Categorical Liveness Checking
by
Corecursive Algebras

Natsuki Urabe, Masaki Hara & Ichiro Hasuo
June 20, 2017

Motivation

ranking function

nondeterministic system

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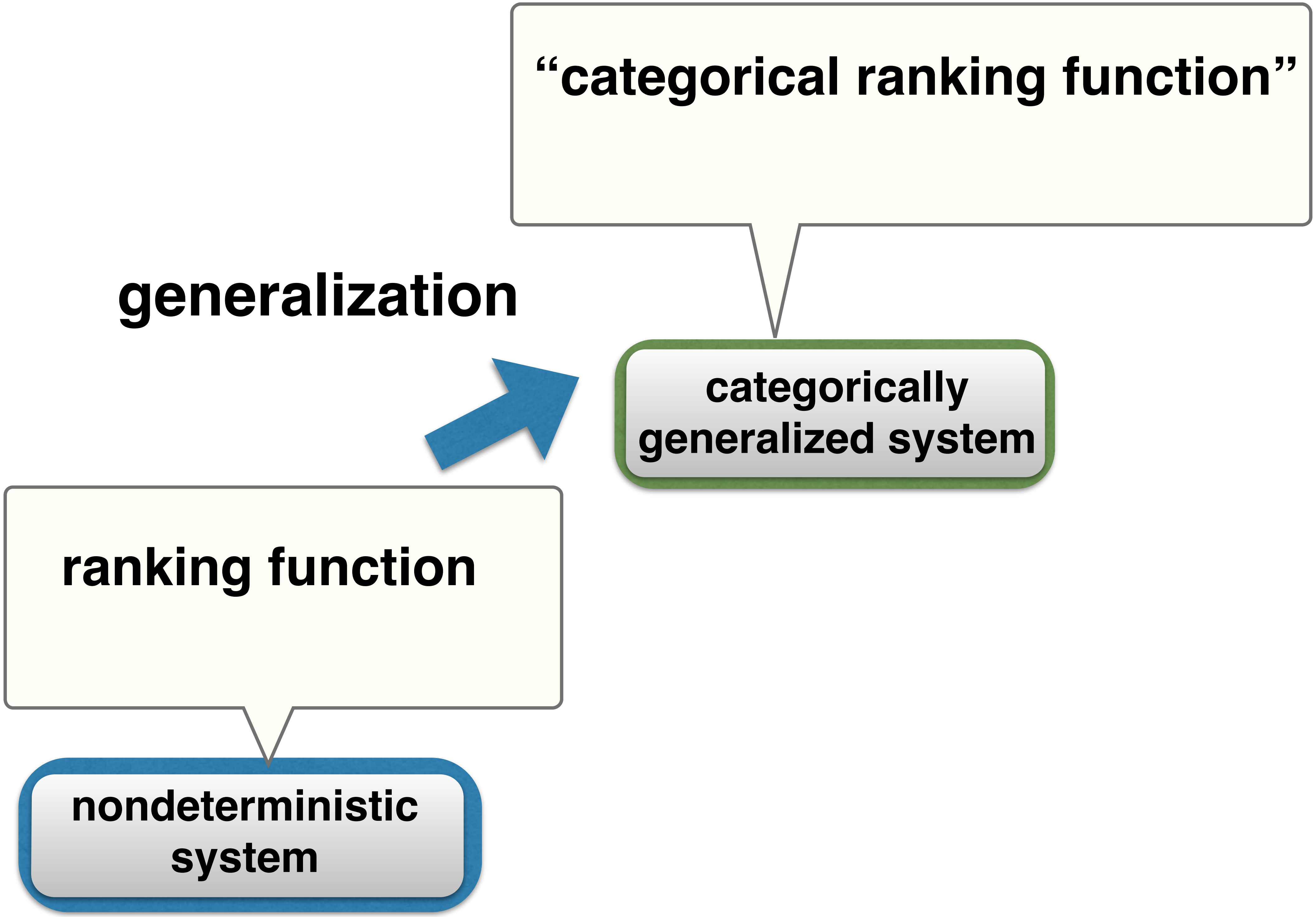

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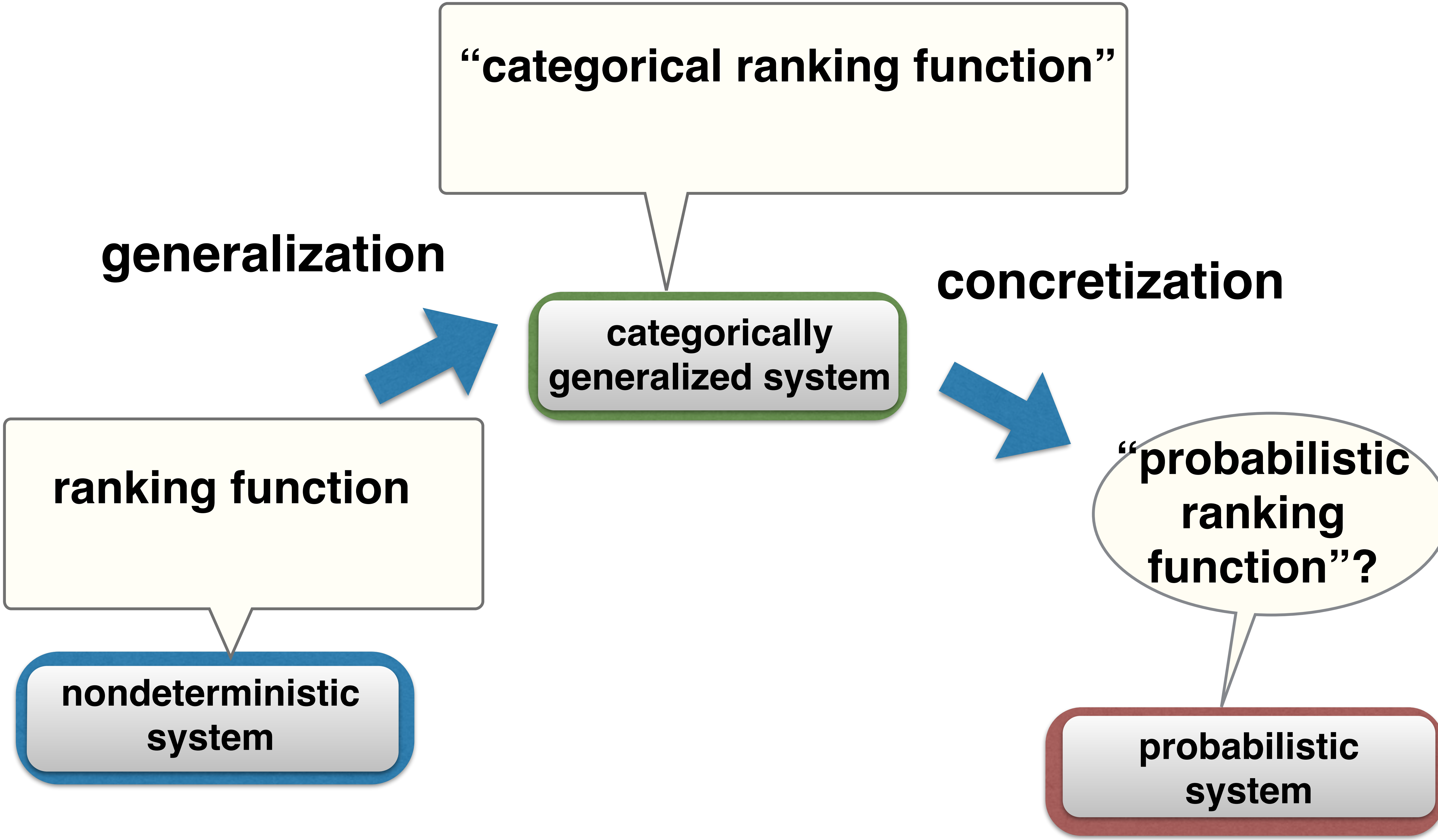
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concretization
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“probabilistic ranking function”?
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probabilistic system

Outline

- Preliminary
  - Ranking Function
  - Coalgebra and Coalgebra-Algebra Homomorphism
- Contribution
  - Coalgebraic Ranking Function
  - Probabilistic Ranking Function
- Conclusion and Future Work

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Ranking Function (see e.g. [Floyd, '67])

- A method for checking reachability

Def:

A function $\boldsymbol{b} : \boldsymbol{X} \to \mathbb{N}_\infty$ is a **ranking function** if:

$$\min_{\boldsymbol{x} \to \boldsymbol{x}'} \boldsymbol{b}(\boldsymbol{x}') + \mathbf{1} \leq \boldsymbol{b}(\boldsymbol{x})$$

for each nonaccepting state $\boldsymbol{x}$ ($\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$)

