a_{1n_1})\ldots \in (\mathbf{A}^+)^\omega$$

$$a_{00}\ a_{01}\ \ldots\ a_{0n_0}\ a_{10}\ a_{11}\ \ldots\ a_{1n_1}\ \ldots$$



$$a_0a_1\ldots a_n(a_{00}a_{01}\ldots a_{0n_0})(a_{10}a_{11}\ldots a_{1n_1})\ldots \in \mathbf{A}^+(\mathbf{A}^+)^\omega$$



Decorated Trace Semantics

- $\mathsf{dtr}_1(\mathcal{A})(x)$ and $\mathsf{dtr}_2(\mathcal{A})(x)$ assign a set of infinite words accepted by $\mathcal{A}$, decorated with $\bigcirc$ and $\circledcirc$, i.e.

$$\begin{aligned} \mathsf{dtr}_1(\mathcal{A}) &: X_1 \longrightarrow \mathcal{P}(\mathsf{A}^+(\mathsf{A}^+)^\omega) \\ \mathsf{dtr}_2(\mathcal{A}) &: X_2 \longrightarrow \mathcal{P}((\mathsf{A}^+)^\omega) \end{aligned} \qquad \begin{pmatrix} X : \text{state space} \\ \mathsf{A} : \text{alphabet} \end{pmatrix}$$

$$x \mapsto \left\{ \bullet_0 \xrightarrow{a_0} \bullet_1 \xrightarrow{a_1} \bullet_2 \rightarrow \cdots \;\middle|\; \begin{array}{l} x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} x_2 \rightarrow \cdots : \text{a run on } \mathcal{A} \\ \bullet_i \in \{\bigcirc, \circledcirc\},\ x_i : \bullet_i, \\ \bullet_i = \circledcirc \text{ for infinitely many } i\text{'s} \end{array} \right\}$$

Decorated Trace Semantics

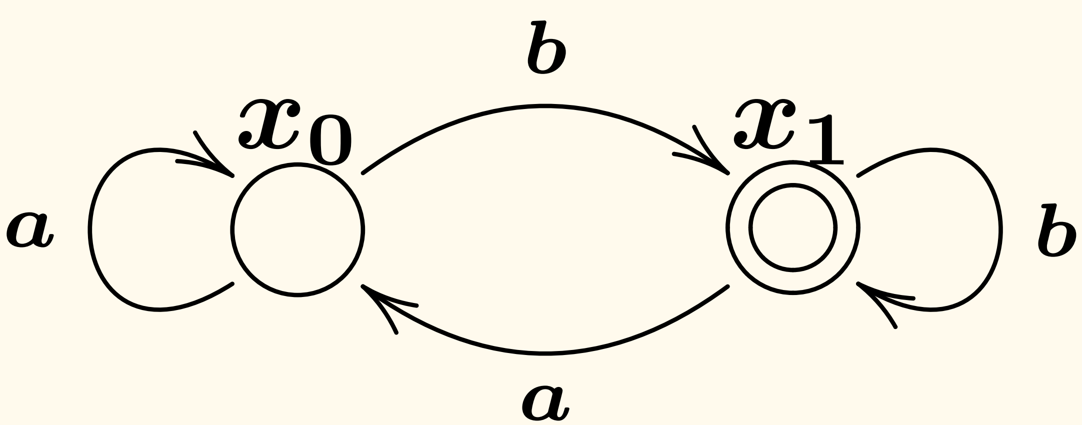
- $\mathsf{dtr}_1(\mathcal{A})(x)$ and $\mathsf{dtr}_2(\mathcal{A})(x)$ assign a set of infinite words accepted by $\mathcal{A}$, decorated with $\bigcirc$ and $\circledcirc$, i.e.

$$\mathsf{dtr}_1(\mathcal{A}) \,:\, X_1 \longrightarrow \mathcal{P}(\mathsf{A}^+(\mathsf{A}^+)^\omega)$$
$$\mathsf{dtr}_2(\mathcal{A}) \,:\, X_2 \longrightarrow \mathcal{P}((\mathsf{A}^+)^\omega)$$

$$\begin{pmatrix} X : \text{state space} \\ \mathsf{A} : \text{alphabet} \end{pmatrix}$$

$$x \mapsto \left\{ \bullet_0 \xrightarrow{a_0} \bullet_1 \xrightarrow{a_1} \bullet_2 \to \cdots \;\middle|\; \begin{array}{l} x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} x_2 \to \cdots : \text{a run on } \mathcal{A} \\ \bullet_i \in \{\bigcirc, \circledcirc\},\; x_i : \bullet_i, \\ \bullet_i = \circledcirc \text{ for infinitely many } i\text{'s} \end{array} \right\}$$

Example:



$$\mathsf{dtr}_1(\mathcal{A})(x_0) =$$

$$\left\{ \bigcirc \xrightarrow{c_0} \bullet_1 \xrightarrow{c_1} \bullet_2 \to \cdots \;\middle|\; \begin{array}{l} \bullet_0 = \bigcirc, \\ (\xrightarrow{c_i} \bullet_{i+1}) \in \{\xrightarrow{a} \bigcirc, \xrightarrow{b} \circledcirc\} \\ \bullet_i = \circledcirc \text{ for infinitely many } i\text{'s} \end{array} \right\}$$

Another Characterization of Behaviors

- (Ordinary Büchi) trace semantics

$$\mathbf{tr}^{B}(\mathcal{A}) : X \to \mathcal{P}\mathsf{A}^{\omega}$$

$$x \mapsto \left\{ a_0 a_1 \dots \;\middle|\; \begin{array}{l} x = x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} \cdots \\ x_i : \circledcirc \text{ for infinitely many } i\text{'s} \end{array} \right\}$$

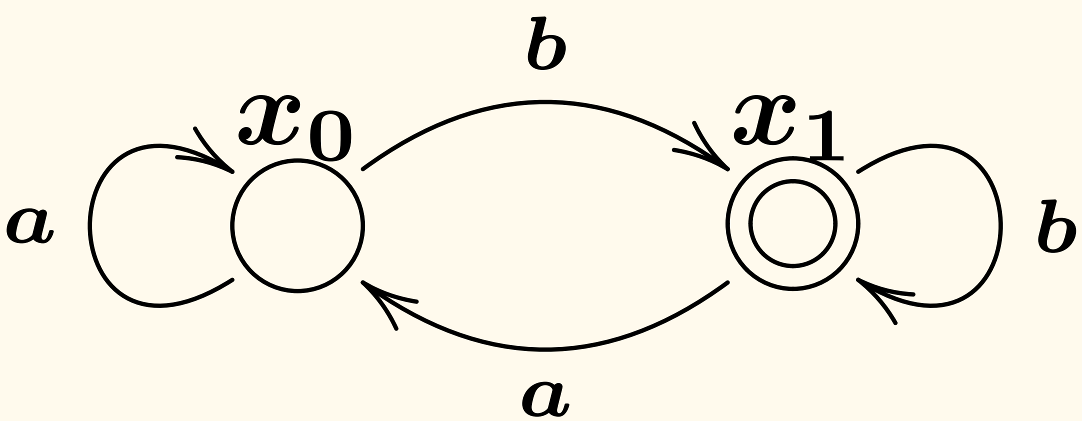
Another Characterization of Behaviors

- (Ordinary Büchi) trace semantics

$$\mathsf{tr}^B(\mathcal{A}) \;:\; X \;\to\; \mathcal{P}\mathsf{A}^\omega$$

$$x \;\mapsto\; \left\{ a_0 a_1 \ldots \;\middle|\; \begin{array}{l} x = x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} \cdots \\ x_i : \circledcirc \text{ for infinitely many } i\text{'s} \end{array} \right\}$$

Example:



$$\mathsf{tr}^B(\mathcal{A})(x_0) = \mathsf{tr}^B(\mathcal{A})(x_1) =$$

$$\left\{ w \;\middle|\; w \text{ contains infinitely many } b\text{'s} \right\}$$

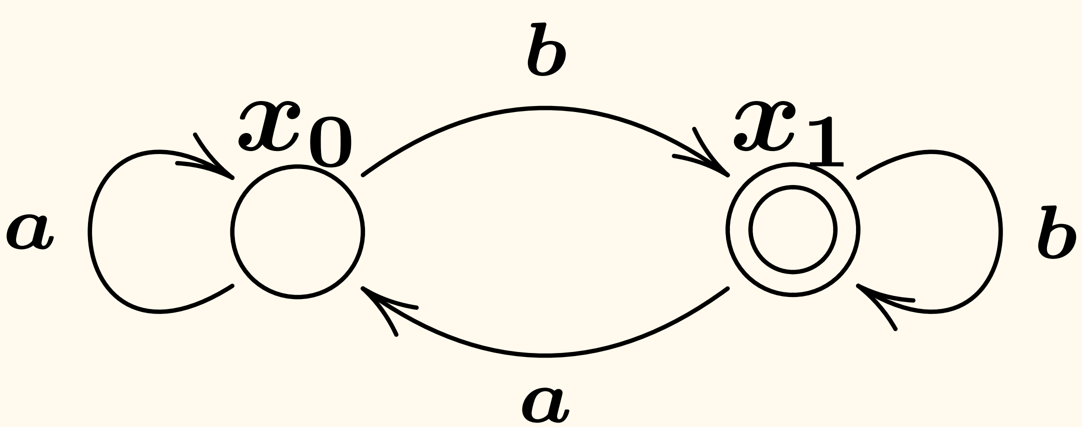
Another Characterization of Behaviors

- (Ordinary Büchi) trace semantics

$$\mathbf{tr}^B(\mathcal{A}) : X \to \mathcal{P}\mathsf{A}^\omega$$

$$x \mapsto \left\{ a_0 a_1 \ldots \;\middle|\; \begin{array}{l} x = x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} \cdots \\ x_i : ◎ \text{ for infinitely many } i\text{'s} \end{array} \right\}$$

