Coalgebraic Approach to Equivalences of Quantum Systems

Hiroshi Ogawa
(Hasuo lab.)

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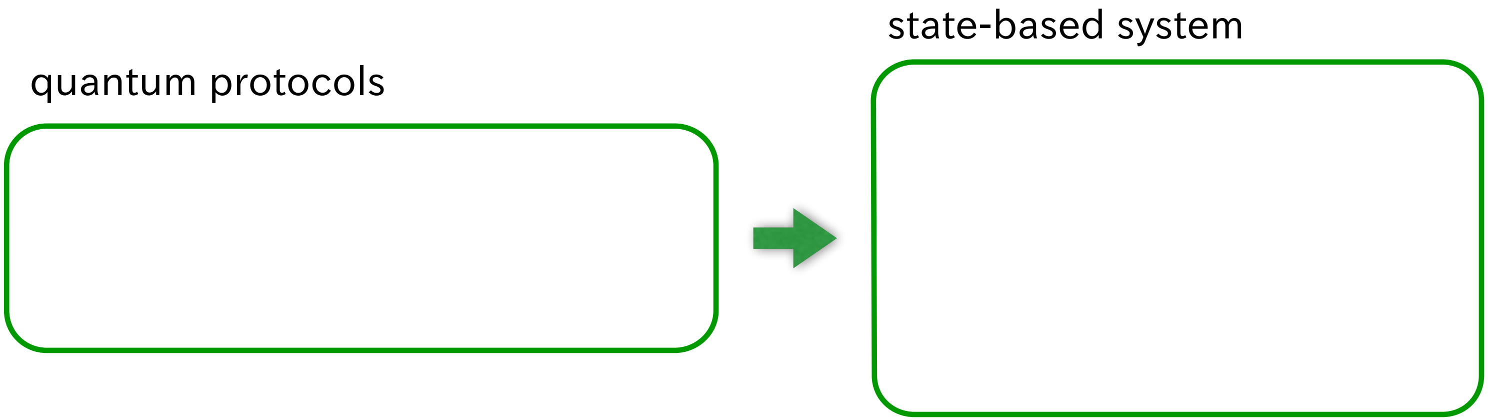

quantum protocols

state-based system

Ogawa (Tokyo)

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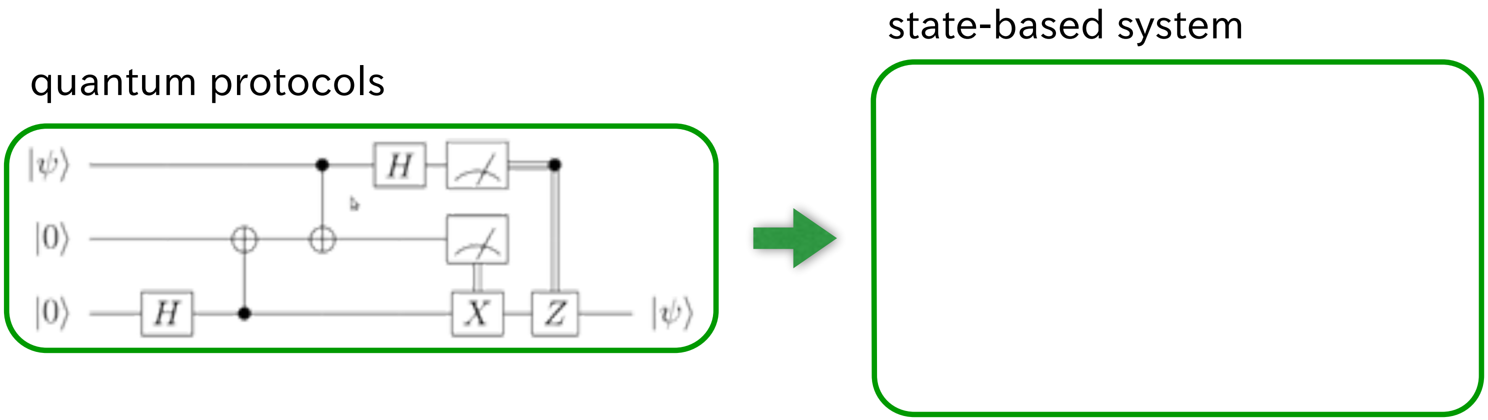

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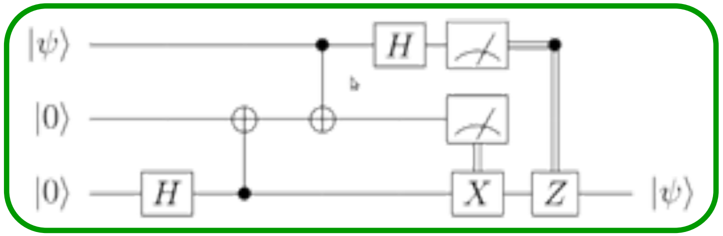

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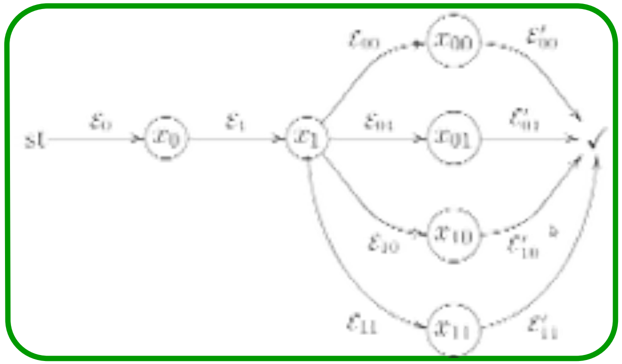

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Coalgebraic Approach to Equivalences of Quantum Systems

$$\begin{tikzcd} \mathcal{Q}X \\ X \arrow[u] \end{tikzcd}$$

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$$\begin{tikzcd} \mathcal{Q}(1+\Sigma\times X) \\ X \arrow[u] \end{tikzcd}$$

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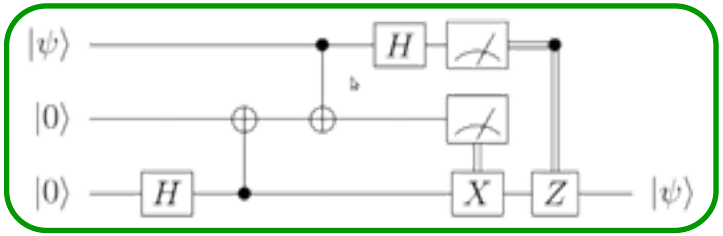

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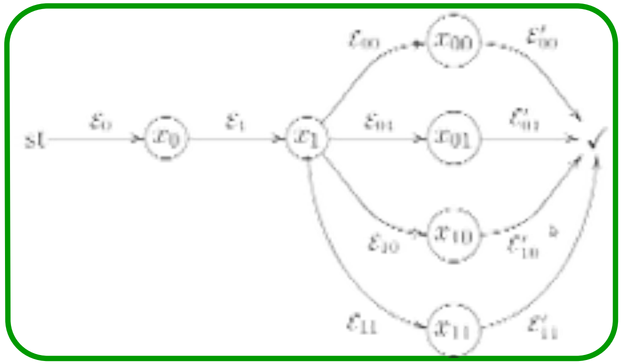

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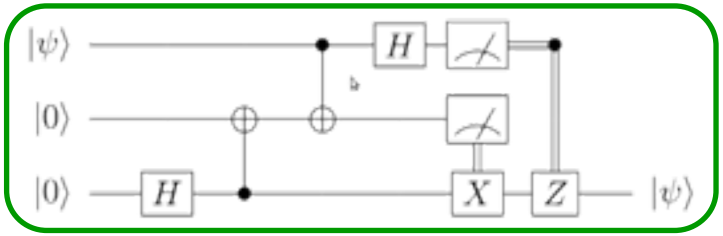

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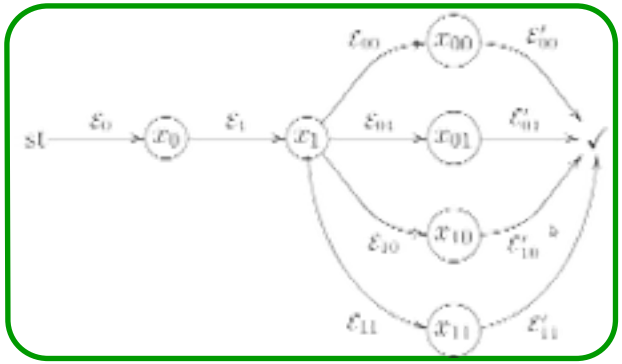

Quantum Branching Monad [Hasuo, Hoshino LICS'11]

$$QX = \{\phi : X \to \prod_{m,n} QO_{m,n} \mid \text{trace condition (1)}\}$$

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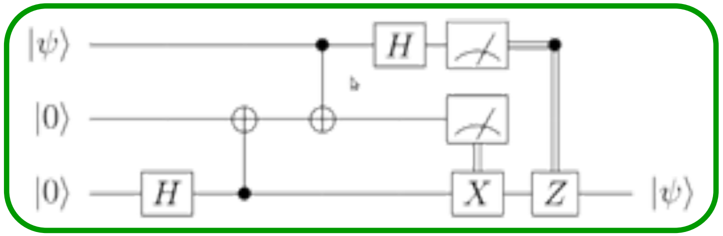

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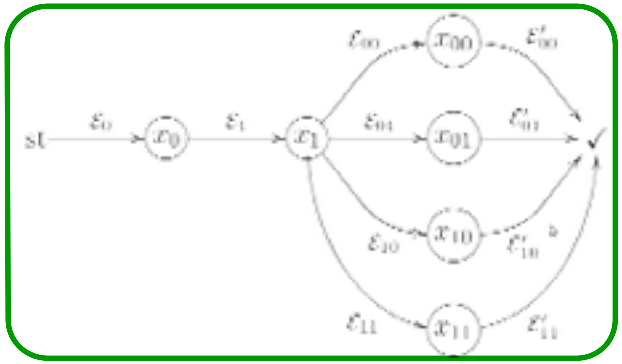

Ogawa (Tokyo)

Why Coalgebra ?