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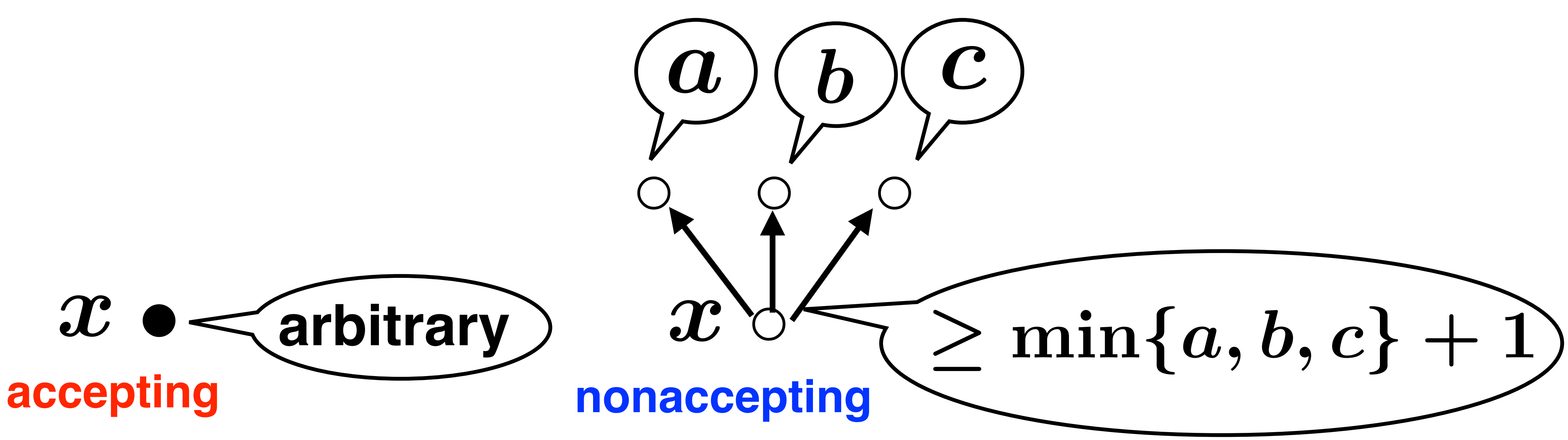

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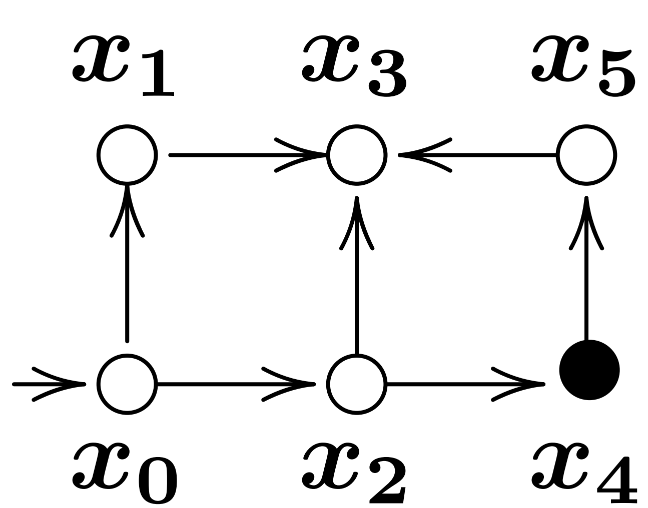

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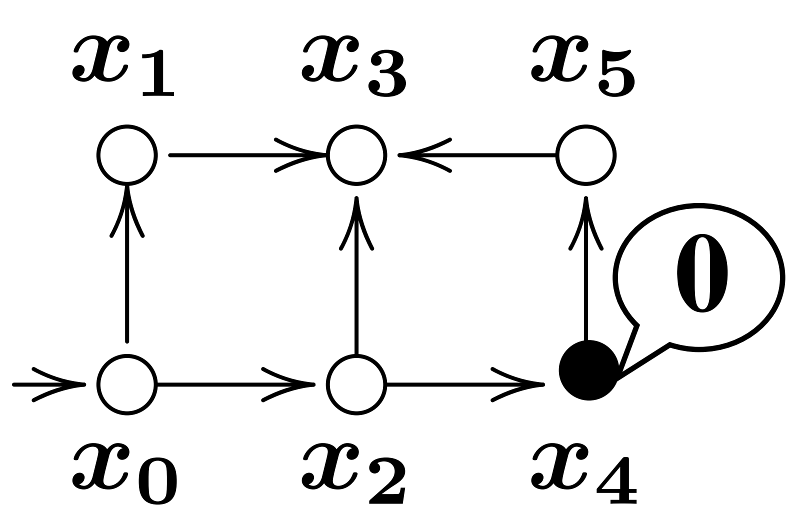

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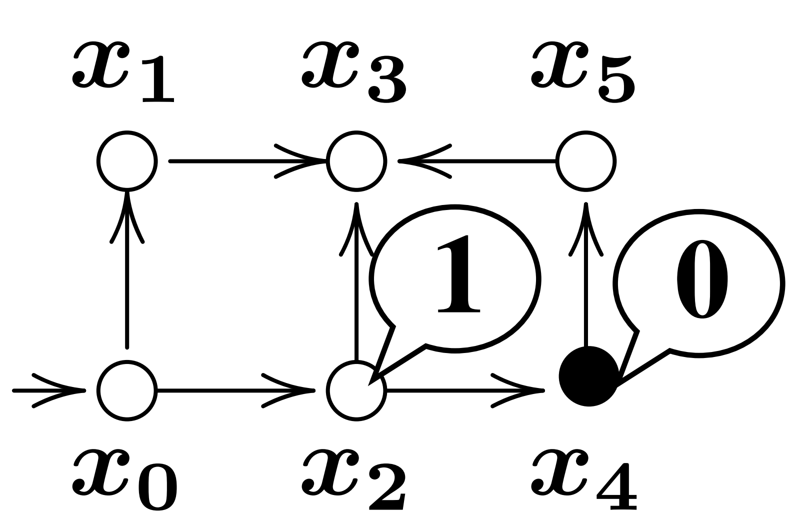

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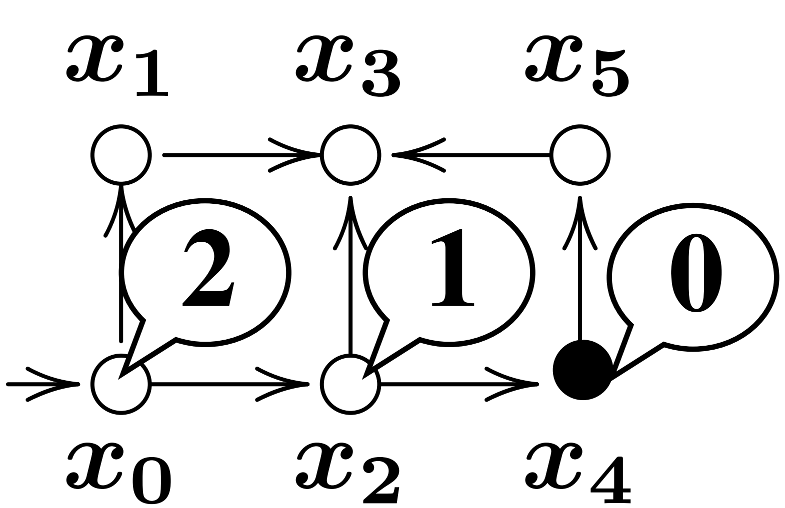

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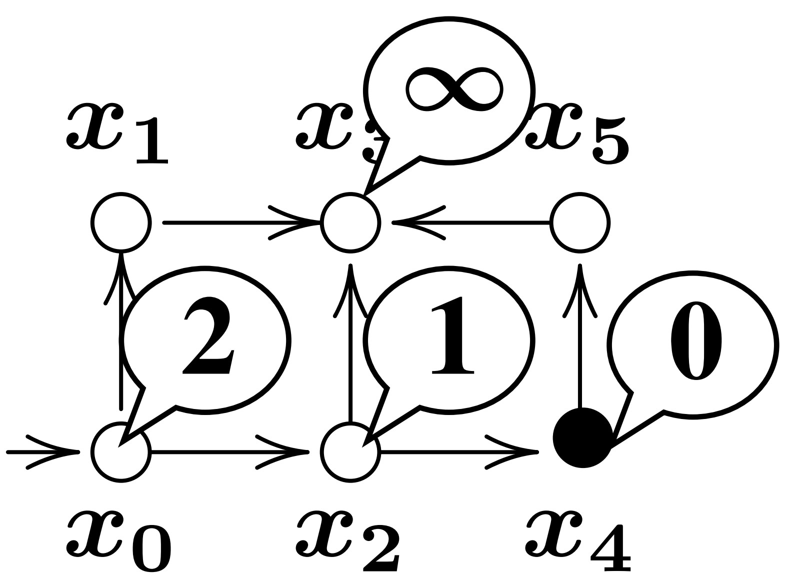