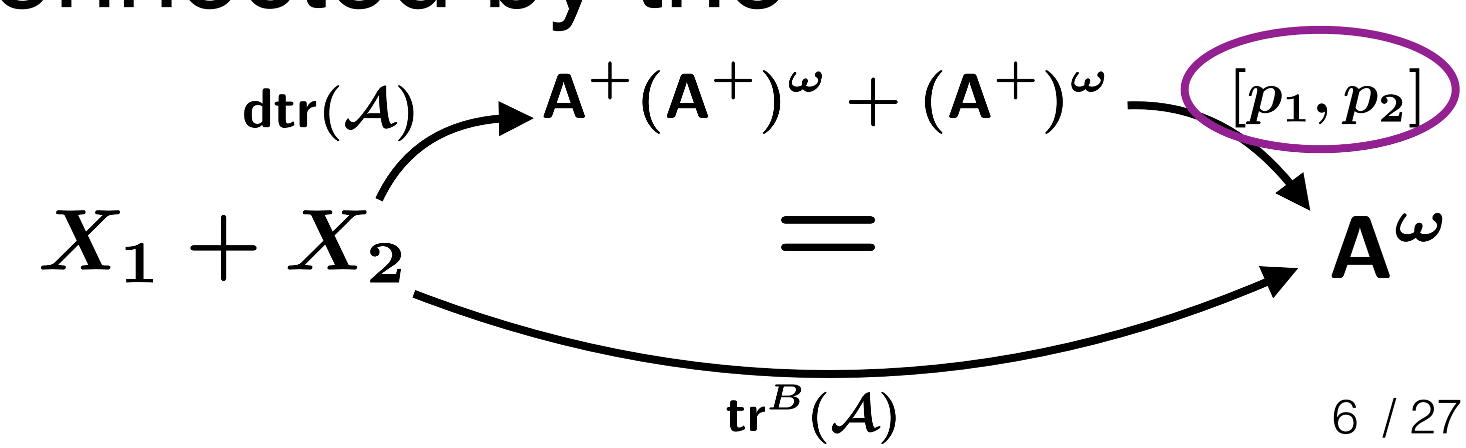
Example:



$$\mathbf{tr}^B(\mathcal{A})(x_0) = \mathbf{tr}^B(\mathcal{A})(x_1) =$$

$$\left\{ w \;\middle|\; w \text{ contains infinitely many } b\text{'s} \right\}$$

- $\mathbf{dtr}(c)$ and $\mathbf{tr}^B(c)$ are connected by the "flattening function"



Contributions

- **Datatypes** $(\mathsf{A}^+)^\omega$ and $\mathsf{A}^+(\mathsf{A}^+)^\omega$ for Büchi condition
- **Decorated trace semantics** $\mathbf{dtr_1}(c)$ and $\mathbf{dtr_2}(c)$
- Relationship to **(ordinary) Büchi trace semantics**

Contributions

- **Datatypes** $(\mathsf{A}^+)^\omega$ and $\mathsf{A}^+(\mathsf{A}^+)^\omega$ for Büchi condition
- **Decorated trace semantics** $\mathbf{dtr_1}(c)$ and $\mathbf{dtr_2}(c)$
- Relationship to **(ordinary) Büchi trace semantics**

Categorical generalization

Contributions

- **Datatypes** $(\mathsf{A}^+)^\omega$ and $\mathsf{A}^+(\mathsf{A}^+)^\omega$ for Büchi condition
- **Decorated trace semantics** $\mathbf{dtr_1}(c)$ and $\mathbf{dtr_2}(c)$
- Relationship to **(ordinary) Büchi trace semantics**

Categorical generalization

- Applicable to **parity condition** (generalization of Büchi cond.)

Contributions

- **Datatypes** $(\mathsf{A}^+)^\omega$ and $\mathsf{A}^+(\mathsf{A}^+)^\omega$ for Büchi condition
- **Decorated trace semantics** $\mathbf{dtr_1}(c)$ and $\mathbf{dtr_2}(c)$
- Relationship to **(ordinary) Büchi trace semantics**

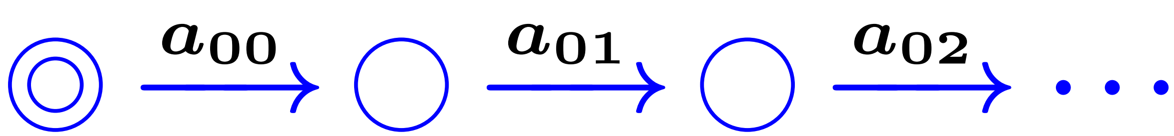
→ **Categorical generalization**

- Applicable to **parity condition** (generalization of Büchi cond.)
- Parameterized by functor F and monad T
  ➡ Applicable to systems with various <u>transition types</u> and <u>branching types</u>

  e.g. words, trees — **e.g. nondeterministic, probabilistic, exception**

Motivation

- In a decorated trace semantics, information about the accepting run is more explicit



$$\in (\mathbf{A}^{+})^{\omega}$$

- Definition of decorated trace semantics is simple compared to ordinary one (gfp vs alternating fixed point)

Motivation

- In a decorated trace semantics, information about the accepting run is more explicit

$$\circledcirc \xrightarrow{a_{00}} \bigcirc \xrightarrow{a_{01}} \bigcirc \xrightarrow{a_{02}} \cdots \quad \in (\mathbf{A}^{+})^{\omega}$$

- Definition of decorated trace semantics is simple compared to ordinary one (gfp vs alternating fixed point)

  ✳ (Possibly) useful for categorically characterizing notions about Büchi/parity systems

  - e.g. fair/delayed/direct simulation for Büchi automata [Etessami, Wilke & Schuller, ICALP '01]

Outline

- Introduction
- Alternating Fixed Point of Functor
- Decorated Trace Semantics
- Trace Semantics vs. Decorated Trace Semantics
- Related Work & Conclusion

Initial Algebra & Final Coalgebra

- For a functor $F : \mathbb{C} \to \mathbb{C}$,

Initial algebra

$$
\begin{array}{ccc}
FI & \dashrightarrow & FX \\
\iota \big\downarrow \cong & & \big\downarrow a \\
I & \dashrightarrow & X
\end{array}
$$

- "Least fixed point" of F
- I collects **finite behaviors** of F-coalgebras

Final coalgebra

$$
\begin{array}{ccc}
FX & \dashrightarrow & FZ \\
c \big\uparrow & & \cong \big\uparrow \zeta \\
X & \dashrightarrow & Z
\end{array}
$$

- "Greatest fixed point" of F
- Z collects **infinitary behaviors** of F-coalgebras

Examples

$\boldsymbol{F}$	$\boldsymbol{F}$-coalgebra	initial algebra	final coalgebra
$\mathbf{A} \times (\_)$			
$\mathbf{1} + \mathbf{A} \times (\_)$			
$\mathbf{A} \times (\_ + \boldsymbol{Y})$			

Examples

F	F-coalgebra	initial algebra	final coalgebra
$\mathbf{A} \times (\_\,)$	$X \to \mathbf{A} \times X$		
$1 + \mathbf{A} \times (\_\,)$			
$\mathbf{A} \times (\_ + Y)$			

Examples

F	F-coalgebra	initial algebra	final coalgebra
$\mathsf{A} \times (\_\,)$	$X \to \mathsf{A} \times X$	$\emptyset$	A^ω
$1 + \mathsf{A} \times (\_\,)$			
$\mathsf{A} \times (\_ + Y)$			

Examples

F	F-coalgebra	initial algebra	final coalgebra
$\mathsf{A} \times (\_)$	$X \to \mathsf{A} \times X$	$\emptyset$	A^ω
$1 + \mathsf{A} \times (\_)$	$X \to 1 + \mathsf{A} \times X$	A^*	$\mathsf{A}^\infty (= \mathsf{A}^* + \mathsf{A}^\omega)$
$\mathsf{A} \times (\_ + Y)$			

Examples

F	F-coalgebra	initial algebra	final coalgebra
$\mathsf{A} \times (\_\,)$	$X \to \mathsf{A} \times X$	$\emptyset$	A^ω
$1 + \mathsf{A} \times (\_\,)$	$X \to 1 + \mathsf{A} \times X$	A^*	$\mathsf{A}^\infty (= \mathsf{A}^* + \mathsf{A}^\omega)$
$\mathsf{A} \times (\_ + Y)$	$X \to \mathsf{A} \times (X + Y)$	$\mathsf{A}^+ Y$	$\mathsf{A}^+ Y + \mathsf{A}^\omega$

F^{+} and $F^{\oplus}$

- For a functor $F : \mathbb{C} \to \mathbb{C}$,
  - $F^{+}Y$ is the carrier of **initial** $F(\_ + Y)$-algebra

$$F(F^{+}Y + Y) \xrightarrow[\text{initial}]{\iota_Y^F \cong} F^{+}Y$$

  - $F^{\oplus}Y$ is the carrier of **final** $F(\_ + Y)$-coalgebra

$$F^{\oplus}Y \xrightarrow[\text{final}]{\zeta_Y^F \cong} F(F^{\oplus}Y + Y)$$

F^+ and $F^\oplus$

- For a functor $F : \mathbb{C} \to \mathbb{C}$,
  - $F^+ Y$ is the carrier of **initial** $F(\_ + Y)$-algebra
    $$F(F^+Y + Y) \xrightarrow[\text{initial}]{\iota^F_Y \cong} F^+Y$$
  - $F^\oplus Y$ is the carrier of **final** $F(\_ + Y)$-coalgebra
    $$F^\oplus Y \xrightarrow[\text{final}]{\zeta^F_Y \cong} F(F^\oplus Y + Y)$$