- coalgebra is a general theory for state-based systems
  - cover nondet. or prob. systems

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$$\begin{tikzcd} \overline{F}X \arrow[r, dashed, "\overline{F}(\mathrm{tr}_c)"] & \overline{F}A \\ X \arrow[u, "c"] \arrow[r, dashed, "\mathrm{tr}_c"'] & A \arrow[u, "\cong"'] \end{tikzcd}$$

- apply existing coalgebra theory to monad Q

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- trace semantics

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- bisimilarity, behavioral equivalence

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FX & & FY \\
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FX \arrow[dr, "Fh"] & & FY \arrow[dl, "Fk"'] \\
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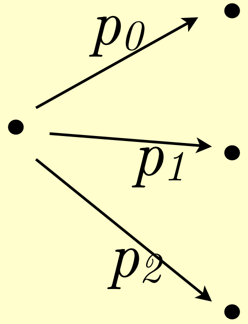
$$\begin{tikzcd}
\mathbf{Sets}^{\mathrm{op}} \arrow[loop left, "\nu_O"] \arrow[r, bend left, "\mathcal{P}"] \arrow[r, phantom, "\top"] & \mathbf{MSL} \arrow[l, bend left, "\mathcal{F}"] \arrow[loop right, "\kappa_O"]
\end{tikzcd}$$

Distribution Monad vs. Quantum Branching Monad

$$\begin{tikzcd}\mathcal{D}X \\ X \arrow[u]\end{tikzcd}$$



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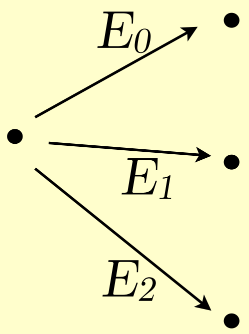
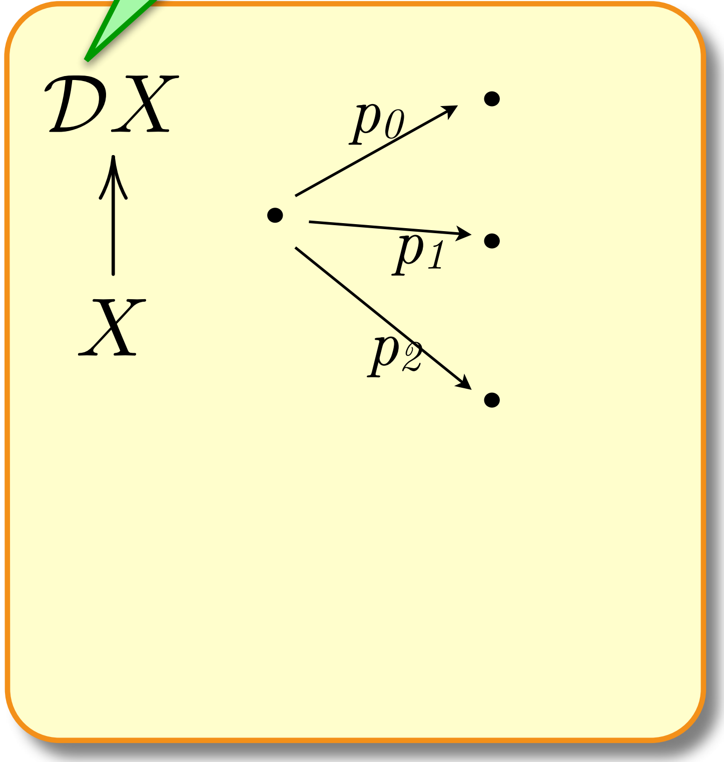
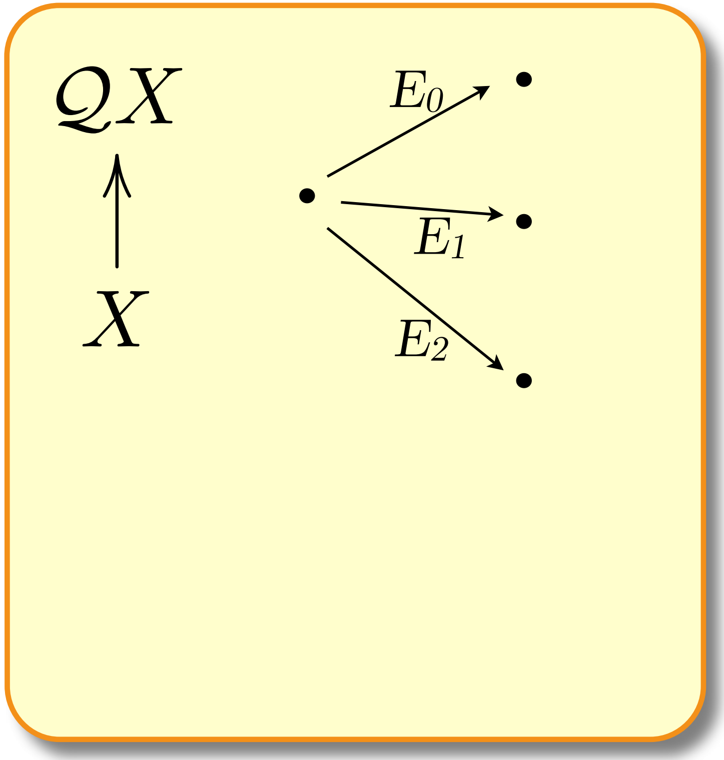


Distribution Monad vs. Quantum Branching Monad

$$\mathcal{D}X := \{\phi : X \to [0,1] \mid \sum_{x\in X} \phi(x) \leq 1\}$$





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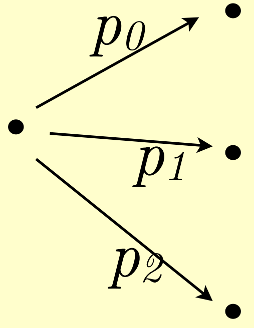

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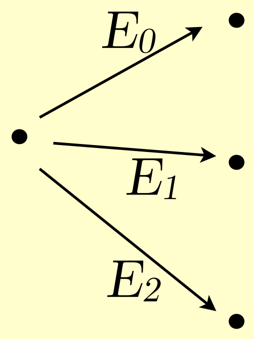

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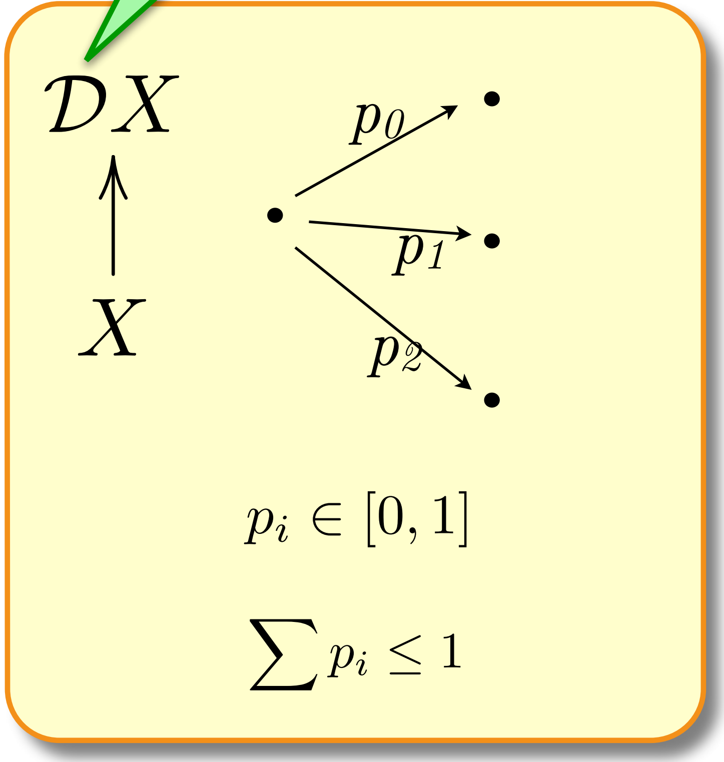

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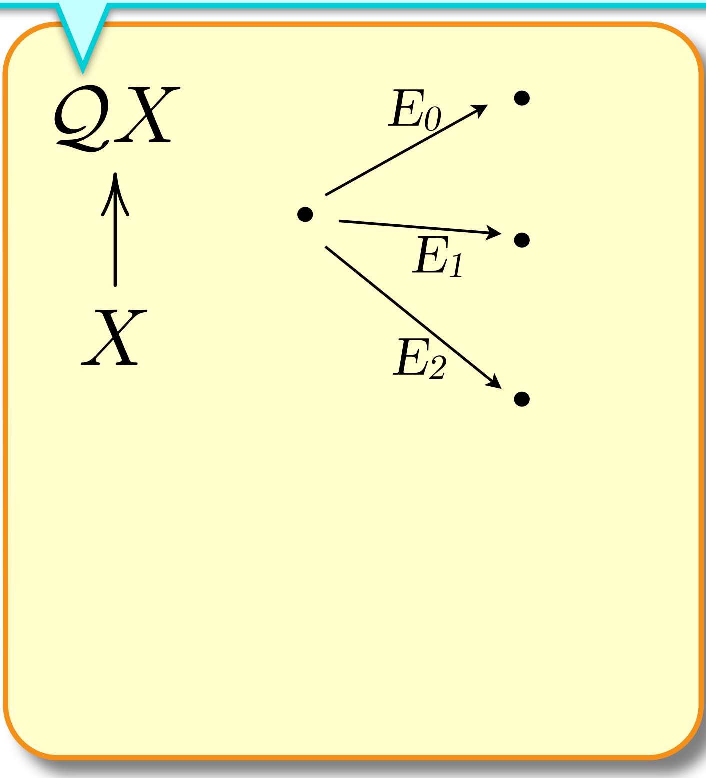

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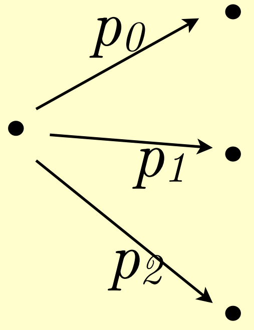

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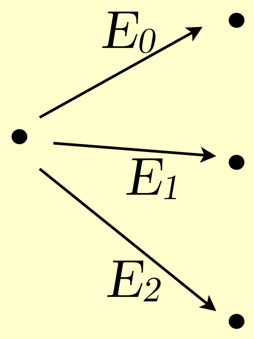

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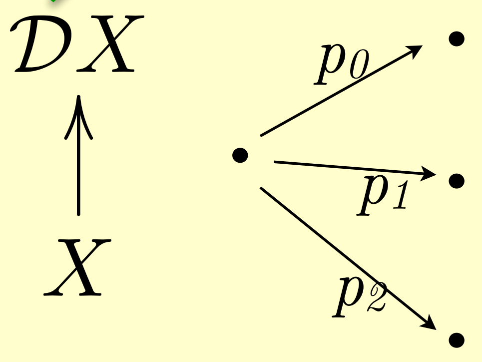

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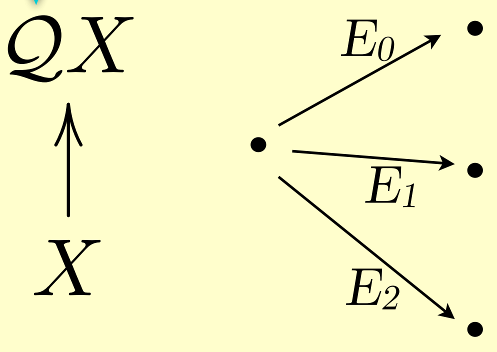

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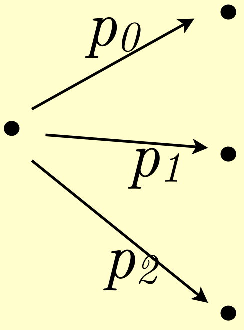

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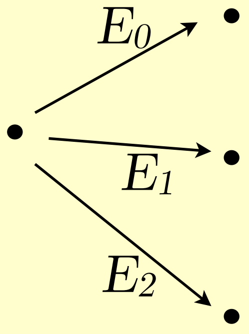

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- preparation
- unitary transf.
- measurement

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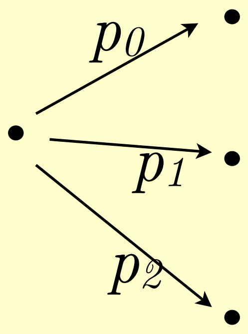

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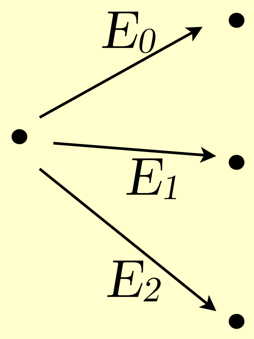