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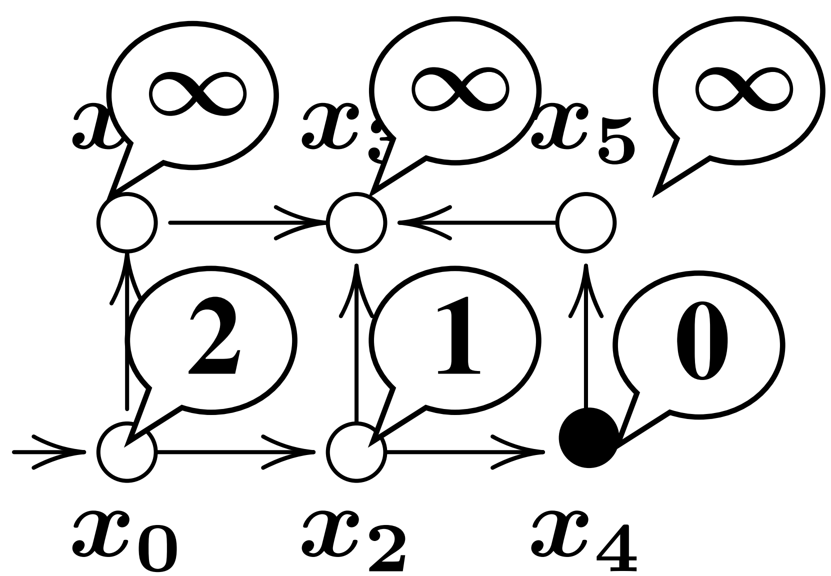
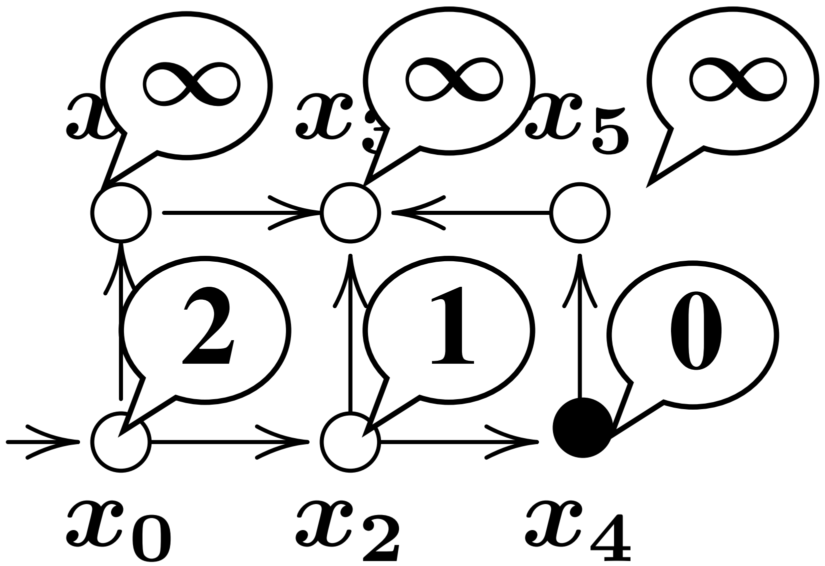


Soundness of Ranking Functions



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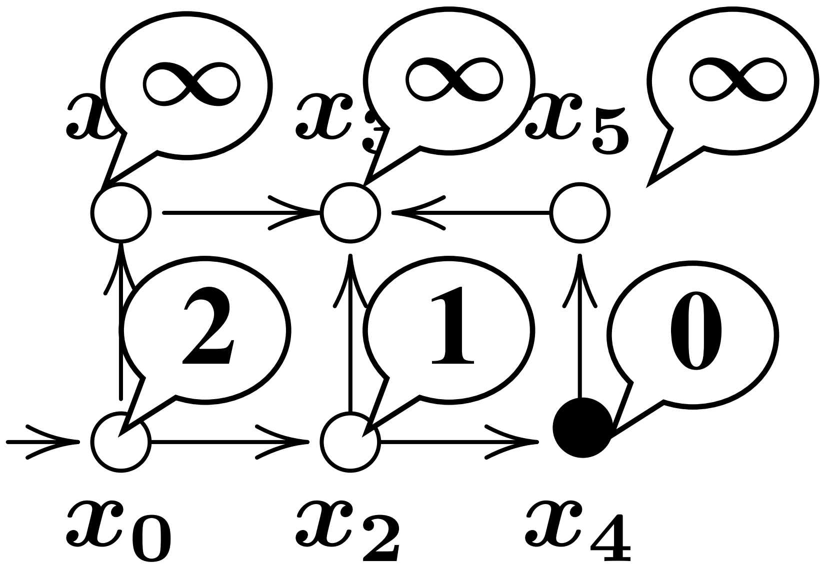

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$\boldsymbol{b}$ is a ranking function and $\boldsymbol{b}(\boldsymbol{x}) < \infty$ $\Rightarrow$ an accepting state is reachable from $\boldsymbol{x}$

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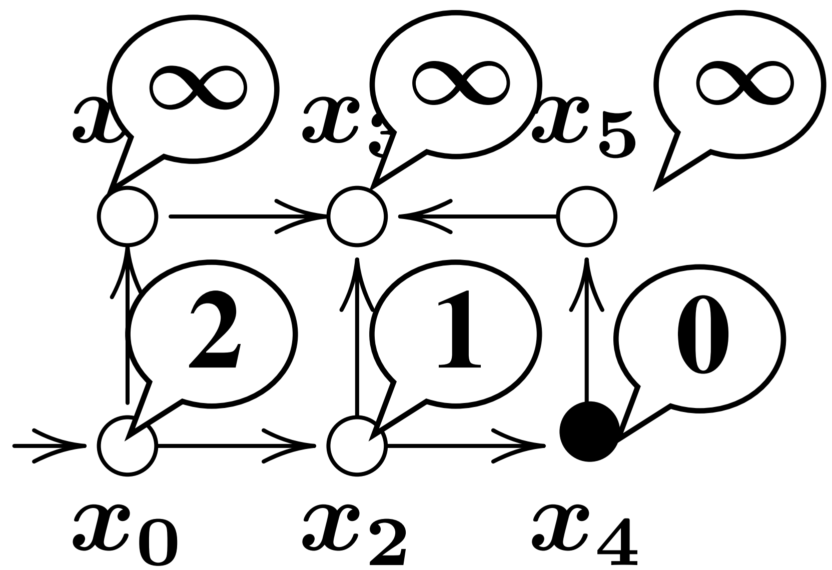

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➡ under-approximates the **reaching set**

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Towards Categorical Generalization

- Our first goal:
  categorical generalization of **ranking function**

- We have to categorically characterize:
  - a transition system
  - a reachability to accepting states

Coalgebra

- An ($\boldsymbol{F}$-)*coalgebra* is a function of the following form:

$$\boldsymbol{X \to FX}$$

$\boldsymbol{F}$: a functor

$$\begin{array}{rcl} \boldsymbol{X} & \longmapsto & \boldsymbol{FX} \\ (\boldsymbol{f : X \to Y}) & \longmapsto & (\boldsymbol{Ff : FX \to FY}) \end{array}$$

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- Coalgebras model **transition systems**