- They can be extended to functors $F^+, F^\oplus : \mathbb{C} \to \mathbb{C}$

F^{+} and $F^{\oplus}$

- For a functor $F : \mathbb{C} \to \mathbb{C}$,
  - $F^{+}Y$ is the carrier of **initial** $F(\_ + Y)$-algebra

lfp

$$F(F^{+}Y + Y) \xrightarrow[\text{initial}]{\iota_Y^F \,\cong} F^{+}Y$$

  - $F^{\oplus}Y$ is the carrier of **final** $F(\_ + Y)$-coalgebra

gfp

$$F^{\oplus}Y \xrightarrow[\text{final}]{\zeta_Y^F \,\cong} F(F^{\oplus}Y + Y)$$

- They can be extended to functors $F^{+}, F^{\oplus} : \mathbb{C} \to \mathbb{C}$

F^{+} and $F^{\oplus}$

c.f. free monad

$$F(F^{*}Y)+Y \xrightarrow[\cong]{\text{init}} F^{*}Y$$

e.g. for $F = \mathsf{A} \times (\_\,)$,

$$F^{*}X \cong \mathsf{A}^{*}X$$

- For a functor $F : \mathbb{C} \to \mathbb{C}$,
  - $F^{+}Y$ is the carrier of **initial** $F(\_ + Y)$-algebra

lfp

$$F(F^{+}Y+Y) \xrightarrow[\text{initial}]{\iota^{F}_{Y}\ \cong} F^{+}Y$$

  - $F^{\oplus}Y$ is the carrier of **final** $F(\_ + Y)$-coalgebra

gfp

$$F^{\oplus}Y \xrightarrow[\text{final}]{\zeta^{F}_{Y}\ \cong} F(F^{\oplus}Y+Y)$$

- They can be extended to functors $F^{+}, F^{\oplus} : \mathbb{C} \to \mathbb{C}$

Examples

- For $F = \mathbf{A} \times (\_)$

initial alg./final coalgebra of $F(\_ + Y)$

  - $F^{+}Y \cong \mathbf{A}^{+}Y$, $F^{\oplus}Y \cong \mathbf{A}^{+}Y + \mathbf{A}^{\omega}$

Examples

- For $F = \mathbf{A} \times (\_\,)$
  - $F^{+}Y \cong \mathbf{A}^{+}Y, \ \ F^{\oplus}Y \cong \mathbf{A}^{+}Y + \mathbf{A}^{\omega}$

    initial alg./final coalgebra of $F(\_ + Y)$

  - $F^{+\oplus}\mathbf{0} \cong (\mathbf{A}^{+})^{\omega}$

    final coalgebra of $F^{+}(\_\,)$

Examples

- For $F = \mathbf{A} \times (\_)$
  - $F^{+}Y \cong \mathbf{A}^{+}Y$, $F^{\oplus}Y \cong \mathbf{A}^{+}Y + \mathbf{A}^{\omega}$

    initial alg./final coalgebra of $F(\_ + Y)$
  - $F^{+\oplus}\mathbf{0} \cong (\mathbf{A}^{+})^{\omega}$

    final coalgebra of $F^{+}(\_)$
  - $F^{+}(F^{+\oplus}\mathbf{0}) \cong \mathbf{A}^{+}(\mathbf{A}^{+})^{\omega}$

    initial algebra of $F(\_ + F^{+\oplus}\mathbf{0})$

Examples

- For $F = \mathbf{A} \times (\_)$

initial alg./final coalgebra of $F(\_ + Y)$

  - $F^{+}Y \cong \mathbf{A}^{+}Y, \quad F^{\oplus}Y \cong \mathbf{A}^{+}Y + \mathbf{A}^{\omega}$
  - $F^{+\oplus}0 \cong (\mathbf{A}^{+})^{\omega}$

    final coalgebra of $F^{+}(\_)$

  - $F^{+}(F^{+\oplus}0) \cong \mathbf{A}^{+}(\mathbf{A}^{+})^{\omega}$

    initial algebra of $F(\_ + F^{+\oplus}0)$

- Equipped with (co)algebraic structures

$$F^{+\oplus}0 \xrightarrow[\zeta \text{ final}]{\cong} F^{+}(F^{+\oplus}0) \xleftarrow[\iota \text{ initial}]{\cong} F\big(F^{+}(F^{+\oplus}0) + F^{+\oplus}0\big)$$

Examples

- For $F = \mathsf{A} \times (\_)$

  initial alg./final coalgebra of $F(\_ + Y)$

  - $F^{+}Y \cong \mathsf{A}^{+}Y$, $F^{\oplus}Y \cong \mathsf{A}^{+}Y + \mathsf{A}^{\omega}$
  - $F^{+\oplus}0 \cong (\mathsf{A}^{+})^{\omega}$ — final coalgebra of $F^{+}(\_)$
  - $F^{+}(F^{+\oplus}0) \cong \mathsf{A}^{+}(\mathsf{A}^{+})^{\omega}$ — initial algebra of $F(\_ + F^{+\oplus}0)$

  alternating fixed point

- Equipped with (co)algebraic structures

$$F^{+\oplus}0 \xrightarrow[\zeta \text{ final}]{\cong} F^{+}(F^{+\oplus}0) \xleftarrow[\iota \text{ initial}]{\cong} F\big(F^{+}(F^{+\oplus}0) + F^{+\oplus}0\big)$$

Outline

- Introduction
- Alternating Fixed Point of Functor
- Decorated Trace Semantics
- Trace Semantics vs. Decorated Trace Semantics
- Related Work & Conclusion

Categorical Modeling of Büchi Automaton

- We categorically model a Büchi automaton as:

$$\Big(c : X \to \mathcal{P}FX,\ (X_1, X_2) \Big)$$

where

$$F = \mathsf{A} \times (\_\,)$$

$$\mathcal{P} : \text{ the powerset monad}$$

$$X = X_1 + X_2$$

$$X_1 := \{x : \text{nonaccepting}\}$$

$$X_2 := \{x : \text{accepting}\}$$

Fixed Point Characterization of $\mathsf{tr}^B(c)$

$$\mathsf{tr}^B(\mathcal{A}) \;:\; X \;\to\; \mathcal{P}\mathsf{A}^\omega$$

$$x \;\mapsto\; \left\{ a_0 a_1 \ldots \;\middle|\; \begin{array}{l} x = x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} \cdots \\ x_i : \circledcirc \text{ for infinitely many } i\text{'s} \end{array} \right\}$$

Fixed Point Characterization of $\mathbf{tr}^B(c)$

$$\mathbf{tr}^B(\mathcal{A}) : X \to \mathcal{P}\mathsf{A}^\omega$$

$$x \mapsto \left\{ a_0 a_1 \ldots \;\middle|\; \begin{array}{l} x = x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} \cdots \\ x_i : \circledcirc \text{ for infinitely many } i\text{'s} \end{array} \right\}$$

- $\mathbf{tr}^B(\mathcal{A}) : X \to \mathcal{P}\mathsf{A}^\omega$ is an **alternating fixed point** of:

$$\Phi : (\mathcal{P}\mathsf{A}^\omega)^X \to (\mathcal{P}\mathsf{A}^\omega)^X$$

$$u \mapsto \left[x \mapsto \left\{ aw \in \mathsf{A}^\omega \;\middle|\; \begin{array}{l} x \xrightarrow{a} x', \\ w \in u(x') \end{array} \right\} \right]$$

(weakest precondition)

Fixed Point Characterization of $\mathbf{tr}^B(c)$

$$\mathbf{tr}^B(\mathcal{A}) \;:\; X \to \mathcal{P}\mathsf{A}^\omega$$

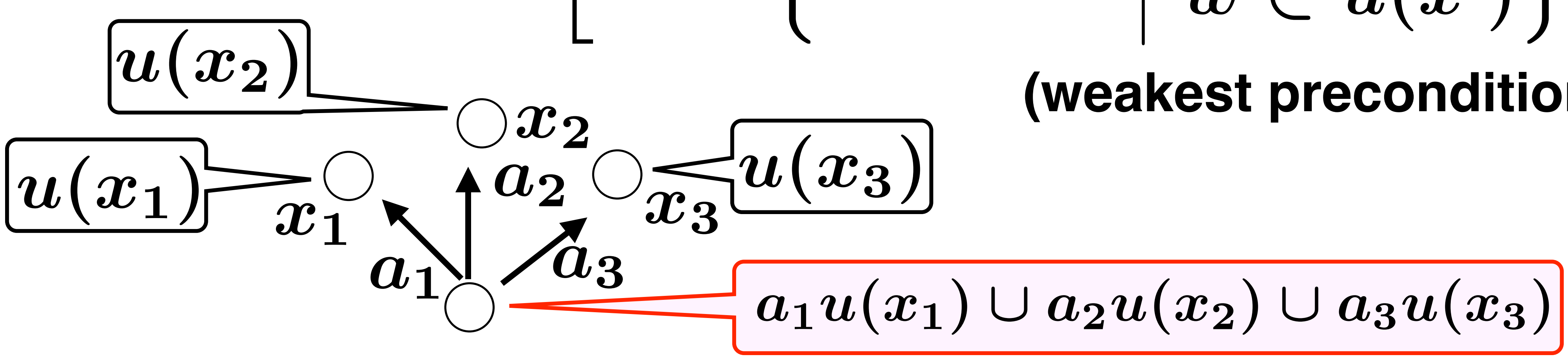
$$x \mapsto \left\{ a_0 a_1 \ldots \;\middle|\; \begin{array}{l} x = x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} \cdots \\ x_i : \circledcirc \text{ for infinitely many } i\text{'s} \end{array} \right\}$$

- $\mathbf{tr}^B(\mathcal{A}) : X \to \mathcal{P}\mathsf{A}^\omega$ is an **alternating fixed point** of:

$$\Phi : (\mathcal{P}\mathsf{A}^\omega)^X \to (\mathcal{P}\mathsf{A}^\omega)^X$$

$$u \mapsto \left[x \mapsto \left\{ aw \in \mathsf{A}^\omega \;\middle|\; \begin{array}{l} x \xrightarrow{a} x', \\ w \in u(x') \end{array} \right\} \right]$$