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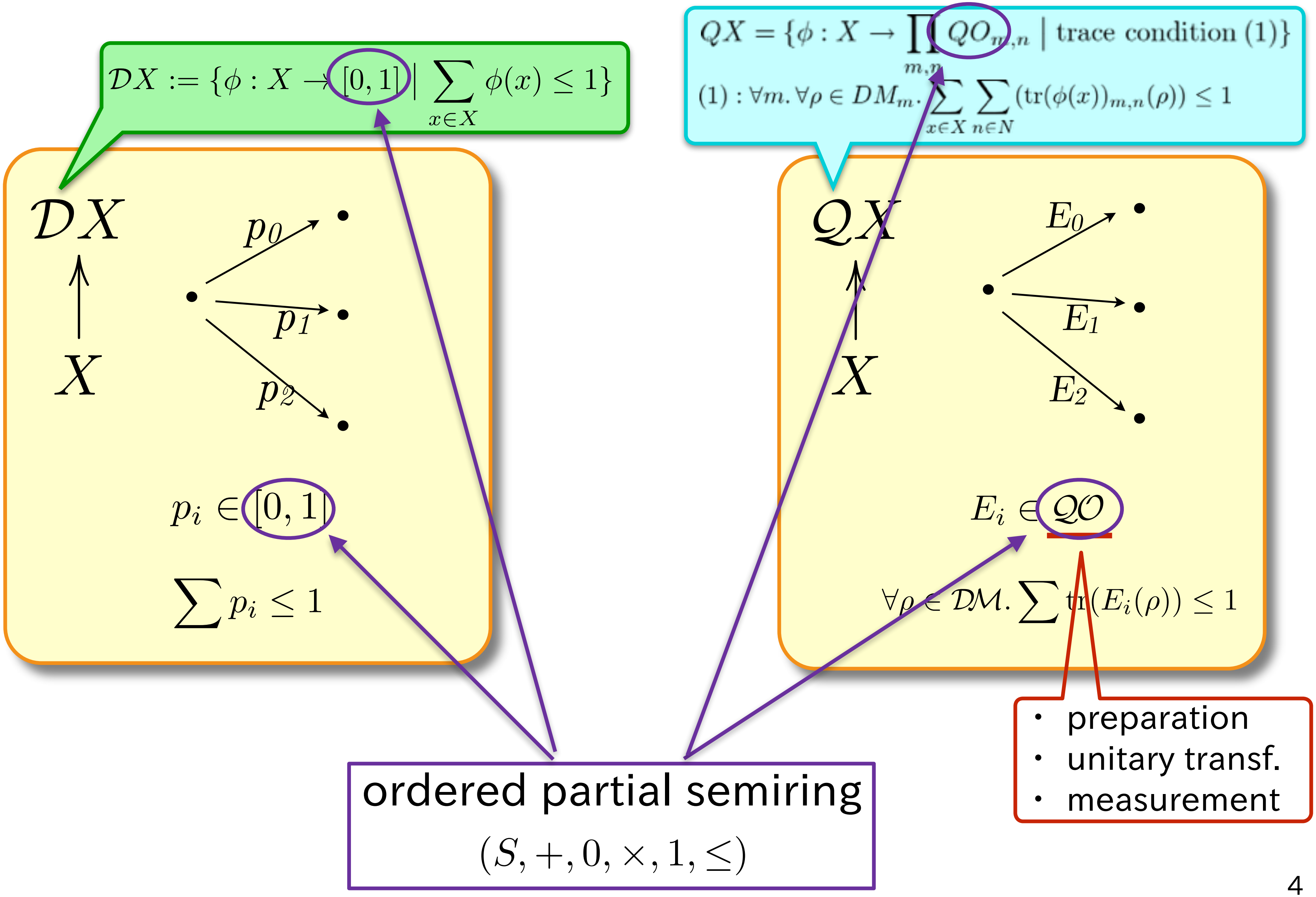

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ordered partial semiring

$(S, +, 0, \times, 1, \le)$

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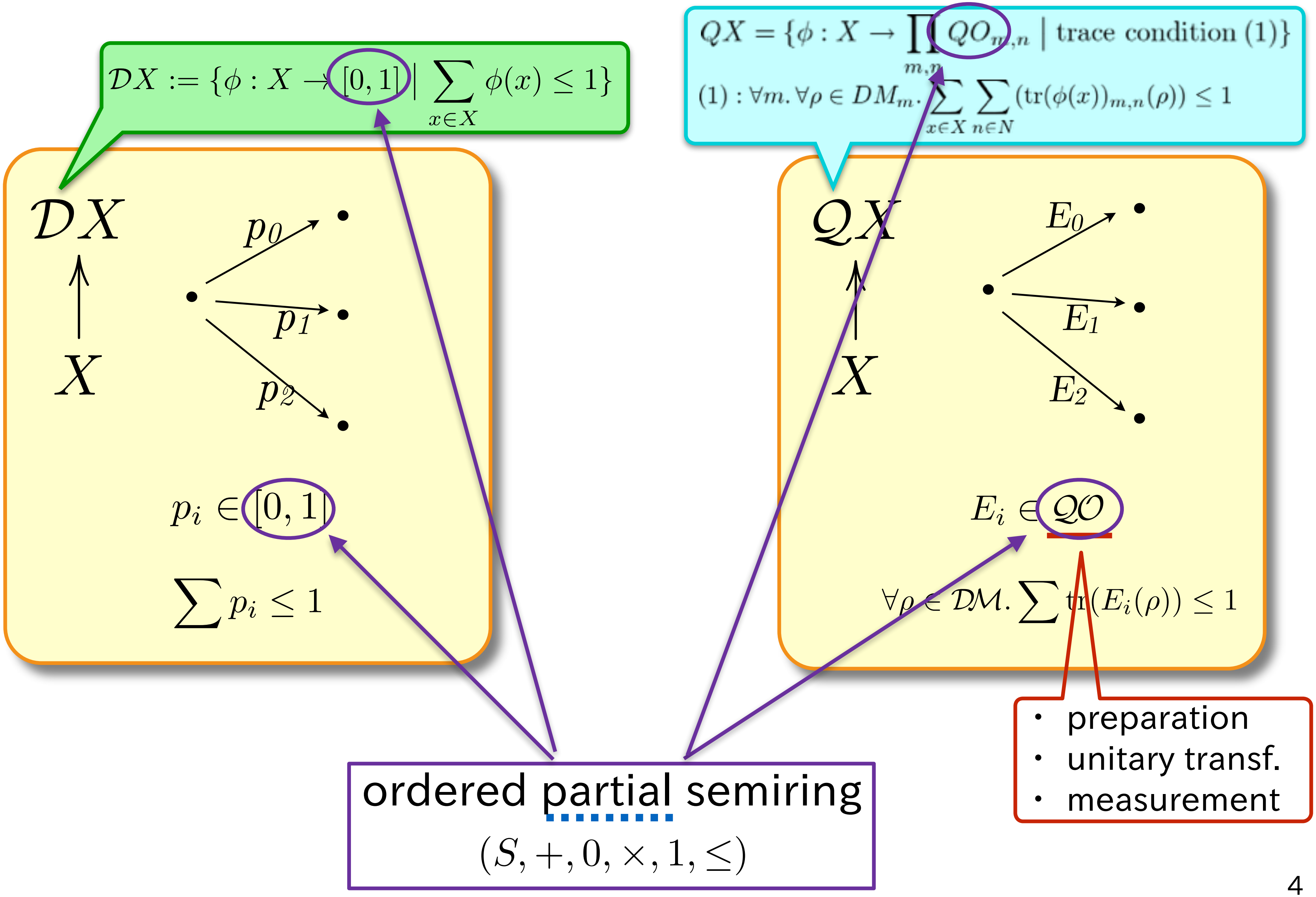

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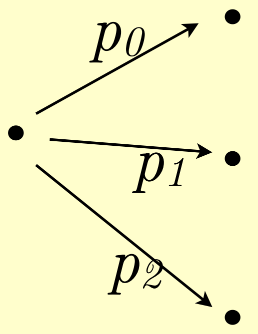

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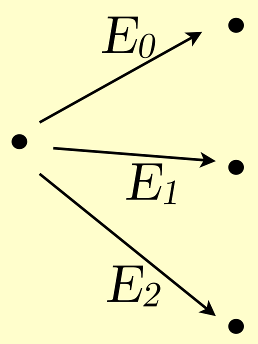

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Contribution

- apply existing coalgebra theory to monad Q

	monad D	monad Q
trace sem.	D is commutative	Q is not commutative
bisim., behav. eq.	[0,1] is refinable	QO is not refinable
coalgebraic modal logic	[0,1] is cancellative	QO is cancellative

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expressive modal logic for Q-coalgebra

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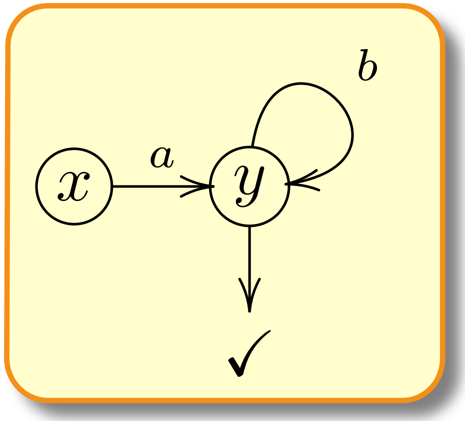
$F ::= \mathrm{id} \mid \Sigma \mid \Sigma \times F' \mid \coprod F_i$

Q-bisim.
$\neq$
Q-behav. eq

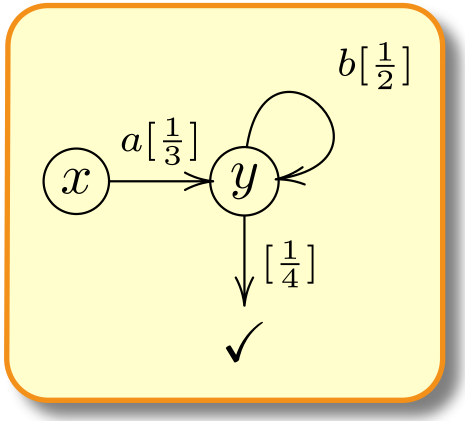
expressive modal logic
for Q-coalgebra

Trace Semantics

- whole behavior of a system, not caring each branch
  - accepted language of nondet. autom.



  - prob. distribution on accepted lang.