- Dual notion: *algebra*

$$FX \to X$$

- Algebras model **modalities**

Example I: Nondeterministic Transition System with Accepting States

$$c : X \to \mathcal{P}X \times \{0,1\} \text{ where } \mathcal{P}X = \{A \subseteq X\}$$

$$F = \mathcal{P}(\_) \times \{0,1\}$$

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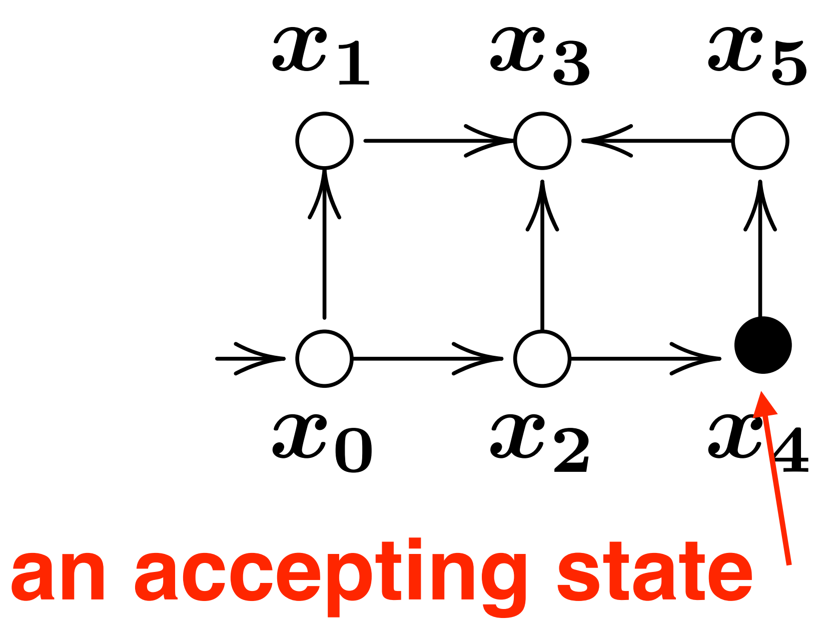

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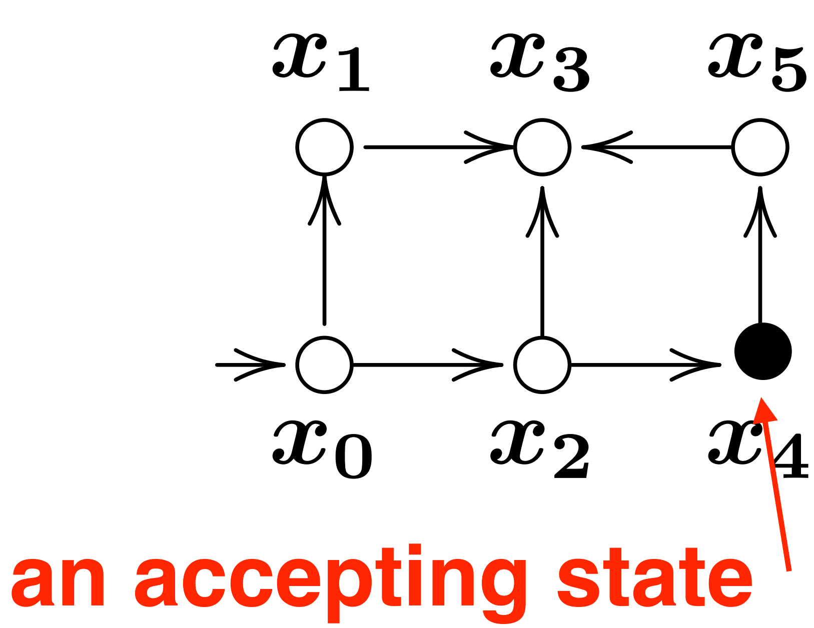

$$X = \{x_0, x_1, x_2, x_3, x_4, x_5\}$$

$$c : \begin{cases} x_0 \mapsto (\{x_1, x_2\}, 0) \\ x_1 \mapsto (\{x_3\}, 0) \\ \vdots \\ x_4 \mapsto (\{x_5\}, 1) \\ \vdots \end{cases}$$

Example II: Probabilistic Transition System with Accepting States

$c : X \to \mathcal{D}X \times \{0,1\}$ where

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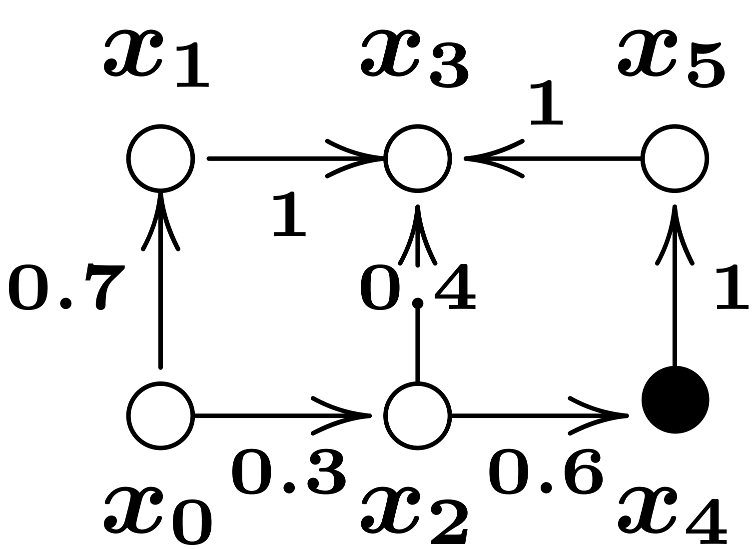

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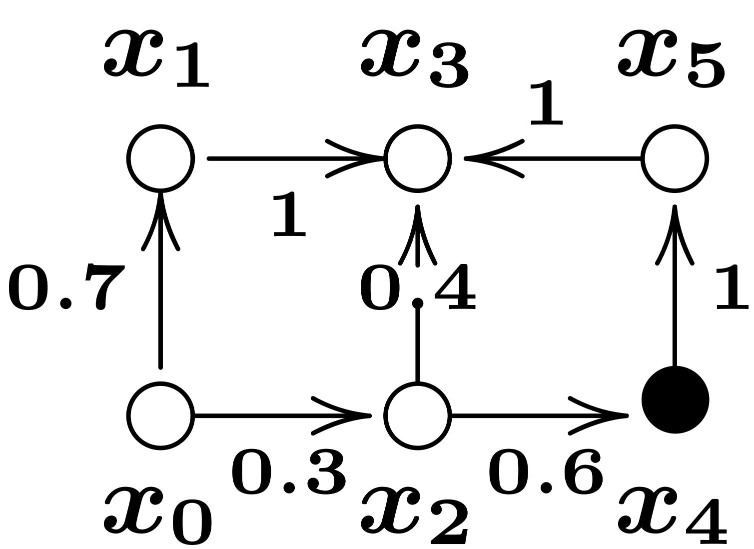

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Coalgebra-Algebra Homomorphism

Def:

A ***coalgebra-algebra homomorphism*** from $c : X \to FX$ to $\sigma : F\Omega \to \Omega$ is $f : X \to \Omega$ s.t. $\sigma \circ Ff \circ c = f$

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & F\Omega \\ {\scriptstyle c}\uparrow & = & \downarrow{\scriptstyle \sigma} \\ X & \xrightarrow{f} & \Omega \end{array}$$

- Especially, the **least coalgebra-algebra homomorphism** $[\![\mu\sigma]\!]_c : X \to \Omega$ captures **reachability**

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Example:

- For nondeterministic systems, $^\exists\sigma : F\{0,1\} \to \{0,1\}$ s.t. $[\![\mu\sigma]\!]_c(x) = 1 \iff \text{an accepting state is reachable from } x$
- For probabilistic systems, $^\exists\sigma : F[0,1] \to [0,1]$ s.t. $[\![\mu\sigma]\!]_c(x) = \mathbf{Prob}(\text{reach an accepting state from } x)$

Remark: Coalgebra-Algebra Homomorphism is Fixed Point

(see e.g. [Jacobs, LMCS 2015])

Def:

$$\Phi_{c,\sigma} : \left(X \xrightarrow{f} \Omega \right) \mapsto \left(X \xrightarrow{c} FX \xrightarrow{Ff} F\Omega \xrightarrow{\sigma} \Omega \right)$$

(predicate lifting + precomposing c => weakest precondition)

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- Reachability as the **least fixed point** (see e.g. [Baier & Katoen])

reachability as the
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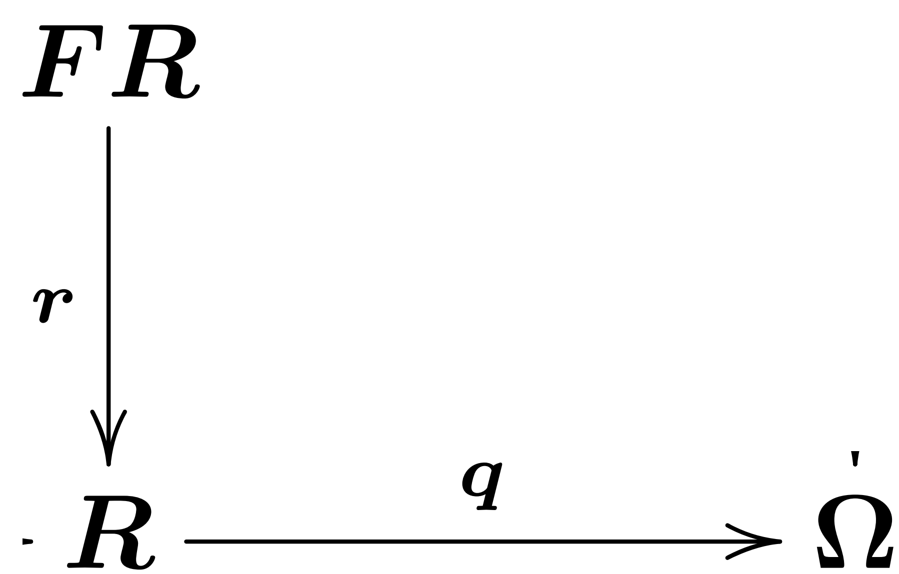