(weakest precondition)



Fixed Point in $(\mathcal{P}(\mathsf{A}^\omega)^X, \subseteq)$

- Let $X_1 := \{x : \text{nonaccepting}\}$ and $X_2 := \{x : \text{accepting}\}$

$$\Phi : (\mathcal{P}\mathsf{A}^\omega)^X \to (\mathcal{P}\mathsf{A}^\omega)^X \mapsto \begin{cases} \Phi_1 : (\mathcal{P}\mathsf{A}^\omega)^X \to (\mathcal{P}\mathsf{A}^\omega)^{X_1} \\ \Phi_2 : (\mathcal{P}\mathsf{A}^\omega)^X \to (\mathcal{P}\mathsf{A}^\omega)^{X_2} \end{cases}$$

Fixed Point in $(\mathcal{P}(\mathbf{A}^\omega)^X, \subseteq)$

- Let $X_1 := \{x : \text{nonaccepting}\}$ and $X_2 := \{x : \text{accepting}\}$

$$\Phi : (\mathcal{P}\mathbf{A}^\omega)^X \to (\mathcal{P}\mathbf{A}^\omega)^X$$
$$\mapsto \begin{cases} \Phi_1 : (\mathcal{P}\mathbf{A}^\omega)^X \to (\mathcal{P}\mathbf{A}^\omega)^{X_1} \\ \Phi_2 : (\mathcal{P}\mathbf{A}^\omega)^X \to (\mathcal{P}\mathbf{A}^\omega)^{X_2} \end{cases}$$

- We take **lfp** for Φ_1 and **gfp** for Φ_2:

$$\begin{cases} u_1 =_\mu \Phi_1(u_1, u_2) \in ((\mathcal{P}\mathbf{A}^\omega)^{X_1}, \subseteq) \\ u_2 =_\nu \Phi_2(u_1, u_2) \in ((\mathcal{P}\mathbf{A}^\omega)^{X_2}, \subseteq) \end{cases}$$

(hierarchical equation system, HES)

Fixed Point in $(\mathcal{P}(\mathsf{A}^\omega)^X, \subseteq)$

- Let $X_1 := \{x : \text{nonaccepting}\}$ and $X_2 := \{x : \text{accepting}\}$

$$\Phi : (\mathcal{P}\mathsf{A}^\omega)^X \to (\mathcal{P}\mathsf{A}^\omega)^X$$
$$\mapsto \begin{cases} \Phi_1 : (\mathcal{P}\mathsf{A}^\omega)^X \to (\mathcal{P}\mathsf{A}^\omega)^{X_1} \\ \Phi_2 : (\mathcal{P}\mathsf{A}^\omega)^X \to (\mathcal{P}\mathsf{A}^\omega)^{X_2} \end{cases}$$

- We take **lfp** for Φ_1 and **gfp** for Φ_2:

$$\begin{cases} u_1 & =_\mu & \Phi_1(u_1, u_2) & \in ((\mathcal{P}\mathsf{A}^\omega)^{X_1}, \subseteq) \\ u_2 & =_\nu & \Phi_2(u_1, u_2) & \in ((\mathcal{P}\mathsf{A}^\omega)^{X_2}, \subseteq) \end{cases}$$

(hierarchical equation system, HES)

Theorem: $\mathsf{tr}^B(\mathcal{A}) = [u_1^{\mathrm{sol}}, u_2^{\mathrm{sol}}] : X_1 + X_2 \to \mathcal{P}\mathsf{A}^\omega$

Weakest Precondition, Categorically

- Note that:

$$\frac{\Phi : (\mathcal{P}\mathsf{A}^{\omega})^{X} \to (\mathcal{P}\mathsf{A}^{\omega})^{X}}{\Phi : \mathcal{K}\ell(\mathcal{P})(X, F^{\oplus}0) \to \mathcal{K}\ell(\mathcal{P})(X, F^{\oplus}0)}$$

the Kleisli category of the powerset monad $\mathcal{P}$

$$\frac{f : X \nrightarrow Y \text{ in } \mathcal{K}\ell(\mathcal{P})}{f : X \to \mathcal{P}Y \text{ in } \mathbf{Sets}}$$

carrier of the final $F(\_ + 0) = \mathsf{A} \times (\_)$ -coalgebra

$$\zeta : F^{\oplus}0 \xrightarrow{\cong} FF^{\oplus}0$$

Weakest Precondition, Categorically

- Note that:

$$\frac{\Phi : (\mathcal{P}\mathsf{A}^{\omega})^{X} \to (\mathcal{P}\mathsf{A}^{\omega})^{X}}{\Phi \;:\; \mathcal{K}\ell(\mathcal{P})(X, F^{\oplus}0) \;\to\; \mathcal{K}\ell(\mathcal{P})(X, F^{\oplus}0)}$$

the Kleisli category of the powerset monad $\mathcal{P}$

$$\frac{f : X \mathrel{\ooalign{$\to$\cr\hfil$\mapstochar$\hfil}} Y \text{ in } \mathcal{K}\ell(\mathcal{P})}{f : X \to \mathcal{P}Y \text{ in } \mathbf{Sets}}$$

carrier of the final $F(\_ + 0) = \mathsf{A} \times (\_)$ -coalgebra

$$\zeta : F^{\oplus}0 \xrightarrow{\cong} FF^{\oplus}0$$

- $\Phi \;:\; \mathcal{K}\ell(\mathcal{P})(X, F^{\oplus}0) \;\to\; \mathcal{K}\ell(\mathcal{P})(X, F^{\oplus}0)$ is modeled by:

$$X \xrightarrow{f} F^{\oplus}0 \quad\longmapsto\quad \begin{array}{ccc} FX & \xrightarrow{\overline{F}f} & F(F^{\oplus}0) \\ \uparrow{\scriptstyle c} & & \cong\downarrow{\scriptstyle J\zeta^{-1}} \\ X & & F^{\oplus}0 \end{array} \quad \text{in } \mathcal{K}\ell(\mathcal{P})$$

where

$\overline{F} : \mathcal{K}\ell(\mathcal{P}) \to \mathcal{K}\ell(\mathcal{P})$: lifting of F

$J : \mathbb{C} \to \mathcal{K}\ell(\mathcal{P})$: the lifting functor

Categorical Trace Semantics of Büchi Automata

[U., Shimizu & Hasuo, CONCUR '16]

$$\begin{cases} u_1 & =_{\mu} & \Phi_1(u_1, u_2) & \in ((\mathcal{P}\mathsf{A}^{\omega})^{X_1}, \subseteq) \\ u_2 & =_{\nu} & \Phi_2(u_1, u_2) & \in ((\mathcal{P}\mathsf{A}^{\omega})^{X_2}, \subseteq) \end{cases}$$

Categorical Trace Semantics of Büchi Automata

[U., Shimizu & Hasuo, CONCUR '16]

$$\begin{cases} u_1 & =_{\mu} & \Phi_1(u_1, u_2) & \in ((\mathcal{P}\mathsf{A}^{\omega})^{X_1}, \subseteq) \\ u_2 & =_{\nu} & \Phi_2(u_1, u_2) & \in ((\mathcal{P}\mathsf{A}^{\omega})^{X_2}, \subseteq) \end{cases}$$

$$\begin{array}{ccc} F(X_1 + X_2) & \xrightarrow{\overline{F}[\mathrm{tr}_1(c), \mathrm{tr}_2(c)]} & F(F^{\oplus}0) \\ {\scriptstyle c_1}\uparrow & =_{\mu} & \uparrow{\scriptstyle J\zeta}\ \cong \\ X_1 & \xrightarrow{\mathrm{tr}_1(c)} & F^{\oplus}0 \end{array} \qquad \begin{array}{ccc} F(X_1 + X_2) & \xrightarrow{\overline{F}[\mathrm{tr}_1(c), \mathrm{tr}_2(c)]} & F(F^{\oplus}0) \\ {\scriptstyle c_2}\uparrow & =_{\nu} & \uparrow{\scriptstyle J\zeta}\ \cong \\ X_2 & \xrightarrow{\mathrm{tr}_2(c)} & F^{\oplus}0 \end{array}$$

where $\begin{array}{l} c_1 = c \circ \pi_1 \\ c_2 = c \circ \pi_2 \end{array}$

Categorical Trace Semantics of Büchi Automata

[U., Shimizu & Hasuo, CONCUR '16]