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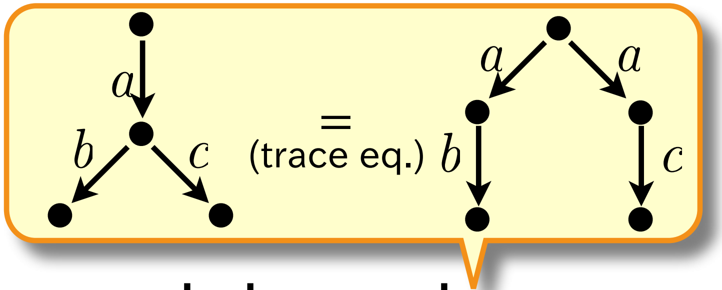

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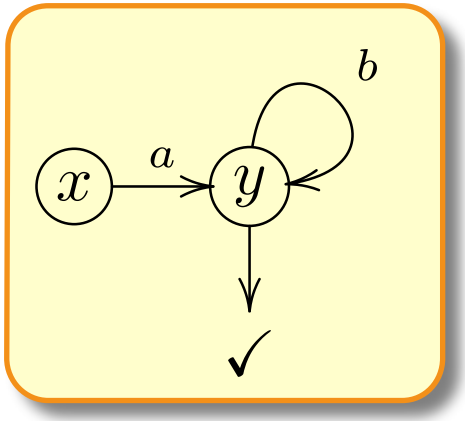

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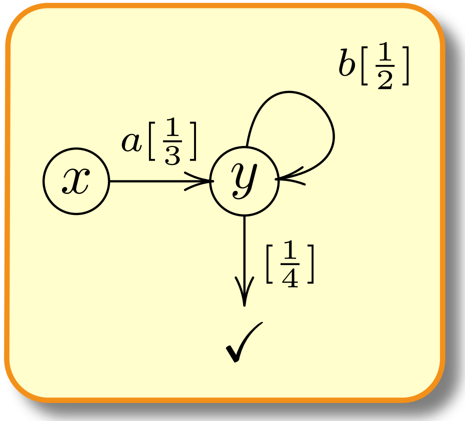

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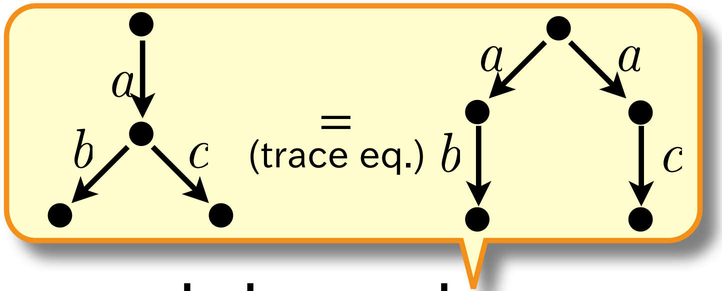

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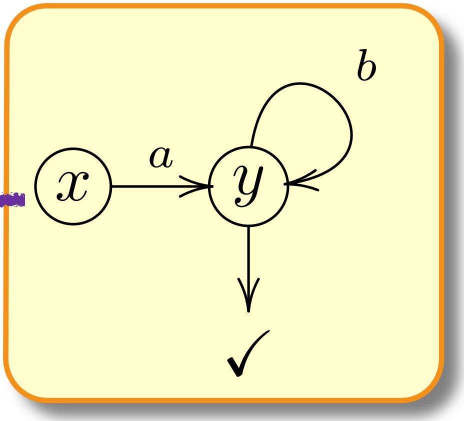
$$\mathrm{tr}(x) = \{a, ab, abb, \ldots\}$$



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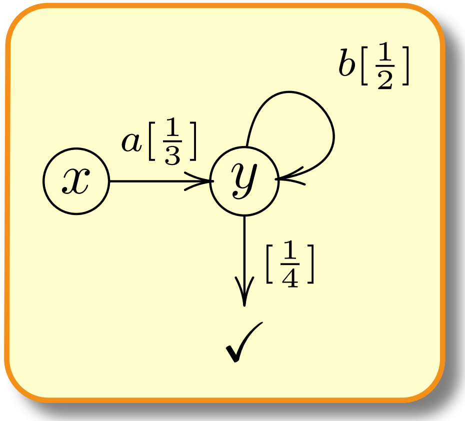

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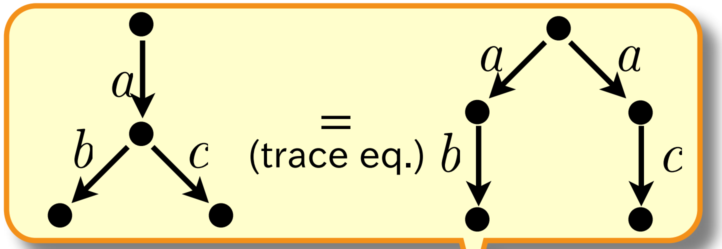

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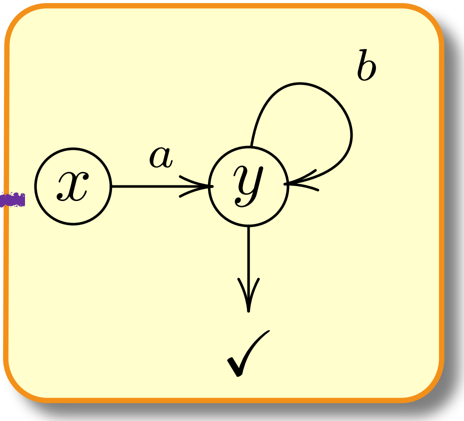
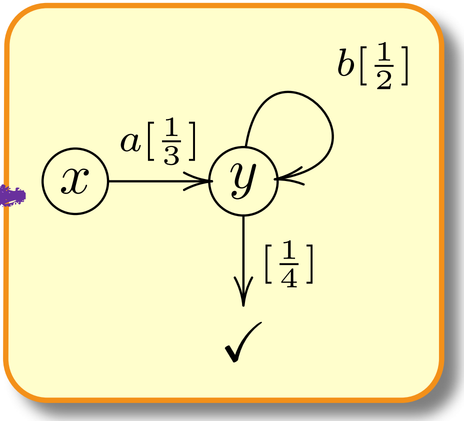


  - prob. distribution on accepted lang.

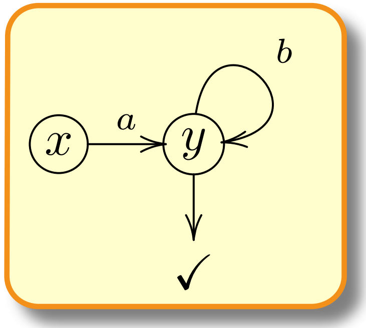
$$\mathrm{tr}(x) = \overset{\Sigma^* \to [0,1]}{\begin{bmatrix} a \mapsto 1/3 \cdot 1/2 \\ a \cdot b \mapsto 1/3 \cdot 1/2 \cdot 1/4 \\ \vdots \end{bmatrix}}$$

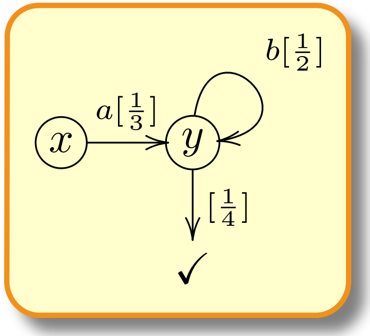


Generic Trace Semantics [Hasuo, Jacobs, Sokolova 2006]

$$\begin{tikzcd} TFX \\ X \arrow[u, "c"] \end{tikzcd}$$

$\left\{ \begin{array}{l} \text{Monad } T\text{: branching type} \\ \text{Functor } F\text{: transition type} \end{array} \right.$





trace situation

- $\mathcal{K}\ell(T)$ is Cppo-enriched
- distributive law $\lambda : FT \Rightarrow TF$
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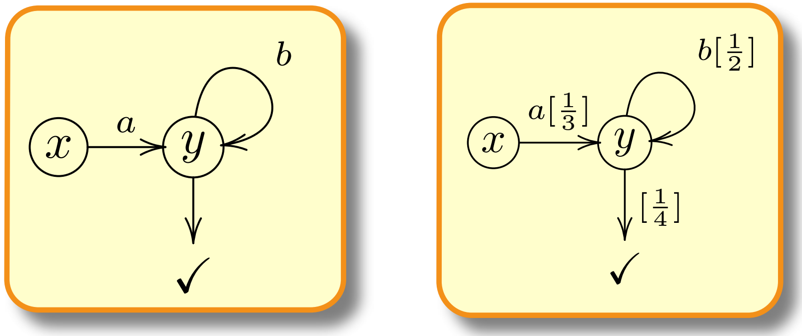

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instanciate!

$$\begin{cases} T \in \{\mathcal{P}, \mathcal{D}\} \\ F ::= \mathrm{id} \mid \Sigma \mid F_1 \times F_2 \mid \coprod_{i \in I} F_i \end{cases}$$

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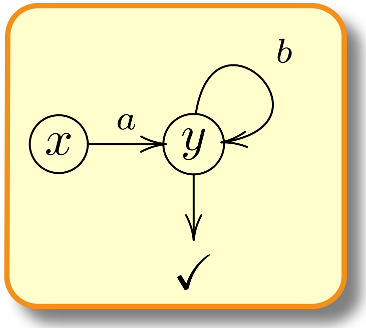
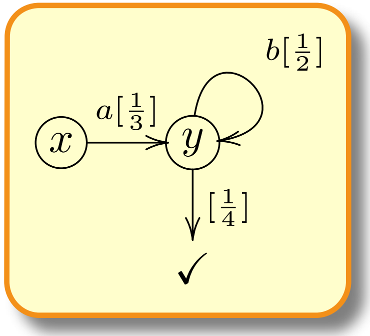


$X \to \mathcal{P}(1 + \Sigma \times X)$



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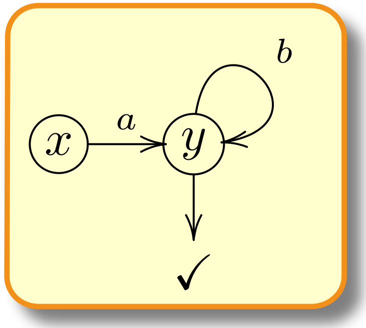

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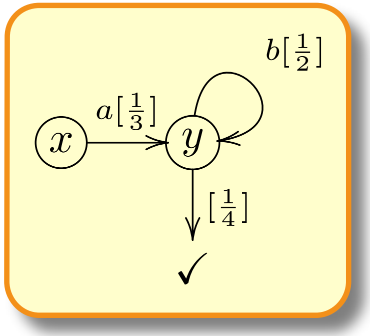

$X \to \mathcal{D}(1 + \Sigma \times X)$

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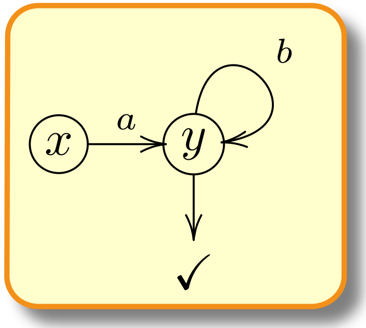

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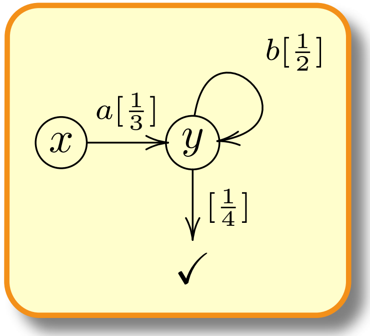
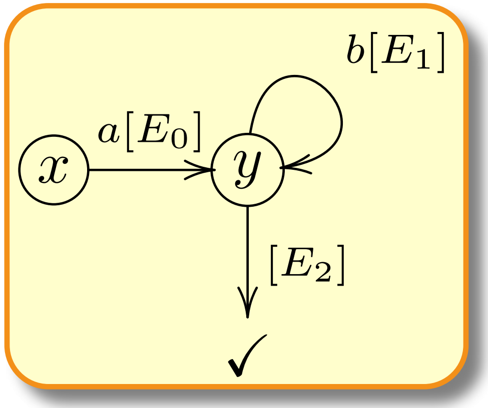


$X \to \mathcal{D}(1 + \Sigma \times X)$

$$X \to \mathcal{Q}(1 + \Sigma \times X)$$



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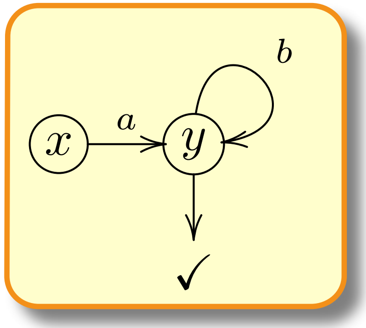

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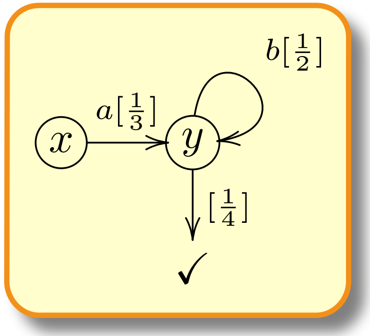

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$$\Sigma^* \to \mathcal{QO}$$

$$\mathrm{tr}(x) = \begin{bmatrix} a \mapsto E_0 \cdot E_2 \\ a \cdot b \mapsto E_0 \cdot E_1 \cdot E_2 \\ \vdots \end{bmatrix}$$

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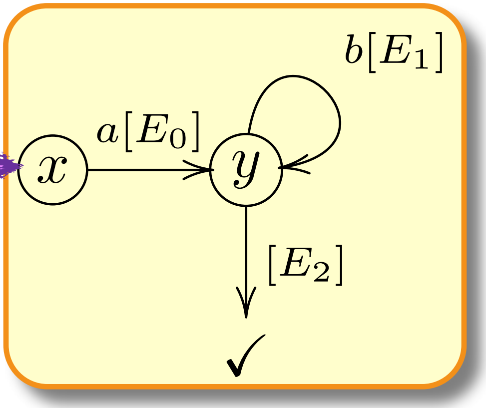

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$F = 1 + \Sigma \times \mathrm{id}$ ◎

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$F = 1 + \mathrm{id} \times \mathrm{id}$ ×

why?