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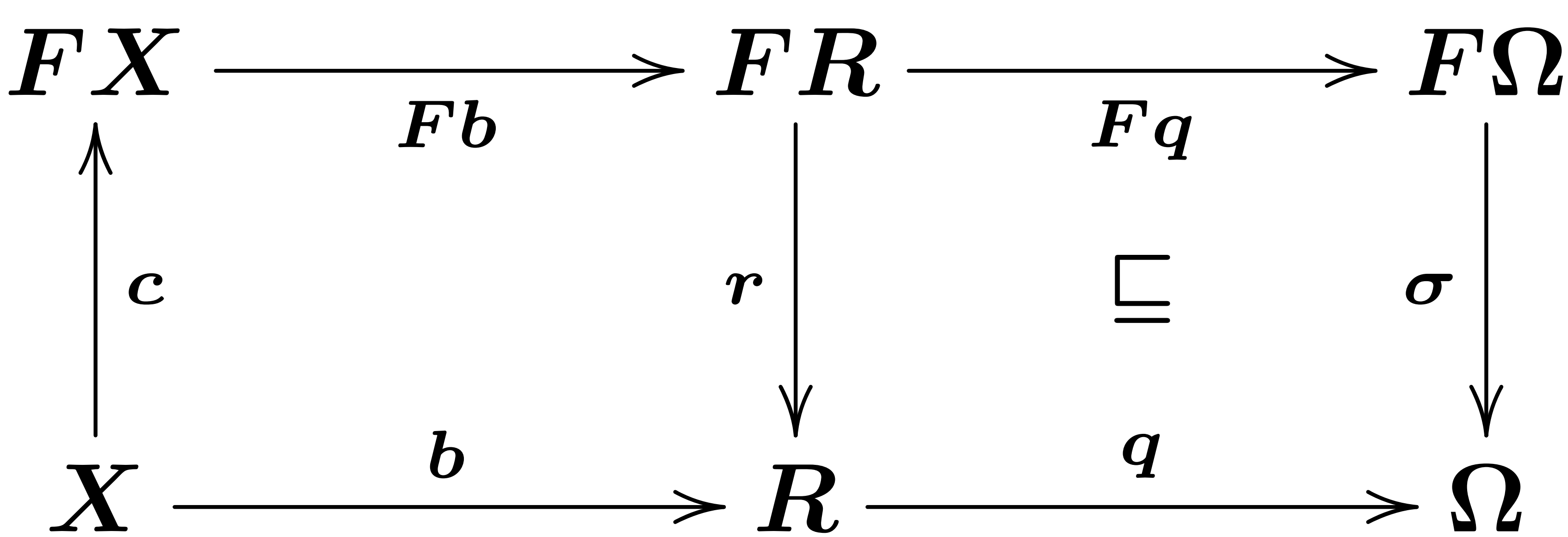

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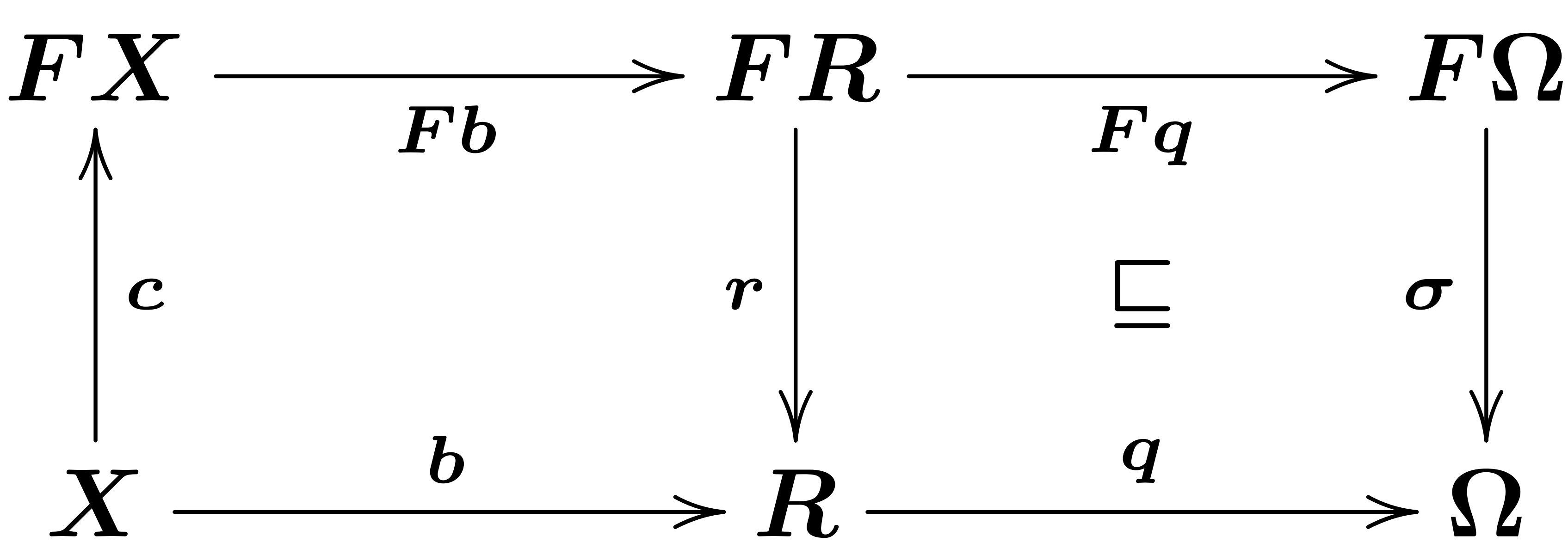

Corecursive Algebra

Def:

An algebra $r : FR \to R$ is **corecursive** if for all coalgebra $c : X \to FX$, a coalgebra-algebra homomorphism from c to r uniquely exists.

$$\begin{array}{ccc} FX & \xrightarrow{F(\!|r|\!)_c} & FR \\ c \uparrow & = & \downarrow r \\ X & \xrightarrow{(\!|r|\!)_c} & R \end{array}$$

- It has been used to ensure **productivity** of general structured corecursion [Capretta et al., SBMF '09]
- We use it to ensure **reachability** (termination)

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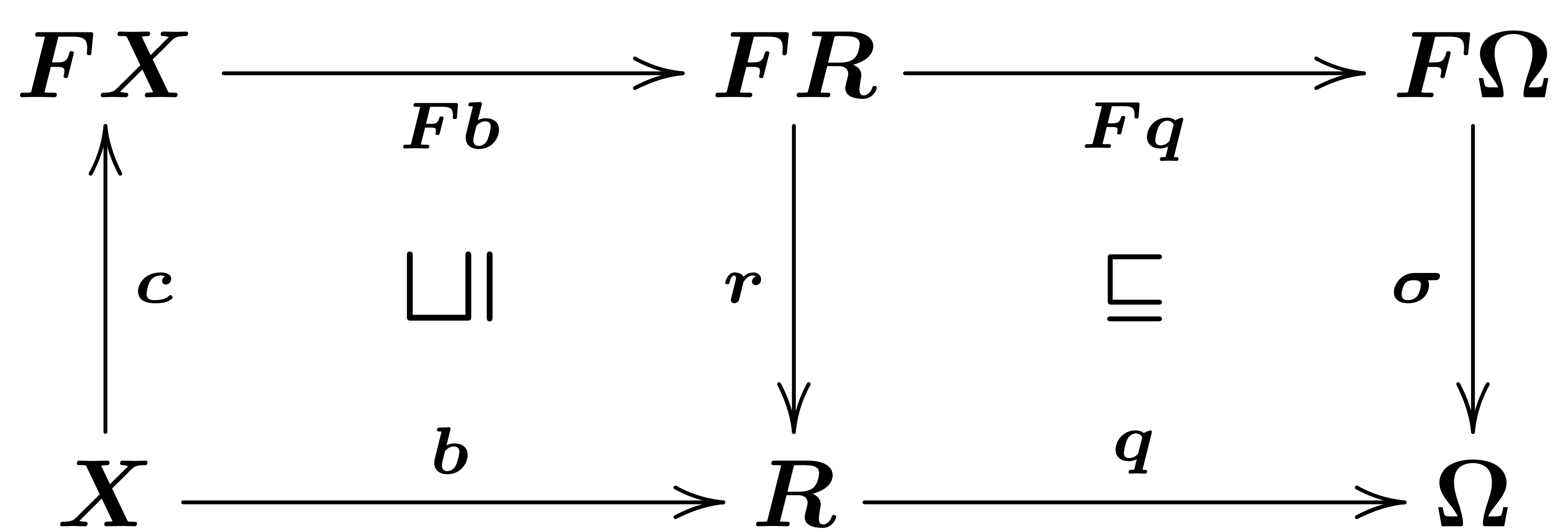