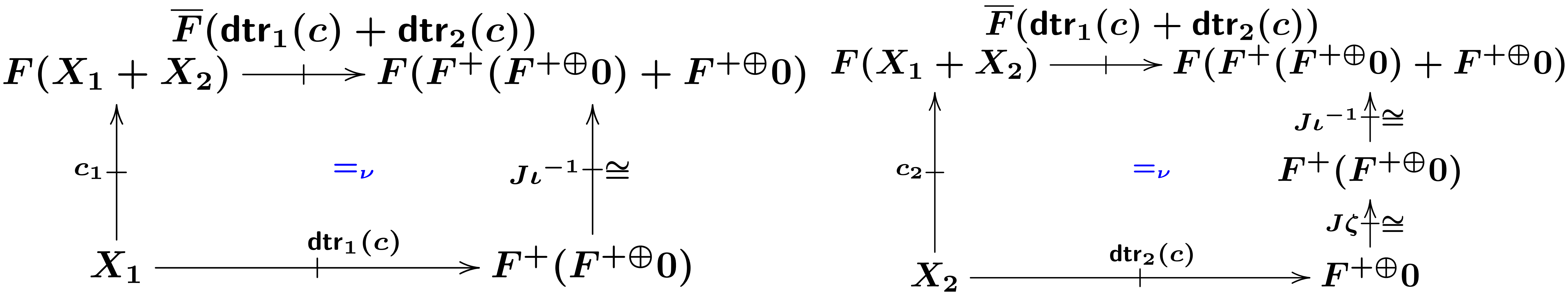
$$\begin{cases} u_1 \;\; =_{\mu} \;\; \Phi_1(u_1, u_2) \;\; \in ((\mathcal{P}\mathsf{A}^{\omega})^{X_1}, \subseteq) \\ u_2 \;\; =_{\nu} \;\; \Phi_2(u_1, u_2) \;\; \in ((\mathcal{P}\mathsf{A}^{\omega})^{X_2}, \subseteq) \end{cases}$$

$$\begin{array}{ccc} F(X_1+X_2) & \xrightarrow{\overline{F}[\mathsf{tr}_1(c),\mathsf{tr}_2(c)]} & F(F^{\oplus}0) \\ {\scriptstyle c_1}\uparrow & =_{\mu} & \uparrow{\scriptstyle J\zeta}\,\cong \\ X_1 & \xrightarrow{\mathsf{tr}_1(c)} & F^{\oplus}0 \end{array} \qquad \begin{array}{ccc} F(X_1+X_2) & \xrightarrow{\overline{F}[\mathsf{tr}_1(c),\mathsf{tr}_2(c)]} & F(F^{\oplus}0) \\ {\scriptstyle c_2}\uparrow & =_{\nu} & \uparrow{\scriptstyle J\zeta}\,\cong \\ X_2 & \xrightarrow{\mathsf{tr}_2(c)} & F^{\oplus}0 \end{array}$$

where $\begin{array}{l} c_1 = c \circ \pi_1 \\ c_2 = c \circ \pi_2 \end{array}$

Theorem: $[\mathsf{tr}_1(c), \mathsf{tr}_2(c)] = [\mathsf{tr}_1(\mathcal{A}), \mathsf{tr}_2(\mathcal{A})]$

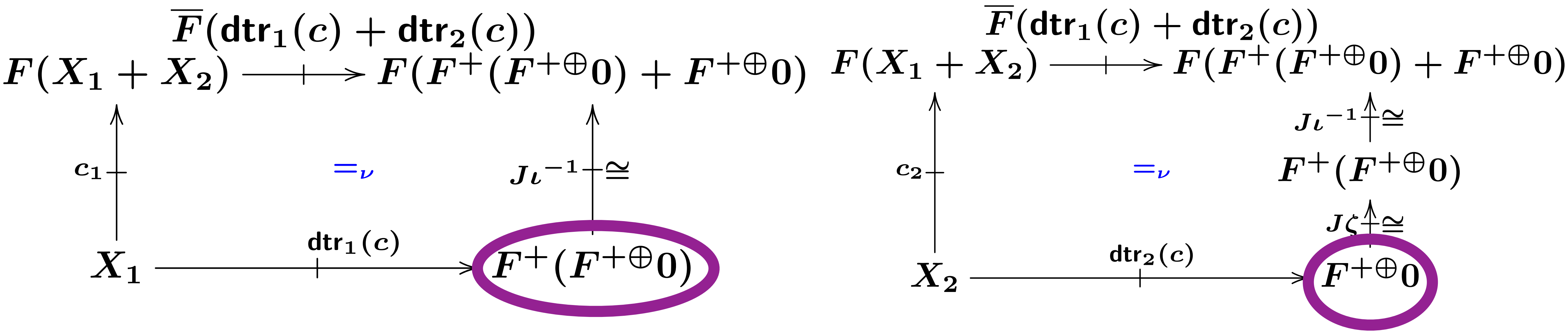
Categorical Decorated Trace Semantics



- Recall:

$$F^{+\oplus}\mathbf{0} \xrightarrow[\zeta \text{ final}]{\cong} F^+(F^{+\oplus}\mathbf{0}) \xleftarrow[\iota \text{ initial}]{\cong} F\big(F^+(F^{+\oplus}\mathbf{0}) + F^{+\oplus}\mathbf{0}\big)$$

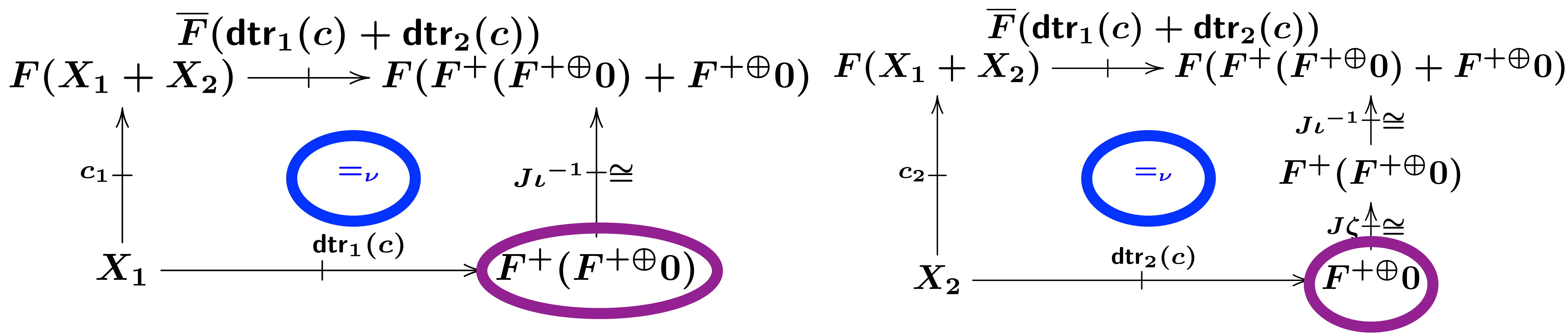
Categorical Decorated Trace Semantics



- Recall:

$$F^{+\oplus}0 \xrightarrow[\zeta \text{ final}]{\cong} F^{+}(F^{+\oplus}0) \xleftarrow[\iota \text{ initial}]{\cong} F\big(F^{+}(F^{+\oplus}0) + F^{+\oplus}0\big)$$

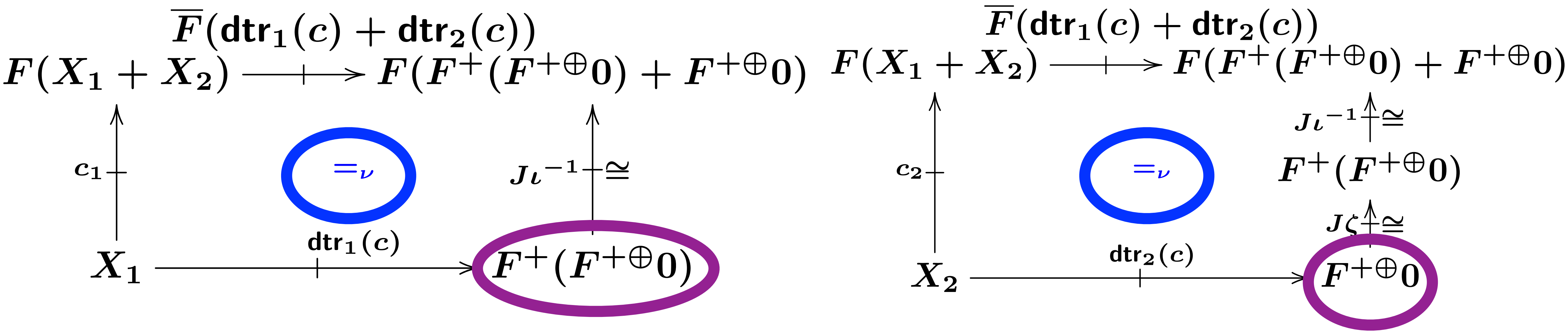
Categorical Decorated Trace Semantics



- Recall:

$$F^{+\oplus}0 \xrightarrow[\zeta \text{ final}]{\cong} F^+(F^{+\oplus}0) \xleftarrow[\iota \text{ initial}]{\cong} F\big(F^+(F^{+\oplus}0) + F^{+\oplus}0\big)$$

Categorical Decorated Trace Semantics



- Recall:

$$F^{+\oplus}0 \xrightarrow[\zeta \text{ final}]{\cong} F^+(F^{+\oplus}0) \xleftarrow[\iota \text{ initial}]{\cong} F\big(F^+(F^{+\oplus}0) + F^{+\oplus}0\big)$$

Theorem: $\mathbf{dtr}_1(c) + \mathbf{dtr}_2(c) \;=\; \mathbf{dtr}_1(\mathcal{A}) + \mathbf{dtr}_2(\mathcal{A})$

For Parity Condition

- For Büchi automata,

$$F^{+\oplus}\mathbf{0} \xrightarrow[\zeta \text{ final}]{\cong} F^{+}(F^{+\oplus}\mathbf{0}) \xleftarrow[\iota \text{ initial}]{\cong} F\big(F^{+}(F^{+\oplus}\mathbf{0}) + F^{+\oplus}\mathbf{0}\big)$$

X_2 X_1

For Parity Condition

- For Büchi automata,

$$F^{+\oplus}\mathbf{0} \xrightarrow[\zeta\ \text{final}]{\cong} F^{+}(F^{+\oplus}\mathbf{0}) \xleftarrow[\iota\ \text{initial}]{\cong} F\big(F^{+}(F^{+\oplus}\mathbf{0}) + F^{+\oplus}\mathbf{0}\big)$$

X_2 X_1

- For parity automata s.t. $\Omega : X \to \{1, \dots, 2n\}$

$$F^{\overbrace{+\oplus\cdots+\oplus}^{2n}}\mathbf{0} \xrightarrow{\cong} F^{\overbrace{+\oplus\cdots+\oplus+}^{2n-1}}(F^{\overbrace{+\oplus\cdots+\oplus}^{2n}}\mathbf{0}) \xleftarrow{\cong} \cdots$$

$$\xleftarrow{\cong} F\big(F^{\overbrace{+\oplus\cdots+\oplus}^{2n}}\mathbf{0} + F^{\overbrace{+\oplus\cdots+\oplus+}^{2n-1}}(F^{\overbrace{+\oplus\cdots+\oplus}^{2n}}\mathbf{0}) + \cdots\big)$$