Trace Situation

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in case of $\mathcal{Q}$

$$(\mathrm{id} \times \mathrm{id})\mathcal{D} \overset{\lambda}{\Rightarrow} \mathcal{D}(\mathrm{id} \times \mathrm{id})$$

(distributive law)

$$\begin{cases} T = \mathcal{Q} \\ F ::= \mathrm{id} \mid \Sigma \mid \underline{\Sigma \times F'} \mid \coprod_{i \in I} F_i \end{cases}$$

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$F = 1 + \mathrm{id} \times \mathrm{id}$ ✕

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Trace Situation

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Trace Situation

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$F = 1 + \mathrm{id} \times \mathrm{id}$ ×

why?

$$\xcancel{(\mathrm{id} \times \mathrm{id})\mathcal{Q} \overset{\lambda}{\Rightarrow} \mathcal{Q}(\mathrm{id} \times \mathrm{id})}$$

(distributive law)

Trace Situation

commutative monad

$$\begin{cases} \underline{T = \mathcal{P} \text{ or } T = \mathcal{D}} \\ F ::= \mathrm{id} \mid \Sigma \mid \underline{F_1 \times F_2} \mid \coprod_{i \in I} F_i \end{cases}$$

$F = 1 + \Sigma \times \mathrm{id}$ ◎

$F = 1 + \mathrm{id} \times \mathrm{id}$ ◎

$$\begin{tikzcd} TFX \\ X \arrow[u, "c"] \end{tikzcd}$$

in case of $\mathcal{Q}$

$$(\mathrm{id} \times \mathrm{id})\mathcal{D} \overset{\lambda}{\Rightarrow} \mathcal{D}(\mathrm{id} \times \mathrm{id})$$

(distributive law)

$$\begin{cases} \underline{T = \mathcal{Q}} \\ F ::= \mathrm{id} \mid \Sigma \mid \underline{\Sigma \times F'} \mid \coprod_{i \in I} F_i \end{cases}$$

$F = 1 + \Sigma \times \mathrm{id}$ ◎

$F = 1 + \mathrm{id} \times \mathrm{id}$ ×

why?

$$\xcancel{(\mathrm{id} \times \mathrm{id})\mathcal{Q} \overset{\lambda}{\Rightarrow} \mathcal{Q}(\mathrm{id} \times \mathrm{id})}$$

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Trace Situation

commutative monad

$$\begin{cases} \underline{T = \mathcal{P} \text{ or } T = \mathcal{D}} \\ F ::= \mathrm{id} \mid \Sigma \mid \underline{F_1 \times F_2} \mid \coprod_{i\in I} F_i \end{cases}$$

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$F = 1 + \mathrm{id} \times \mathrm{id}$ ◎

$$\begin{tikzcd} TFX \\ X \arrow[u, "c"] \end{tikzcd}$$

in case of $\mathcal{Q}$

$(\mathrm{id} \times \mathrm{id})\mathcal{D} \overset{\lambda}{\Rightarrow} \mathcal{D}(\mathrm{id} \times \mathrm{id})$

(distributive law)

not commutative monad

$$\begin{cases} \underline{T = \mathcal{Q}} \\ F ::= \mathrm{id} \mid \Sigma \mid \underline{\Sigma \times F'} \mid \coprod_{i\in I} F_i \end{cases}$$

$F = 1 + \Sigma \times \mathrm{id}$ ◎

$F = 1 + \mathrm{id} \times \mathrm{id}$ ×

why?

$\xcancel{(\mathrm{id} \times \mathrm{id})\mathcal{Q} \overset{\lambda}{\Rightarrow} \mathcal{Q}(\mathrm{id} \times \mathrm{id})}$

(distributive law)

Case Study $X \to \mathcal{P}(1 + X \times X)$

S ⟶ PQ

P ⟶ PQ

P ⟶ *

Q ⟶ *

Case Study $\underset{\{S, P, Q\}}{\underline{X}} \to \mathcal{P}(1 + X \times X)$

S ⟶ PQ

P ⟶ PQ

P ⟶ *

Q ⟶ *

Case Study $\underset{\{S,\ P,\ Q\}}{\underline{X}} \to \mathcal{P}(\underset{\{\ *\ \}}{\underline{1}} + X \times X)$

$S \longrightarrow PQ$

$P \longrightarrow PQ$

$P \longrightarrow *$

$Q \longrightarrow *$

Case Study $\underset{\{S,\ P,\ Q\}}{\underline{X}} \to \mathcal{P}(\underset{\{\ *\ \}}{\underline{1}} + X \times X)$

S ⟶ PQ
P ⟶ PQ
P ⟶ *
Q ⟶ *

S

Case Study $\underbrace{X}_{\{S,\ P,\ Q\}} \to \mathcal{P}(\underbrace{1}_{\{\ *\ \}} + X \times X)$

S ⟶ PQ

P ⟶ PQ

P ⟶ *

Q ⟶ *

S →

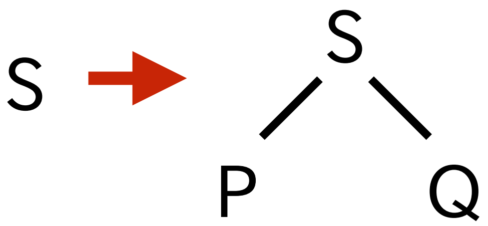
Case Study $\underset{\{S,\ P,\ Q\}}{\underline{X}} \to \mathcal{P}(\underset{\{\ *\ \}}{\underline{1}} + X \times X)$

S ⟶ PQ

P ⟶ PQ

P ⟶ *

Q ⟶ *



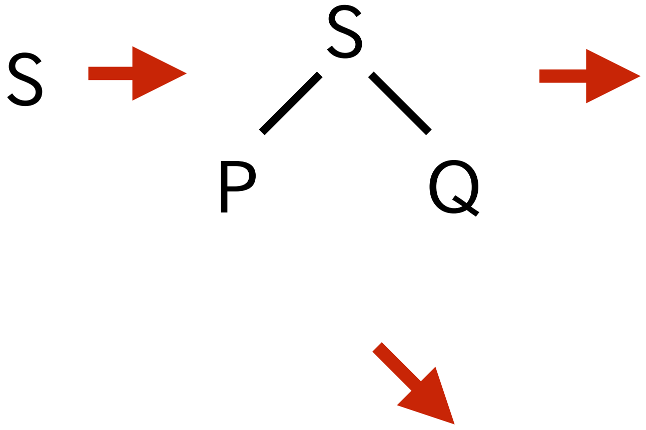
Case Study $\underset{\{S, P, Q\}}{\underline{X}} \to \mathcal{P}(\underset{\{\ *\ \}}{\underline{1}} + X \times X)$

S ⟶ PQ

P ⟶ PQ

P ⟶ *

Q ⟶ *



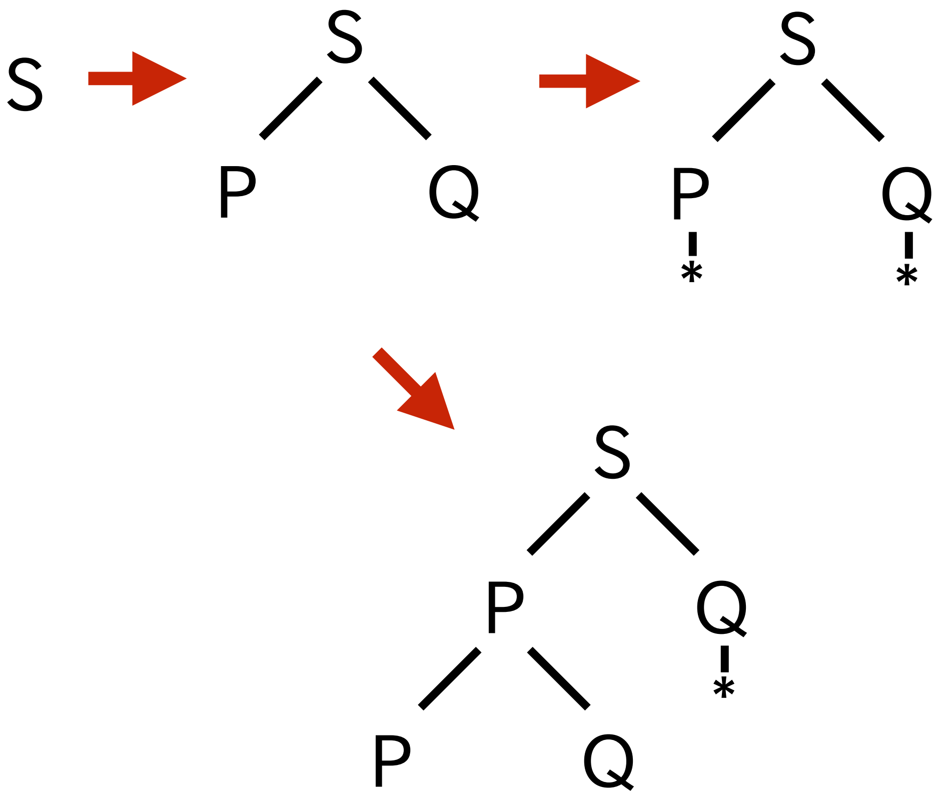
Case Study $\underset{\{S, P, Q\}}{X} \to \mathcal{P}(\underset{\{\,*\,\}}{1} + X \times X)$