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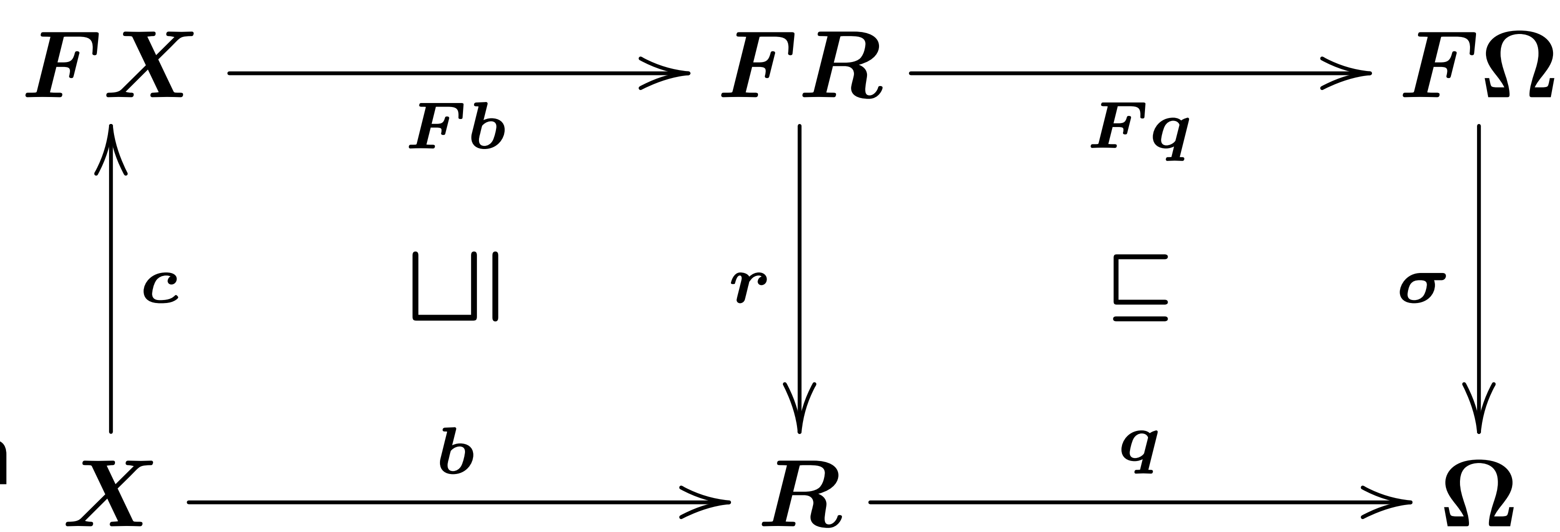
fix a ranking domain

⬇

notion of ranking function



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Categorical Soundness Theorem

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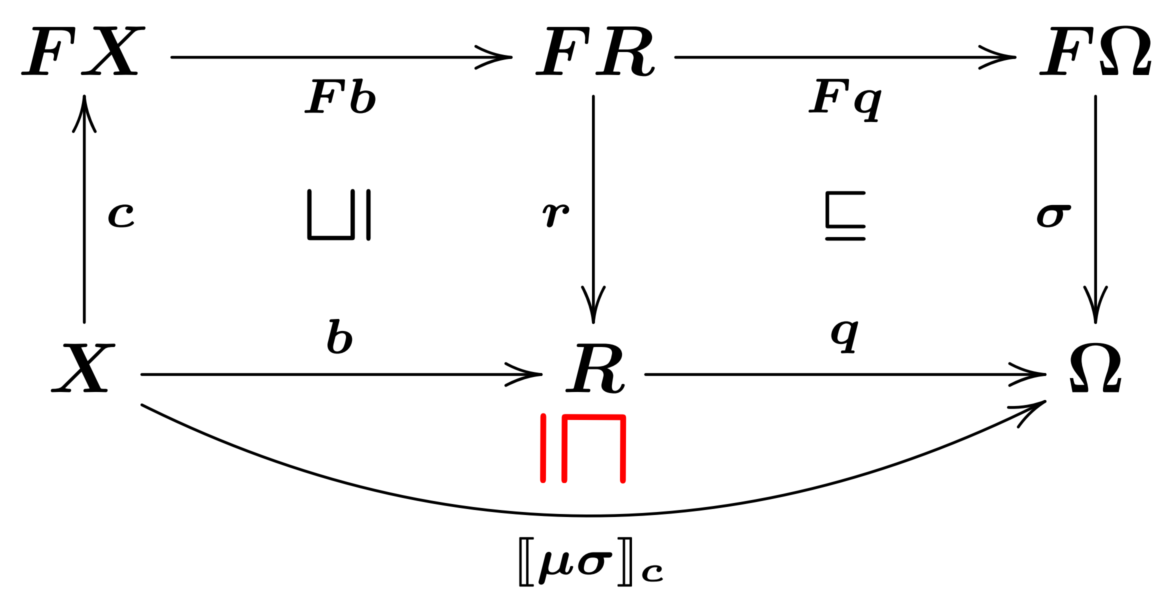

Thm (soundness):

$\boldsymbol{b}$ is a ranking arrow wrt. $(\boldsymbol{r}, \boldsymbol{q}, \sqsubseteq_{\boldsymbol{R}})$ $\Longrightarrow$ $\boldsymbol{q} \circ \boldsymbol{b} \sqsubseteq [\![\boldsymbol{\mu\sigma}]\!]_{\boldsymbol{c}}$

Intuition behind Corecursiveness

$$\begin{array}{ccc} FX & \xrightarrow{F[\![\mu\sigma]\!]_c} & F\Omega \\ {\scriptstyle c}\uparrow & =_\mu & \downarrow{\scriptstyle \sigma} \\ X & \xrightarrow{[\![\mu\sigma]\!]_c} & \Omega \end{array}$$

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  under-approximate the **least** **fixed point**

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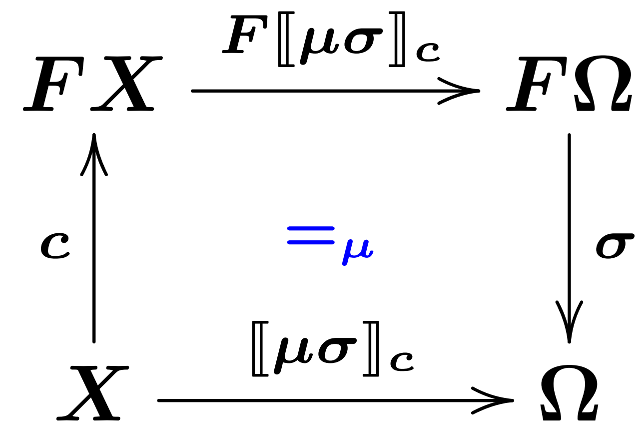

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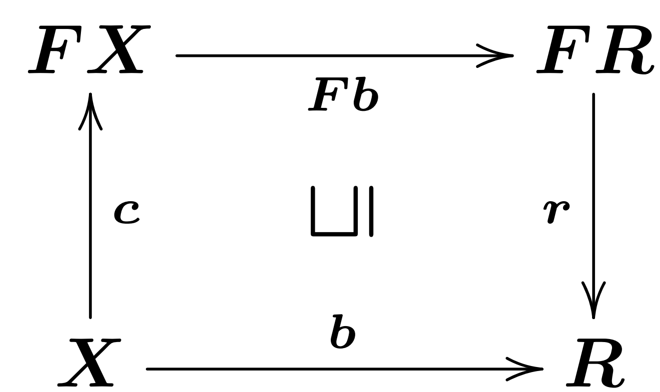