For Parity Condition

- For Büchi automata,

$$\underset{X_2}{F^{+\oplus}0} \xrightarrow[\zeta\ \text{final}]{\cong} \underset{X_1}{F^{+}(F^{+\oplus}0)} \xleftarrow[\iota\ \text{initial}]{\cong} F\big(F^{+}(F^{+\oplus}0) + F^{+\oplus}0\big)$$

- For parity automata s.t. $\Omega : X \to \{1, \ldots, 2n\}$

$$\underset{X_{2n}}{F^{\overbrace{+\oplus\cdots+\oplus}^{2n}}0} \xrightarrow{\cong} \underset{X_{2n-1}}{F^{\overbrace{+\oplus\cdots+\oplus+}^{2n-1}}(F^{\overbrace{+\oplus\cdots+\oplus}^{2n}}0)} \xleftarrow{\cong} \cdots$$

$$\xleftarrow{\cong} F\big(F^{\overbrace{+\oplus\cdots+\oplus}^{2n}}0 + F^{\overbrace{+\oplus\cdots+\oplus+}^{2n-1}}(F^{\overbrace{+\oplus\cdots+\oplus}^{2n}}0) + \cdots\big)$$

Outline

- Introduction
- Alternating Fixed Point of Functor
- Decorated Trace Semantics
- Trace Semantics vs. Decorated Trace Semantics
- Related Work & Conclusion

Removing Decoration, Categorically

$$\mathbf{A}^{+}(\mathbf{A}^{+})^{\omega} + (\mathbf{A}^{+})^{\omega} \xrightarrow{[\boldsymbol{p_1}, \boldsymbol{p_2}]} \mathbf{A}^{\omega}$$

Removing Decoration, Categorically

$$\mathsf{A}^{+}(\mathsf{A}^{+})^{\omega} + (\mathsf{A}^{+})^{\omega} \xrightarrow{[p_1, p_2]} \mathsf{A}^{\omega}$$

- Categorically,

$$F^{+}(F^{+\oplus}0) + F^{+\oplus}0 \xrightarrow{[p_1, p_2]} F^{\oplus}0$$

Removing Decoration, Categorically

$$\mathsf{A}^{+}(\mathsf{A}^{+})^{\omega} + (\mathsf{A}^{+})^{\omega} \xrightarrow{[p_1, p_2]} \mathsf{A}^{\omega}$$

- Categorically,

$$F^{+}(F^{+\oplus}\mathbf{0}) + F^{+\oplus}\mathbf{0} \xrightarrow{[p_1, p_2]} F^{\oplus}\mathbf{0}$$

- We define them by:

$$\begin{array}{ccc}
F(F^{+}(F^{+\oplus}\mathbf{0}) + F^{+\oplus}\mathbf{0}) & & F(F^{\oplus}\mathbf{0}) \\
\cong \Big\uparrow \begin{smallmatrix}\text{initial} \\ \iota^{-1}\end{smallmatrix} & & \Big\uparrow \zeta \\
F(F^{+\oplus}\mathbf{0}) & & \\
\cong \Big\uparrow \begin{smallmatrix}\text{final} \\ \zeta\end{smallmatrix} & & F^{\oplus}\mathbf{0} \\
F^{+\oplus}\mathbf{0} & &
\end{array}$$

Removing Decoration, Categorically

$$\mathsf{A}^{+}(\mathsf{A}^{+})^{\omega} + (\mathsf{A}^{+})^{\omega} \xrightarrow{[p_1, p_2]} \mathsf{A}^{\omega}$$

- Categorically,

$$F^{+}(F^{+\oplus}\mathbf{0}) + F^{+\oplus}\mathbf{0} \xrightarrow{[p_1, p_2]} F^{\oplus}\mathbf{0}$$

- We define them by:

$$\begin{array}{ccc} \multicolumn{2}{c}{F(F^{+}(F^{+\oplus}\mathbf{0}) + F^{+\oplus}\mathbf{0})} & \\ & \uparrow \cong \ \text{initial}, \ \iota^{-1} & \\ \uparrow \cong \ \text{initial}, \ \iota^{-1} & F(F^{+\oplus}\mathbf{0}) & \\ & \uparrow \cong \ \text{final}, \ \zeta & \\ F(F^{+\oplus}\mathbf{0}) & F^{+\oplus}\mathbf{0} & \end{array} \qquad \begin{array}{c} F(F^{\oplus}\mathbf{0}) \\ \uparrow \zeta \\ F^{\oplus}\mathbf{0} \end{array}$$

Removing Decoration, Categorically

$$\mathsf{A}^+(\mathsf{A}^+)^\omega + (\mathsf{A}^+)^\omega \xrightarrow{[p_1,p_2]} \mathsf{A}^\omega$$

- Categorically,

$$F^+(F^{+\oplus}\mathbf{0}) + F^{+\oplus}\mathbf{0} \xrightarrow{[p_1,p_2]} F^{\oplus}\mathbf{0}$$

- We define them by:

$$\begin{array}{ccc} F(F^+(F^{+\oplus}\mathbf{0}) + F^{+\oplus}\mathbf{0}) & & F(F^{\oplus}\mathbf{0}) \\ \uparrow{\scriptstyle [\iota^{-1},\iota^{-1}\circ\zeta]} & & \uparrow{\scriptstyle \zeta} \\ F^+(F^{+\oplus}\mathbf{0}) + F^{+\oplus}\mathbf{0} & & F^{\oplus}\mathbf{0} \end{array}$$

Removing Decoration, Categorically

$$\mathsf{A}^+(\mathsf{A}^+)^\omega + (\mathsf{A}^+)^\omega \xrightarrow{[p_1,p_2]} \mathsf{A}^\omega$$

- Categorically,

$$F^+(F^{+\oplus}\mathbf{0}) + F^{+\oplus}\mathbf{0} \xrightarrow{[p_1,p_2]} F^{\oplus}\mathbf{0}$$

- We define them by:

$$\begin{array}{ccc} F(F^+(F^{+\oplus}\mathbf{0}) + F^{+\oplus}\mathbf{0}) & \dashrightarrow & F(F^{\oplus}\mathbf{0}) \\ \uparrow{\scriptstyle [\iota^{-1},\iota^{-1}\circ\zeta]} & & \uparrow{\scriptstyle \zeta} \\ F^+(F^{+\oplus}\mathbf{0}) + F^{+\oplus}\mathbf{0} & \overset{[p_1,p_2]}{\dashrightarrow} & F^{\oplus}\mathbf{0} \end{array}$$

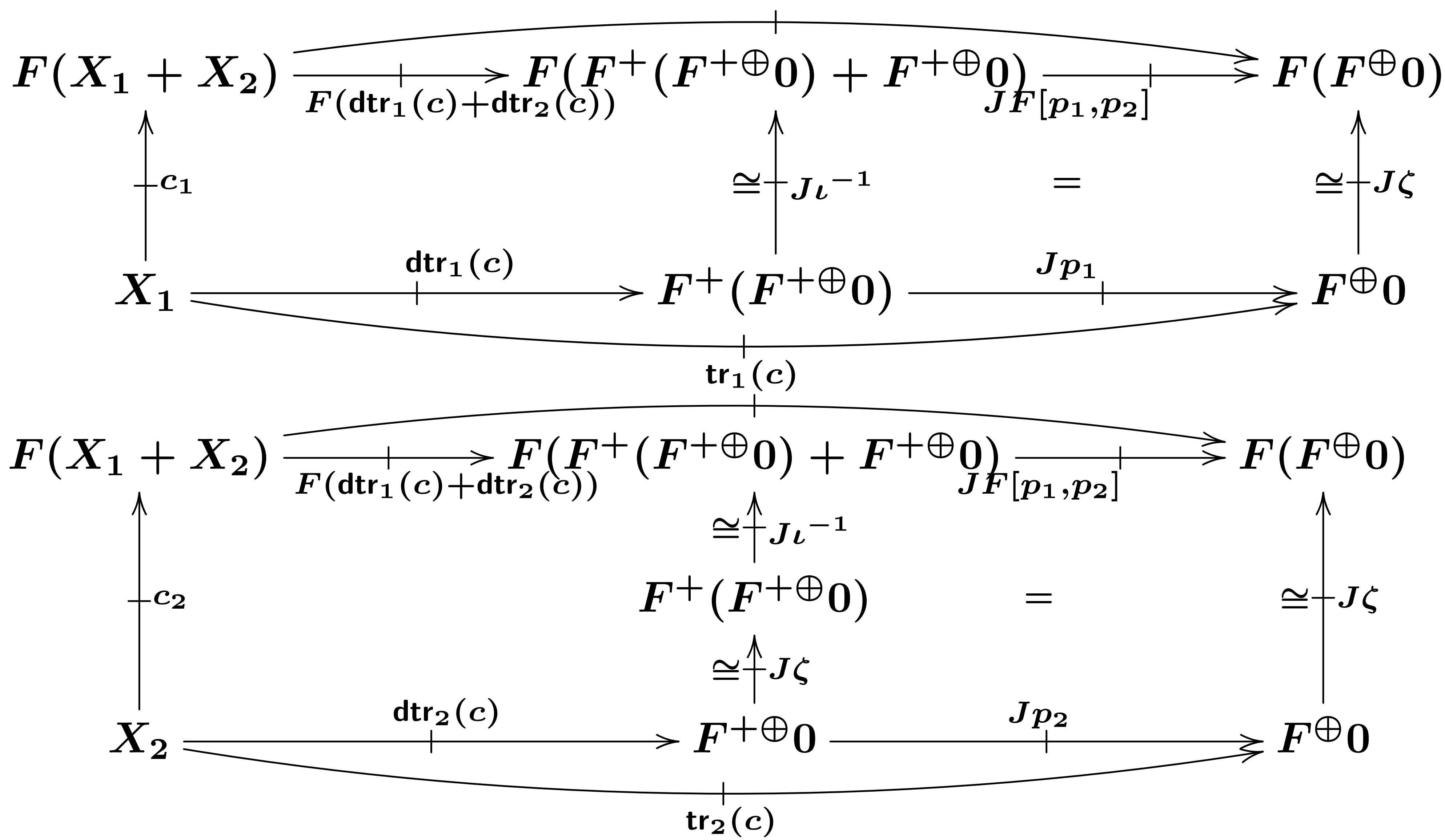
Coincidence

Theorem:

$$\mathsf{tr}^B_1(c) = Jp_1 \odot \mathsf{dtr}_1(c)$$
$$\mathsf{tr}^B_2(c) = Jp_2 \odot \mathsf{dtr}_2(c)$$