S → PQ

P → PQ

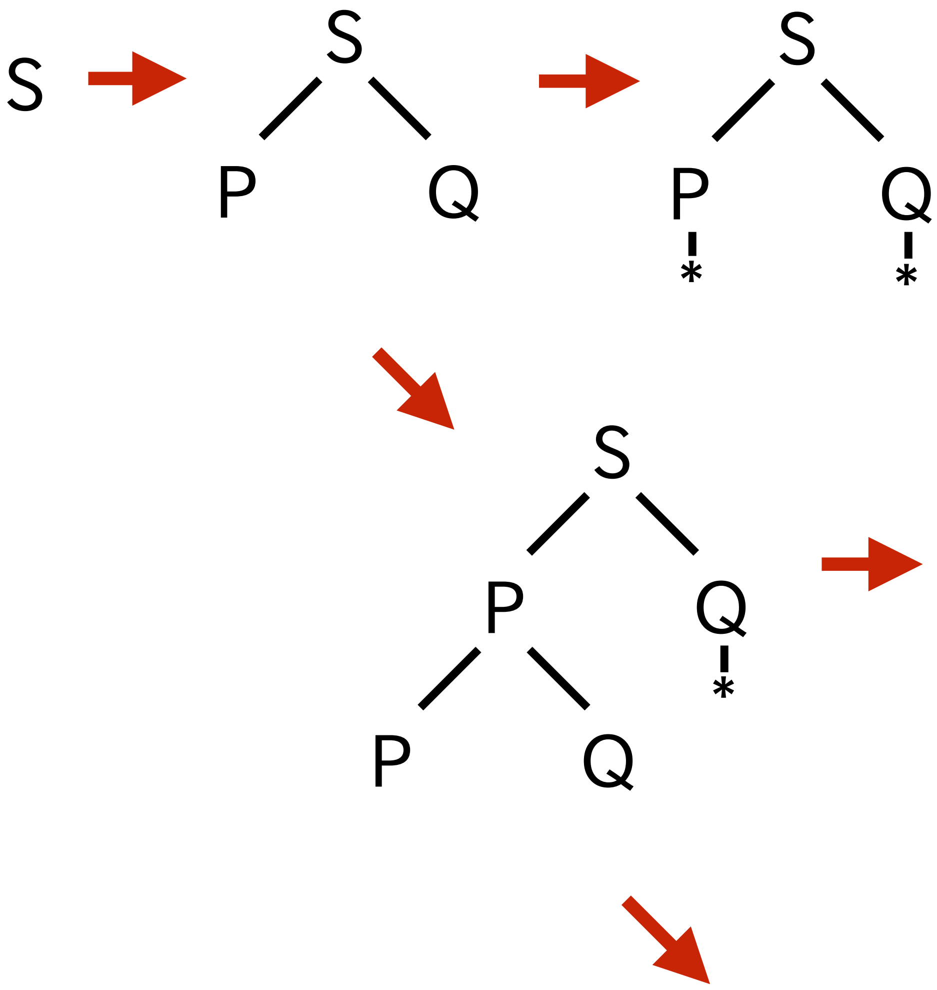
P → *

Q → *



Case Study $\underset{\{S,\ P,\ Q\}}{\underline{X}} \to \mathcal{P}(\underset{\{\ *\ \}}{\underline{1}} + X \times X)$

S ⟶ PQ
P ⟶ PQ
P ⟶ *
Q ⟶ *



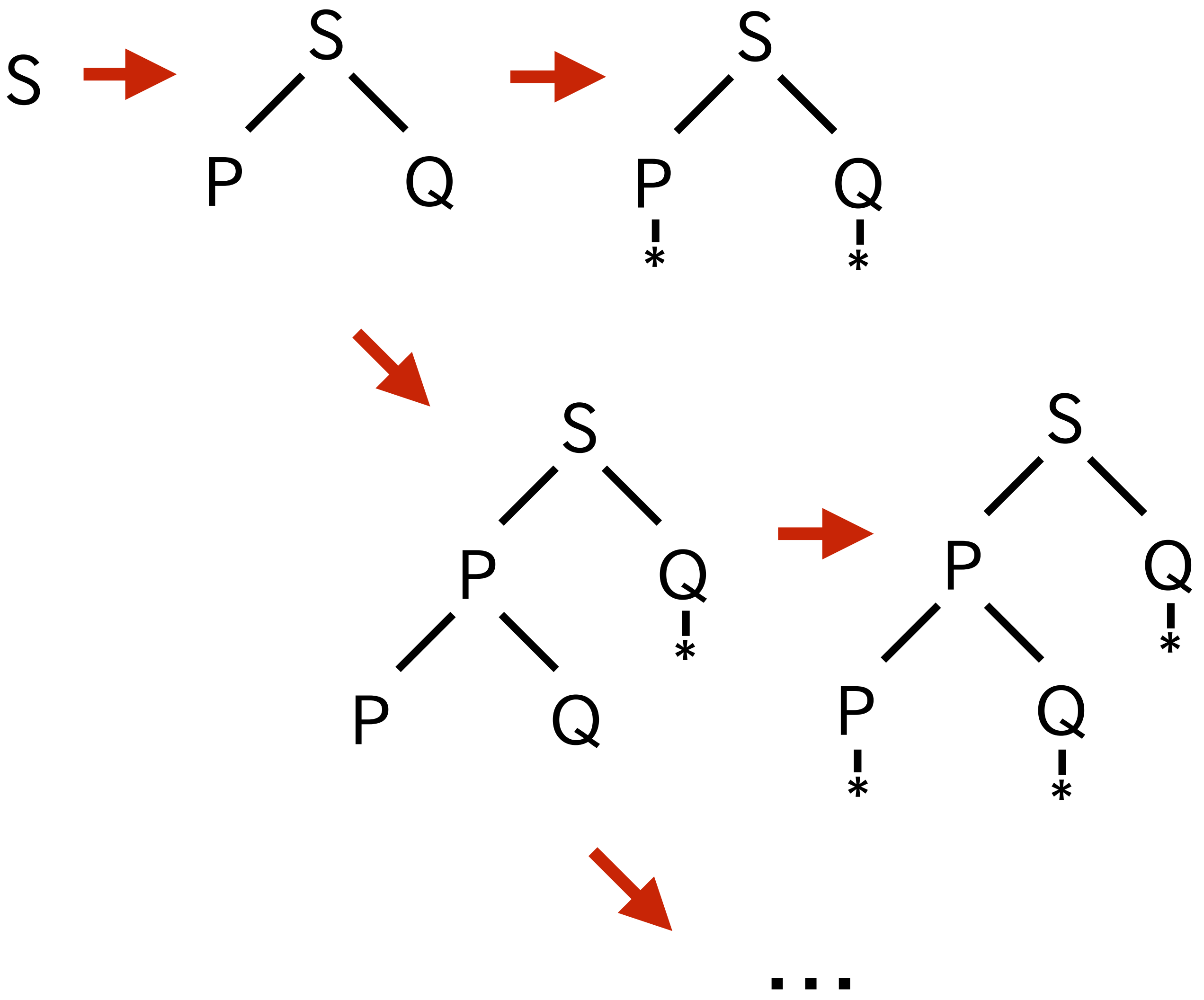
Case Study $\underbrace{X}_{\{S,\ P,\ Q\}} \to \mathcal{P}(\underbrace{1}_{\{\ *\ \}} + X \times X)$

S ⟶ PQ

P ⟶ PQ

P ⟶ *

Q ⟶ *



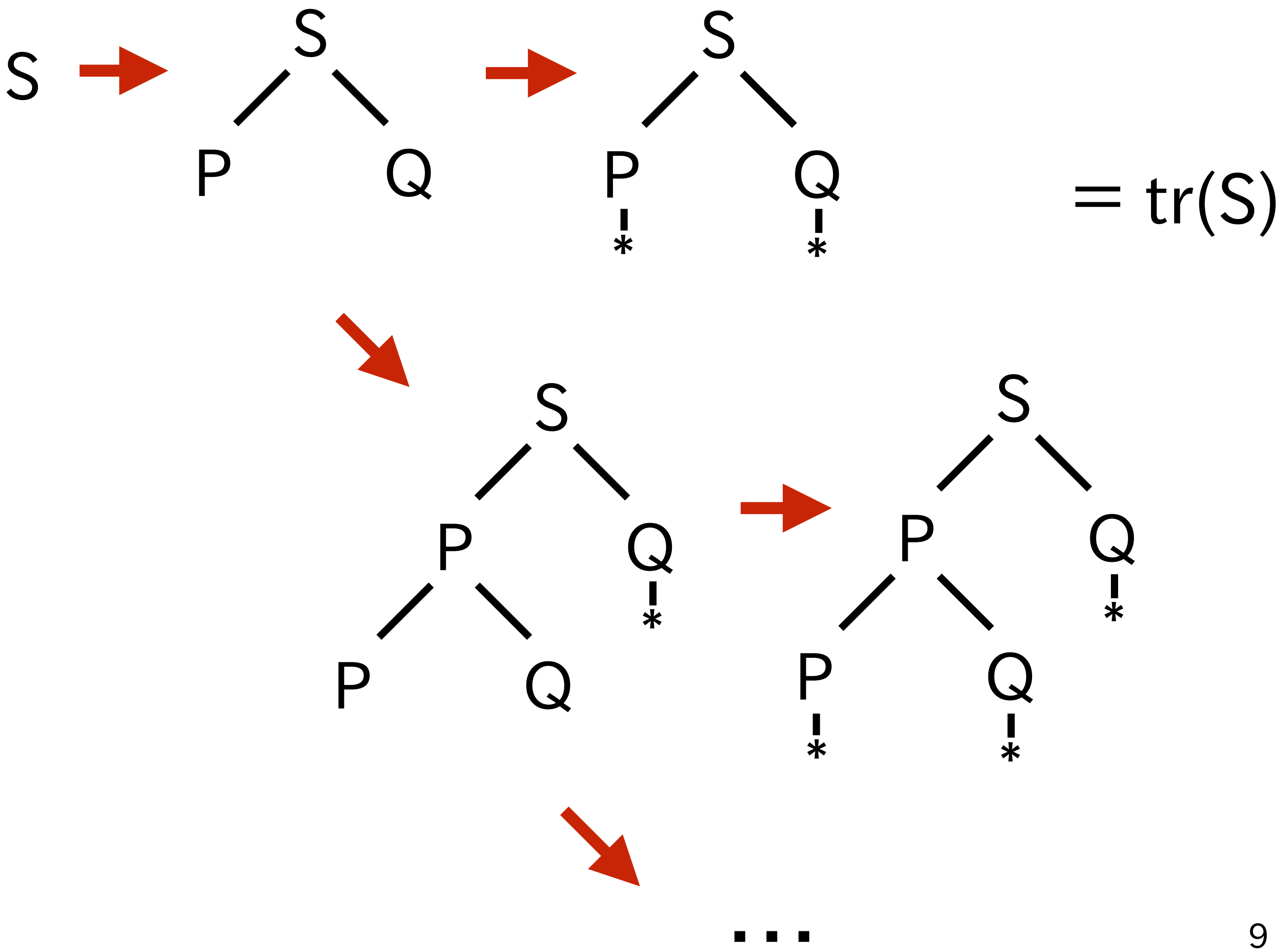
Case Study $\underbrace{X}_{\{S, P, Q\}} \to \mathcal{P}(\underbrace{1}_{\{*\}} + X \times X)$