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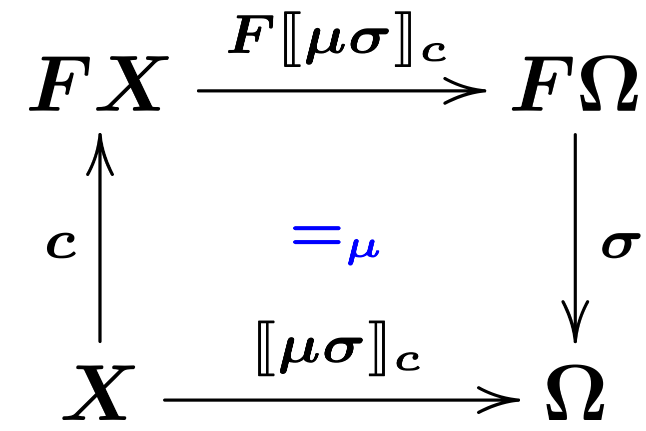

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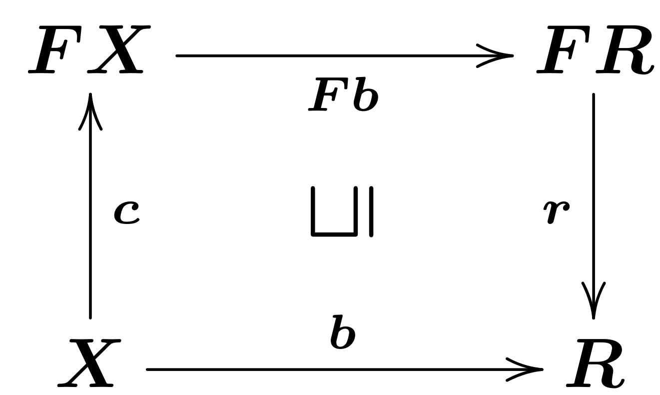

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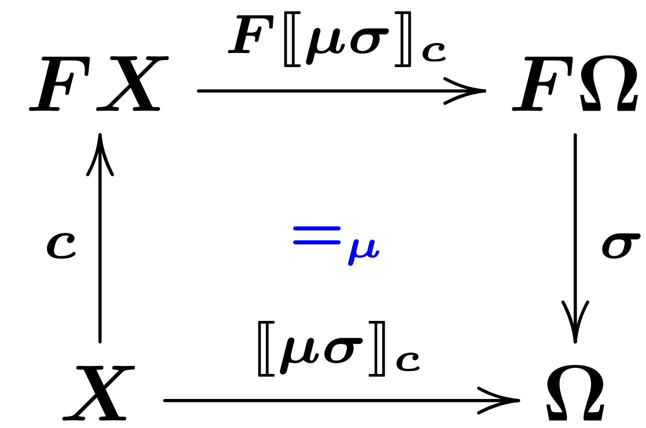

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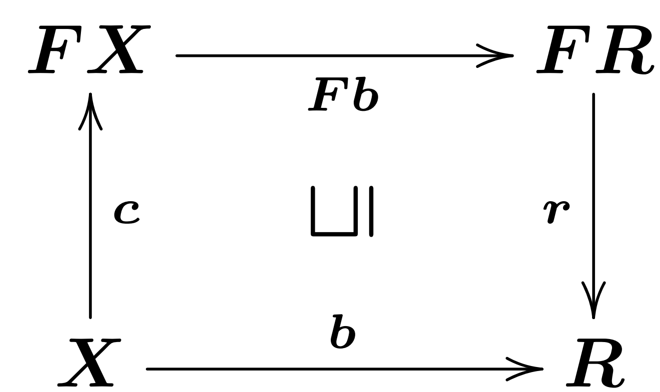

- Ranking arrow is a **post-fixed point**

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we collapse the least and the greatest fixed points
(i.e. unique coalgebra-algebra homomorphism)

Outline

- Preliminary
  - Ranking Function
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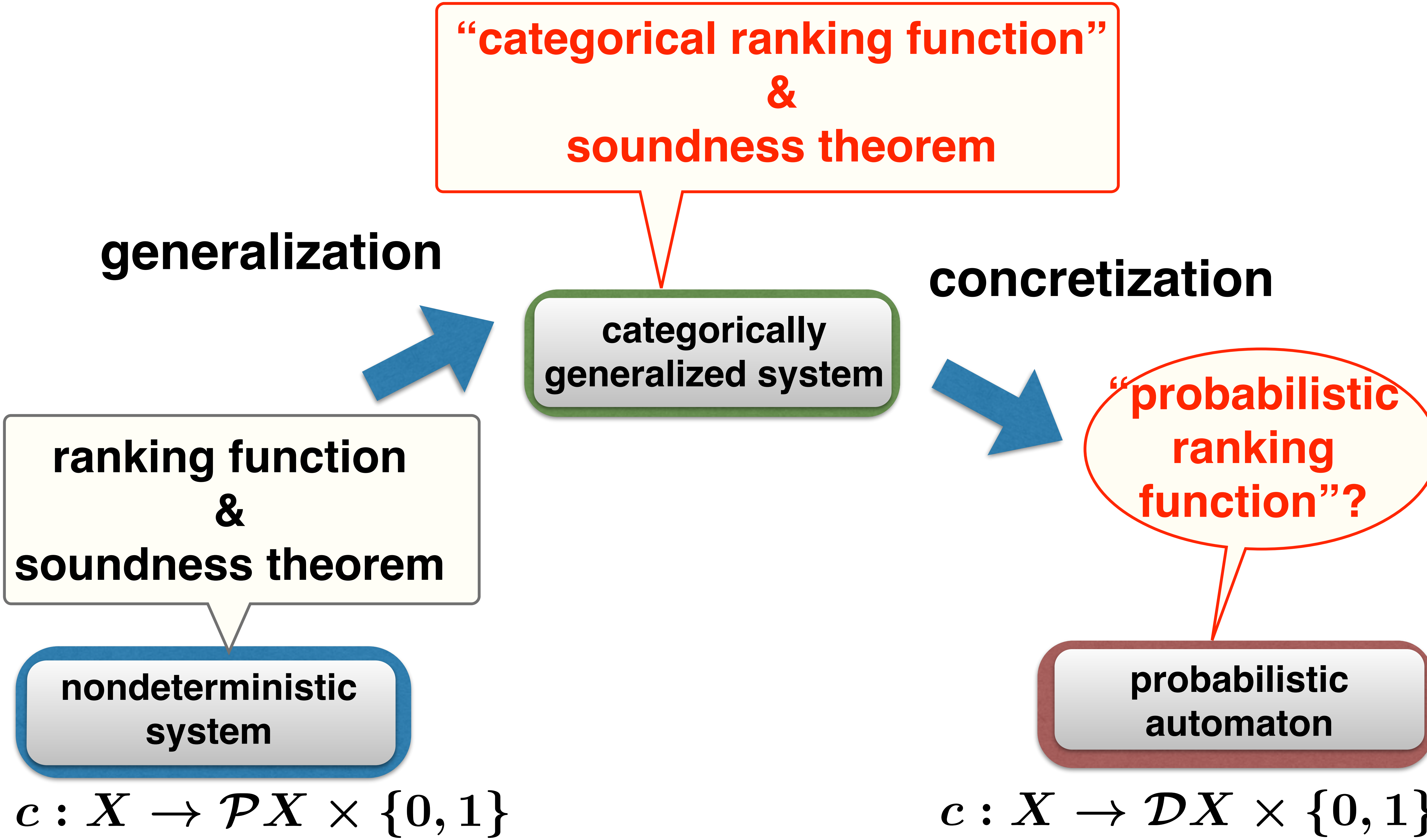
Concretization



Ranking Supermartingale [Chakarov et al., '13]

- A method for checking almost-sure reachability on probabilistic systems

Def:

A function $b : X \to [0, \infty]$ is a **ranking supermartingale** if:

$$\sum_{x' \in X} \mathbf{Prob}(x \to x') \cdot b(x') + 1 \leq b(x)$$

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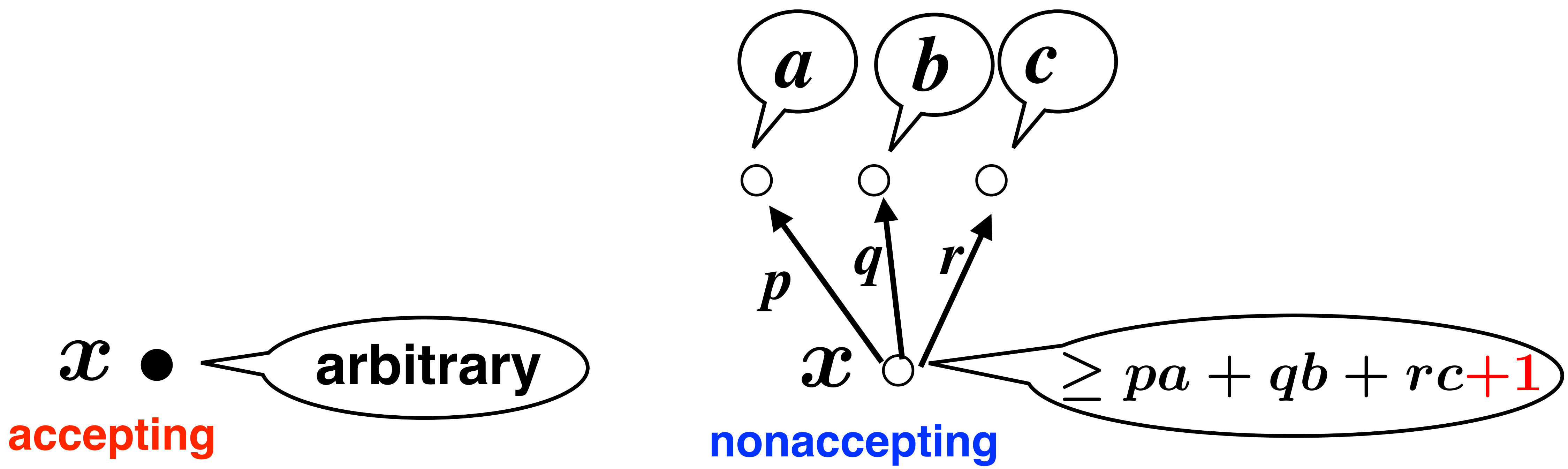

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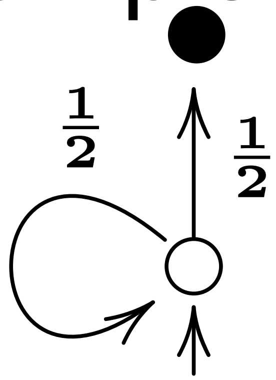

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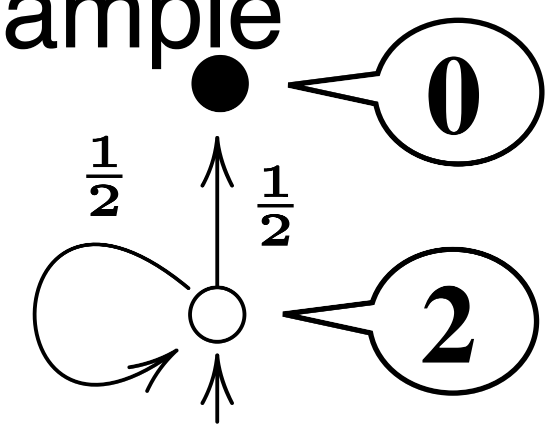

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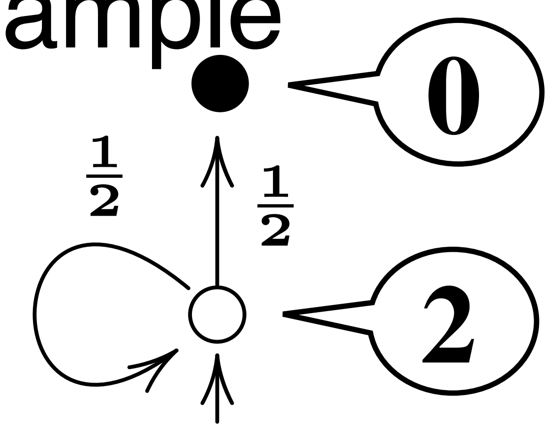

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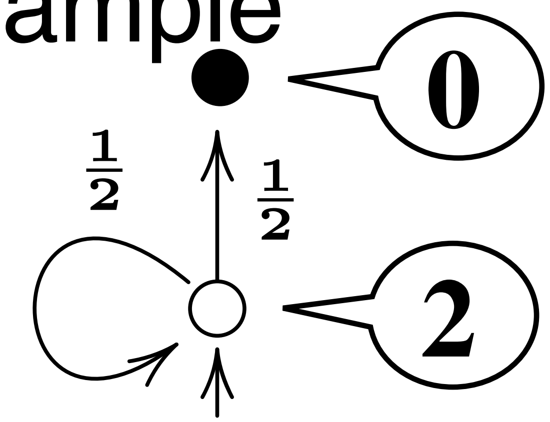

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b is a ranking supermartingale and $b(x) < \infty$ $\Longrightarrow$ $\mathbf{Pr}\left(\begin{array}{r}\text{an accepting state} \\ \text{is reached}\end{array}\right) = 1$

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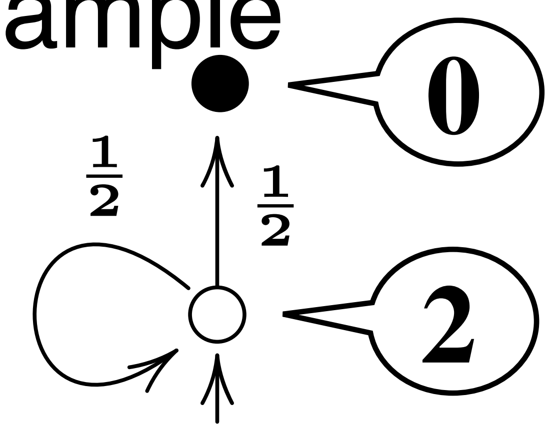

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- Ranking supermartingale resembles to ranking function

  ➡ a ranking domain for ranking supermartingale exists?

Problem and Next Step

- We couldn't find a ranking domain $(\boldsymbol{r}, \boldsymbol{q}, \sqsubseteq_{\boldsymbol{R}})$ s.t.

$$\boldsymbol{b} \text{ is a ranking supermartingale} \iff \begin{array}{l}\boldsymbol{b} \text{ is a ranking arrow} \\ \text{wrt. } (\boldsymbol{r}, \boldsymbol{q}, \sqsubseteq_{\boldsymbol{R}})\end{array}$$

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➡ We decided to **give up** describing ranking supermartingales

- Instead, we found **two ranking domains** for probabilistic systems

➡ They induces **new** definitions of ranking function (to the best of our knowledge)

Distribution-valued Ranking Supermartingale

Def:

For a probabilistic transition system, a function $\boldsymbol{b} : \boldsymbol{X} \to \mathcal{D}\mathbb{N}_\infty$ is a **distribution-valued ranking function** if:

$$\forall \boldsymbol{a} \in \mathbb{N}_\infty . \left(\sum_{\boldsymbol{x}' \in \boldsymbol{X}} \mathbf{Pr}(\boldsymbol{x} \to \boldsymbol{x}') \cdot \boldsymbol{b}(\boldsymbol{x}') \right) \big([\mathbf{0}, \boldsymbol{a} - \mathbf{1}] \big) \;\geq\; \boldsymbol{b}(\boldsymbol{x}) \big([\mathbf{0}, \boldsymbol{a}] \big)$$

By soundness of (categorical) ranking arrows,

Thm:

$$\boldsymbol{b}(\boldsymbol{x}) \big([\mathbf{0}, \infty) \big) \;\leq\; \mathbf{Pr} \left(\begin{array}{c} \text{an accepting state} \\ \text{is reached from } \boldsymbol{x} \end{array} \right)$$

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Thm:

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→ **Quantitative reasoning**

Scaled Noncounting Ranking Supermartingale

Def:

For $\gamma \in (0,1)$, a function $b : X \to [0,1]$ is a γ **-scaled noncounting ranking function** if:

$$\gamma \cdot \sum_{x' \in X} \Pr(x \to x') \cdot b(x') \;\geq\; b(x)$$

By soundness of (categorical) ranking arrows,

Thm:

$$b(x) \;\leq\; \Pr\left(\begin{array}{c}\text{an accepting state}\\ \text{is reached from } x\end{array}\right)$$

→ **Quantitative reasoning**

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Conclusion

- Categorical generalization of **ranking function**
  - **Post-fixed point + corecursive algebra**
  - (Categorical) soundness theorem

- Concretization for probabilistic systems:
  - failed to describe ranking supermartingale
  - induced two new notions for liveness checking

Future Work

- Extension to Büchi/parity systems
- Implementation