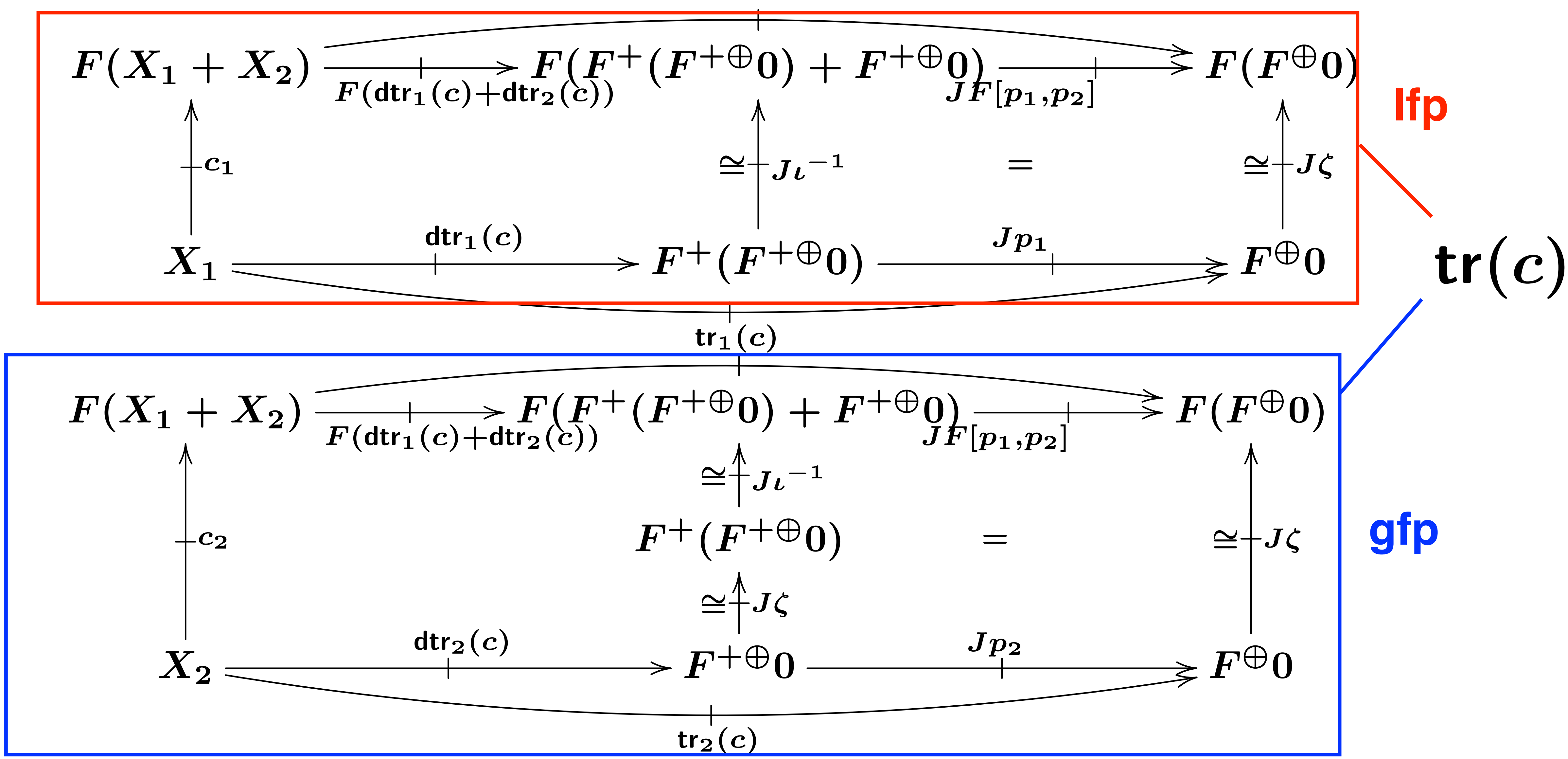
Coincidence

Theorem:
$$\mathbf{tr}_1^B(c) = Jp_1 \odot \mathbf{dtr}_1(c)$$
$$\mathbf{tr}_2^B(c) = Jp_2 \odot \mathbf{dtr}_2(c)$$



Coincidence

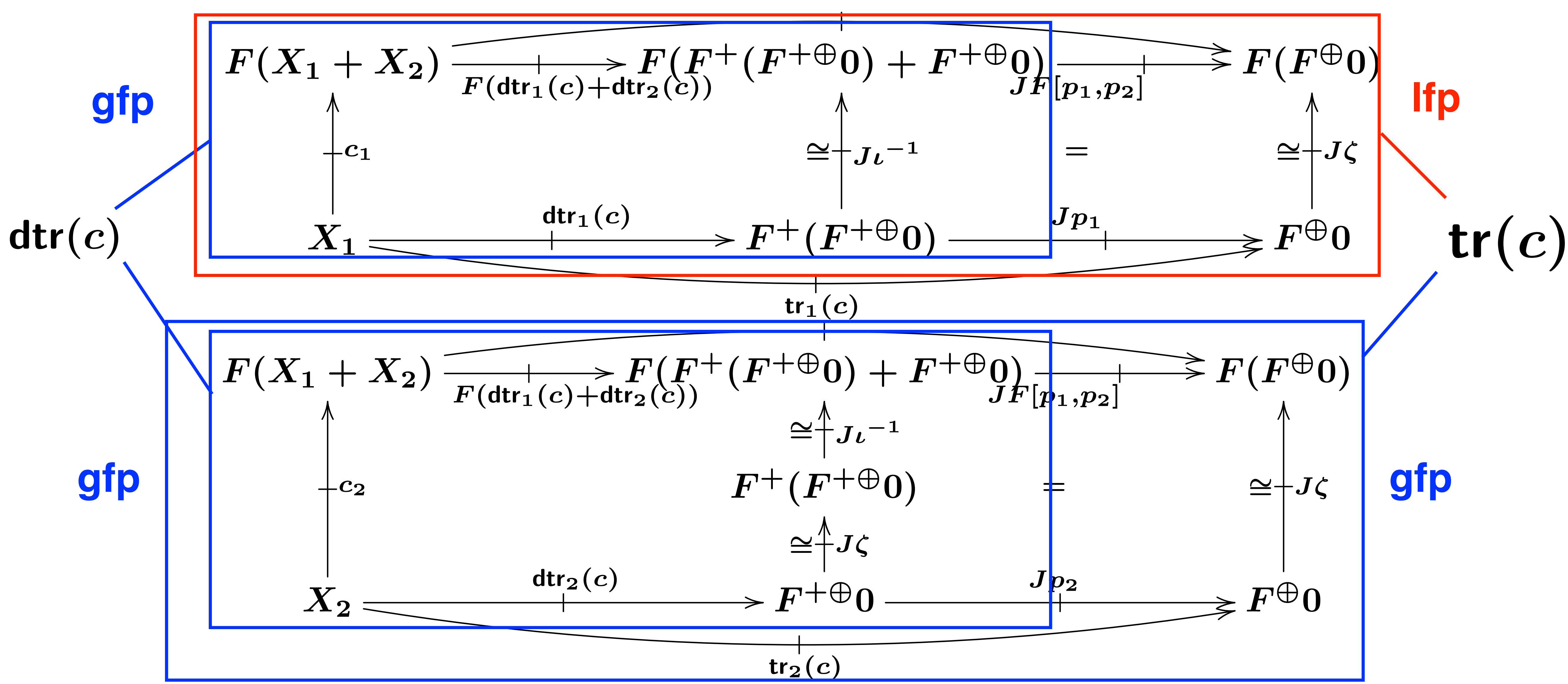
Theorem:
$$\mathbf{tr}_1^B(c) = Jp_1 \odot \mathbf{dtr}_1(c)$$
$$\mathbf{tr}_2^B(c) = Jp_2 \odot \mathbf{dtr}_2(c)$$