S ⟶ PQ

P ⟶ PQ

P ⟶ *

Q ⟶ *



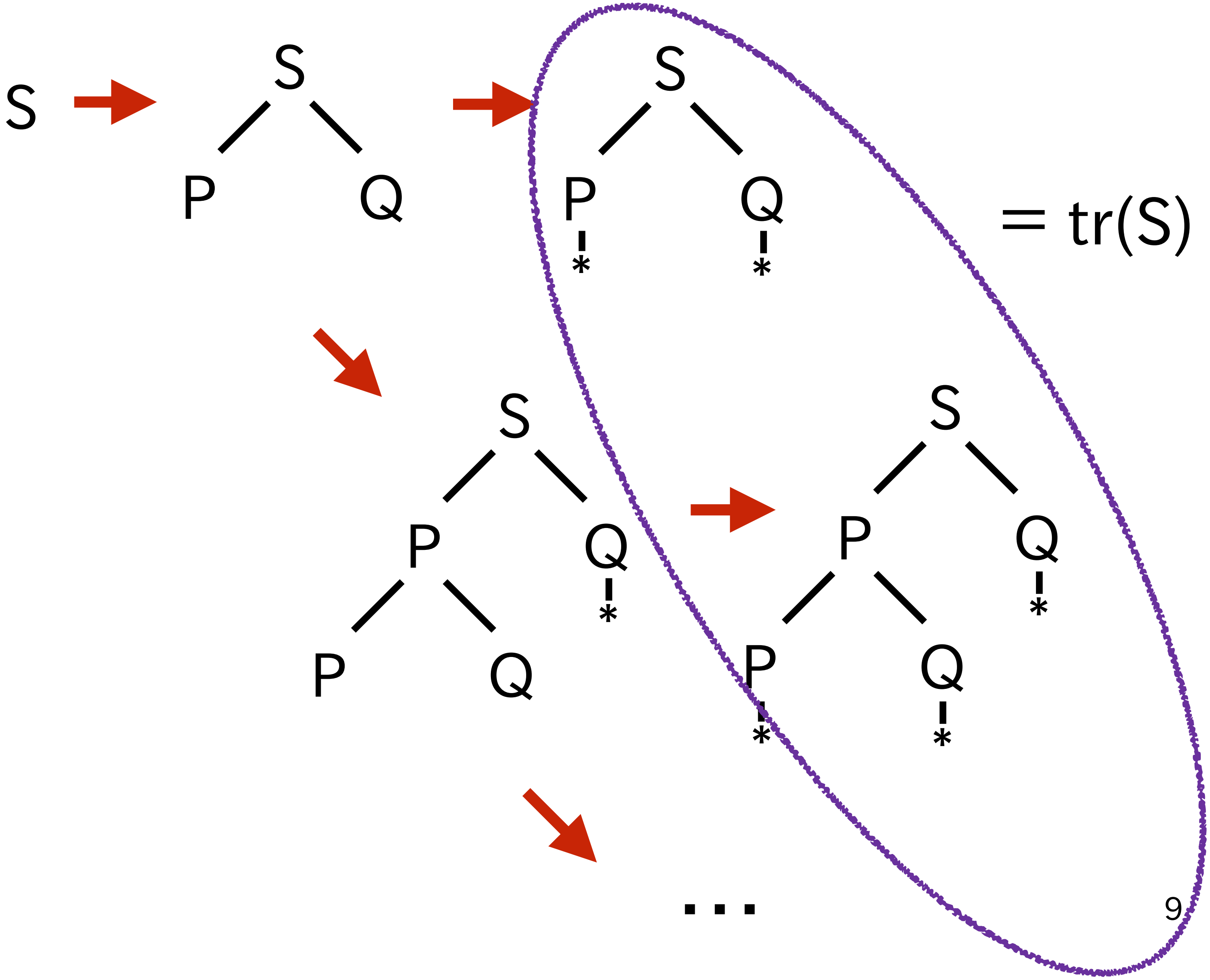
Case Study $\underbrace{X}_{\{S, P, Q\}} \to \mathcal{P}(\underbrace{1}_{\{\ast\}} + X \times X)$

S ⟶ PQ

P ⟶ PQ

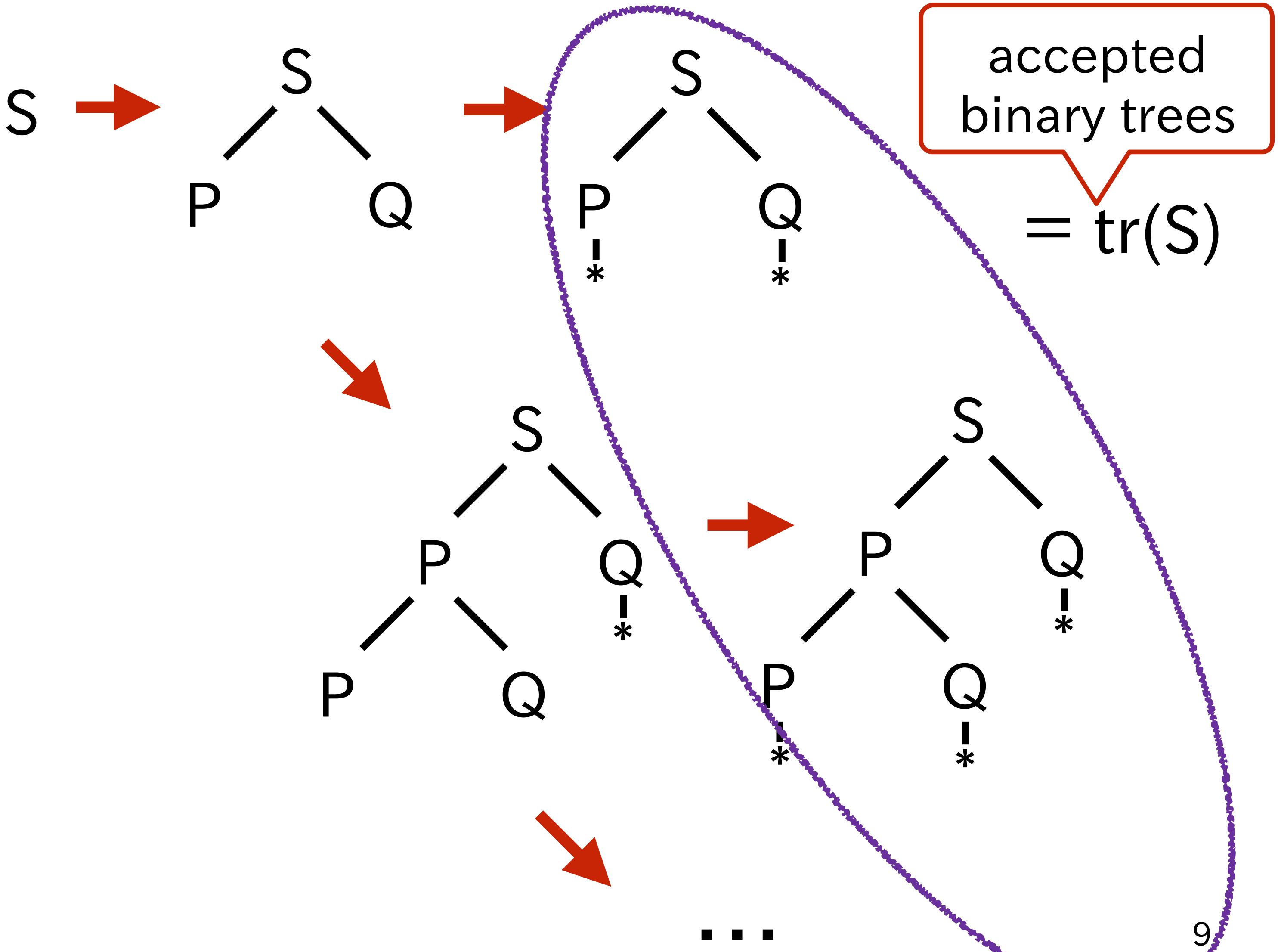
P ⟶ *

Q ⟶ *



Case Study $\underset{\{S,\ P,\ Q\}}{X} \to \mathcal{P}(\underset{\{\ *\ \}}{1} + X \times X)$

S ⟶ PQ
P ⟶ PQ
P ⟶ *
Q ⟶ *

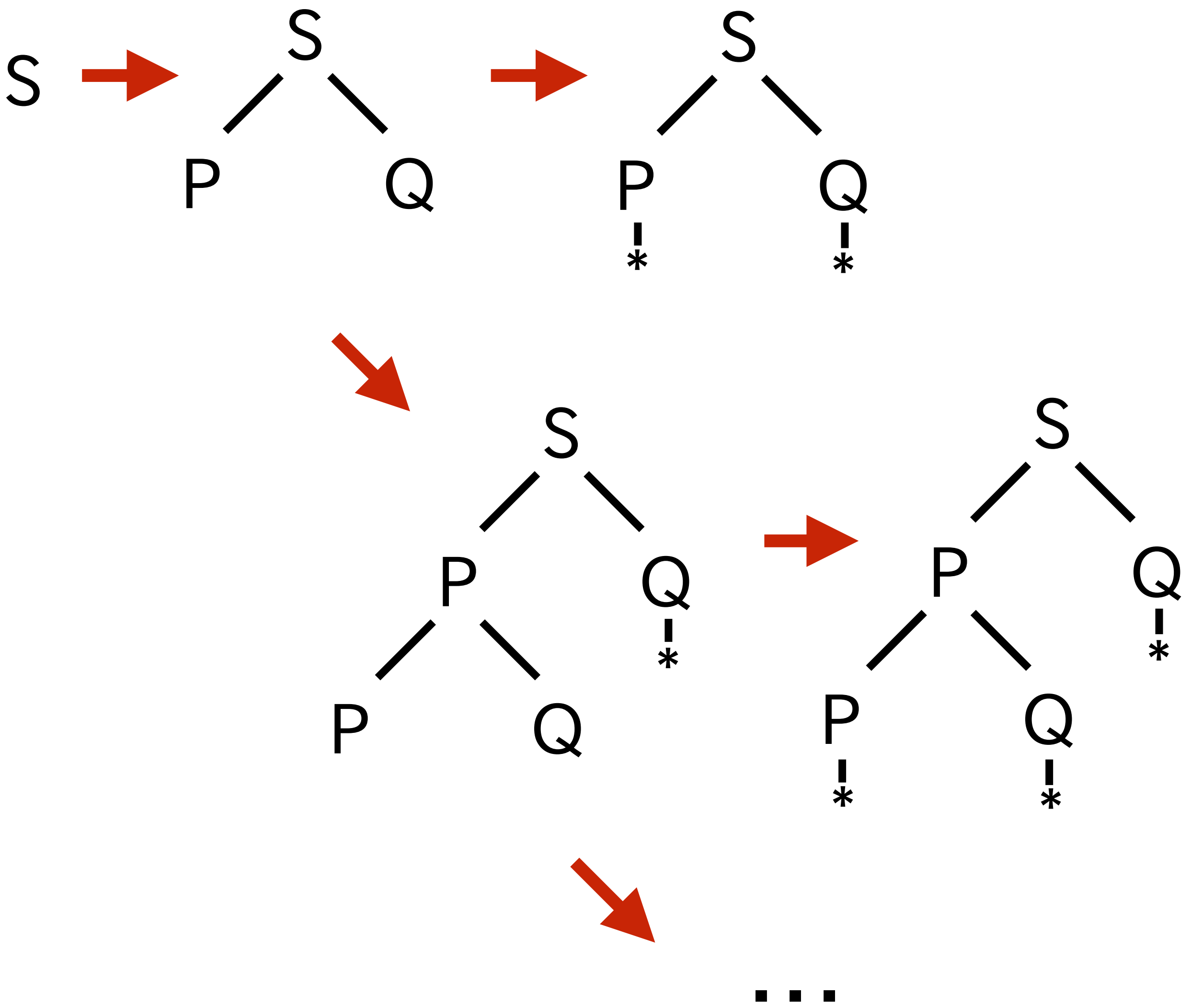


accepted binary trees

= tr(S)

Case Study $\underset{\{S,\ P,\ Q\}}{\underline{X}} \to \mathcal{D}(\underset{\{\ *\ \}}{\underline{1}} + X \times X)$