Coincidence

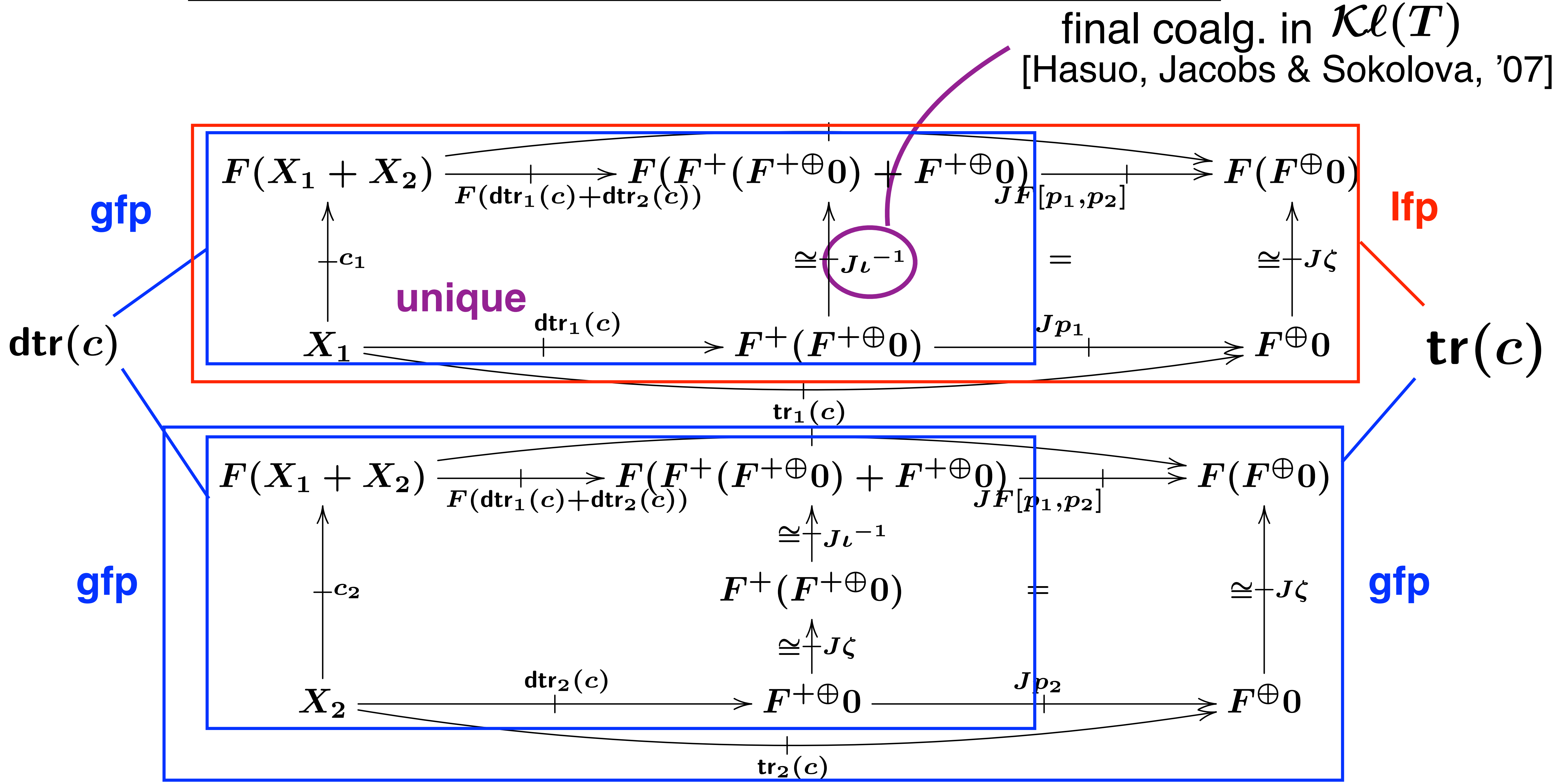
Theorem:

$$\mathbf{tr}_1^B(c) = Jp_1 \odot \mathbf{dtr}_1(c)$$
$$\mathbf{tr}_2^B(c) = Jp_2 \odot \mathbf{dtr}_2(c)$$



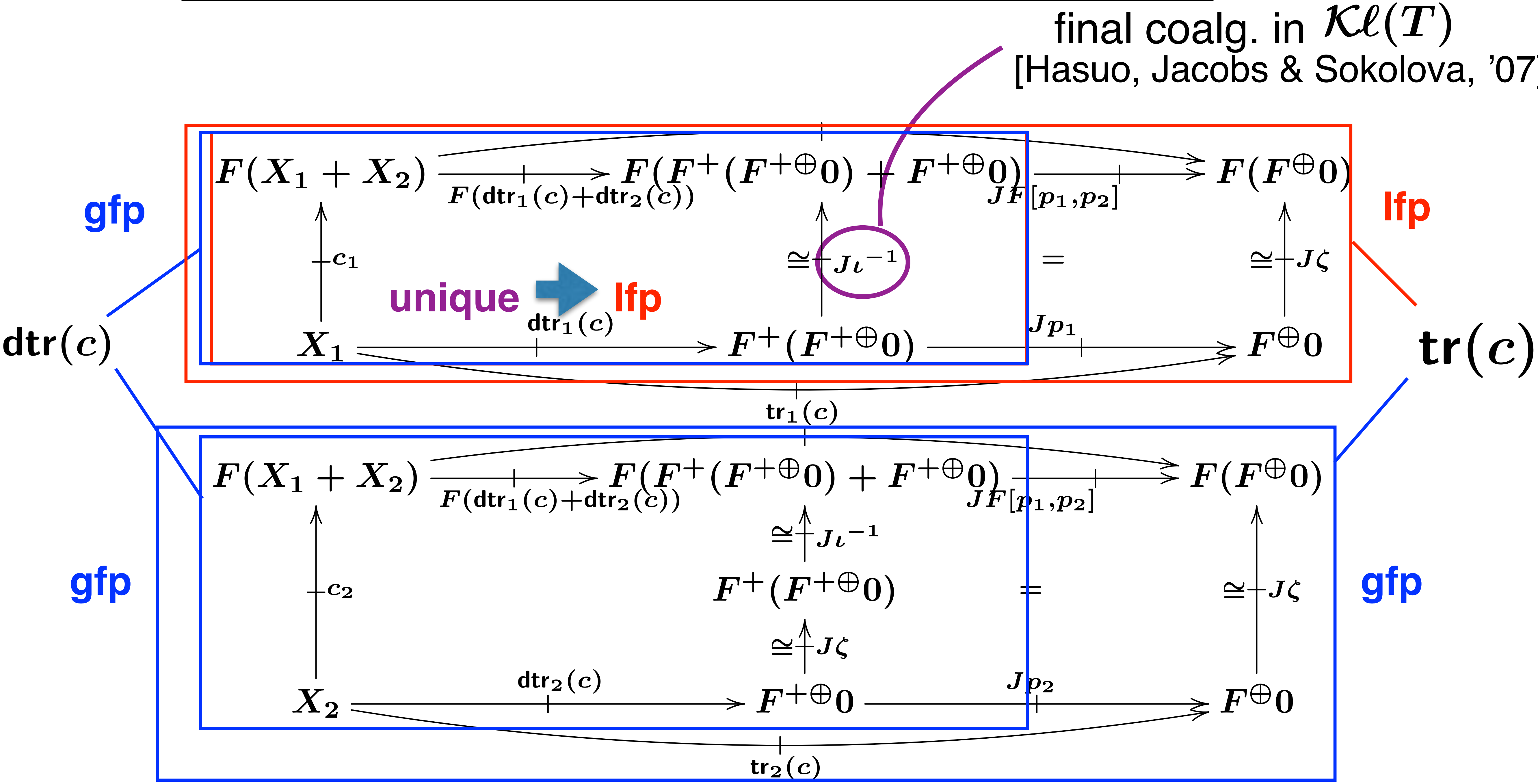
Coincidence

Theorem:
$$\mathbf{tr}_1^B(c) = Jp_1 \odot \mathbf{dtr}_1(c)$$
$$\mathbf{tr}_2^B(c) = Jp_2 \odot \mathbf{dtr}_2(c)$$



Coincidence

Theorem:
$$\mathbf{tr}_1^B(c) = Jp_1 \odot \mathbf{dtr}_1(c)$$
$$\mathbf{tr}_2^B(c) = Jp_2 \odot \mathbf{dtr}_2(c)$$



Outline

- Introduction
- Alternating Fixed Point of Functor
- Decorated Trace Semantics
- Trace Semantics vs. Decorated Trace Semantics
- Related Work & Conclusion

Related Work

- [Urabe, Shimizu & Hasuo, CONCUR '16]
  - Categorical characterization of ordinary trace semantics

Related Work

- [Urabe, Shimizu & Hasuo, CONCUR ’16]
  - Categorical characterization of ordinary trace semantics
- [Ciancia & Venema, CMCS ’12]
  - Categorical characterization of Büchi automata
  - Use of lasso characterization → restricted to finite states
  - Category $\mathbf{Sets^2}$
  - Captures bisimilarity

Related Work

- [Urabe, Shimizu & Hasuo, CONCUR '16]
  - Categorical characterization of ordinary trace semantics
- [Ciancia & Venema, CMCS '12]
  - Categorical characterization of Büchi automata
  - Use of lasso characterization → restricted to finite states
  - Category $\mathbf{Sets^2}$
  - Captures bisimilarity
- [Ghani, Hancock & Pattinson, LMCS '09]
  - Categorical characterization of continuous function by alternating fixed point of functors
  - $(A^\omega \Rightarrow B^\omega) \cong \mu X. \nu Y. (B \times X) + Y^A$
- [Adámek, Milius & Moss, JLAMP '18]
  - Sufficient cond. for an alternating fixed point of a functor exists

Conclusion

- Categorical datatype for characterizing the Büchi condition
  $F^{+}(F^{+\oplus}\mathbf{0})$ & $F^{+\oplus}\mathbf{0}$ as alternating fixed points of a functor

- Categorical definition of decorated trace semantics
  $\mathbf{dtr_1}(c) : X_1 \nrightarrow F^{+}(F^{+\oplus}\mathbf{0})$ and $\mathbf{dtr_2}(c) : X_2 \nrightarrow F^{+\oplus}\mathbf{0}$
  as greatest fixed points in homsets

- Trace semantics vs. decorated trace semantics
  $$\mathbf{tr}^B(c) = J[p_1, p_2] \odot \mathbf{dtr}(c)$$

- Extension to:

  parity condition, tree automata, probabilistic automata