S ⟶ PQ
P ⟶ PQ
P ⟶ *
Q ⟶ *



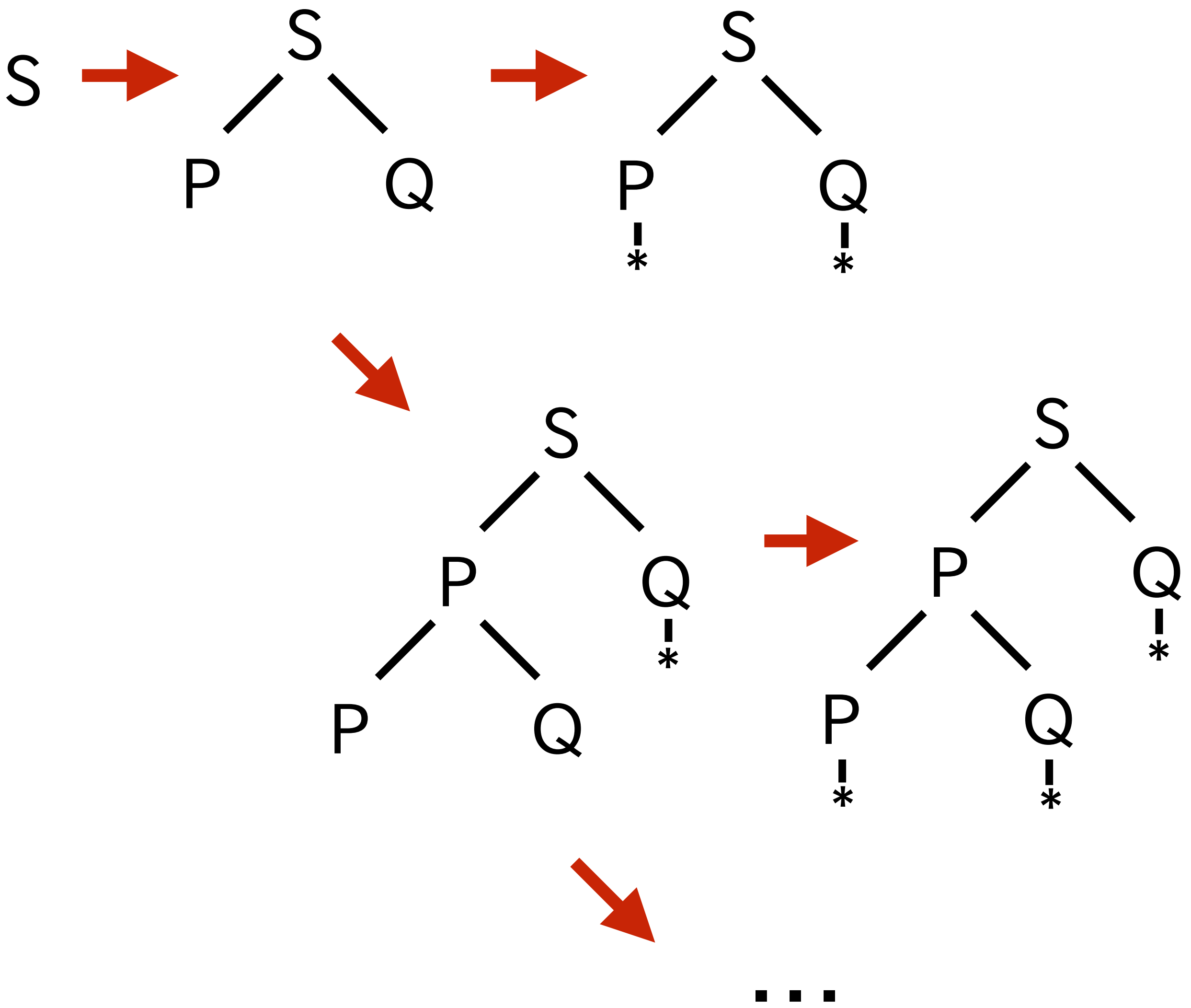
Case Study $\underbrace{X}_{\{S,\, P,\, Q\}} \to \mathcal{D}(\underbrace{1}_{\{\,*\,\}} + X \times X)$

$S \xrightarrow{p_0} PQ$

$P \xrightarrow{p_1} PQ$

$P \xrightarrow{p_2} *$

$Q \xrightarrow{p_3} *$



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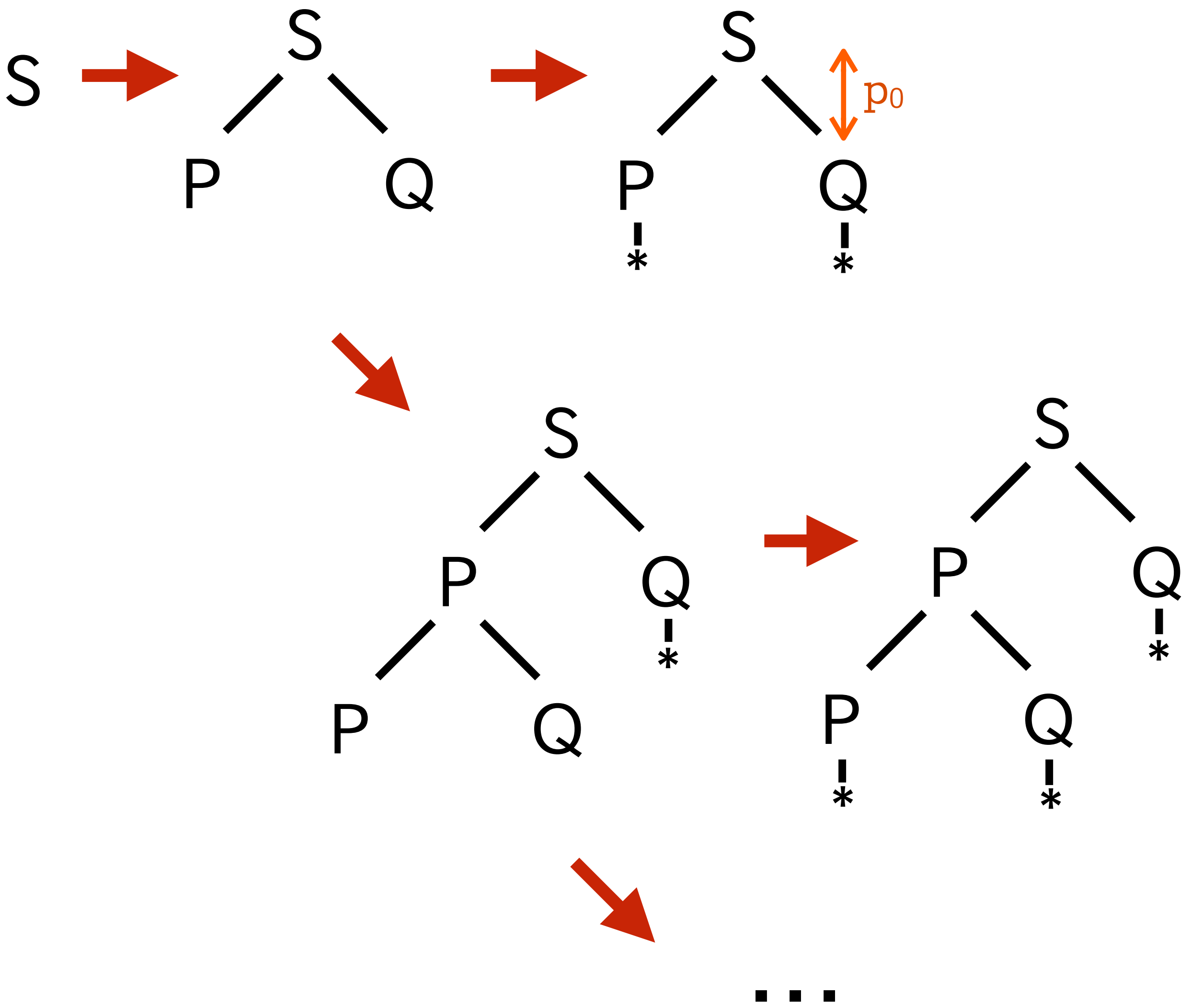

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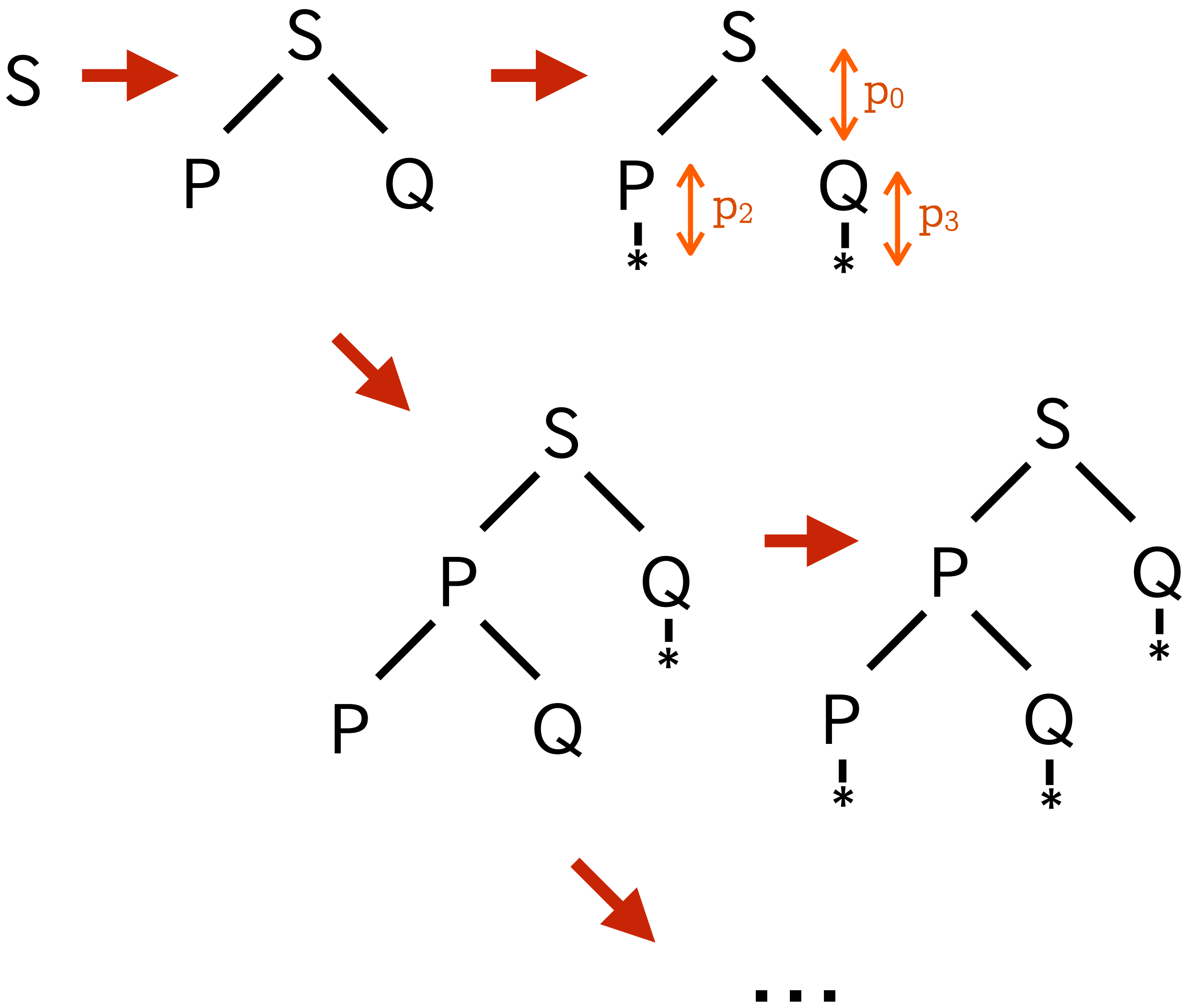

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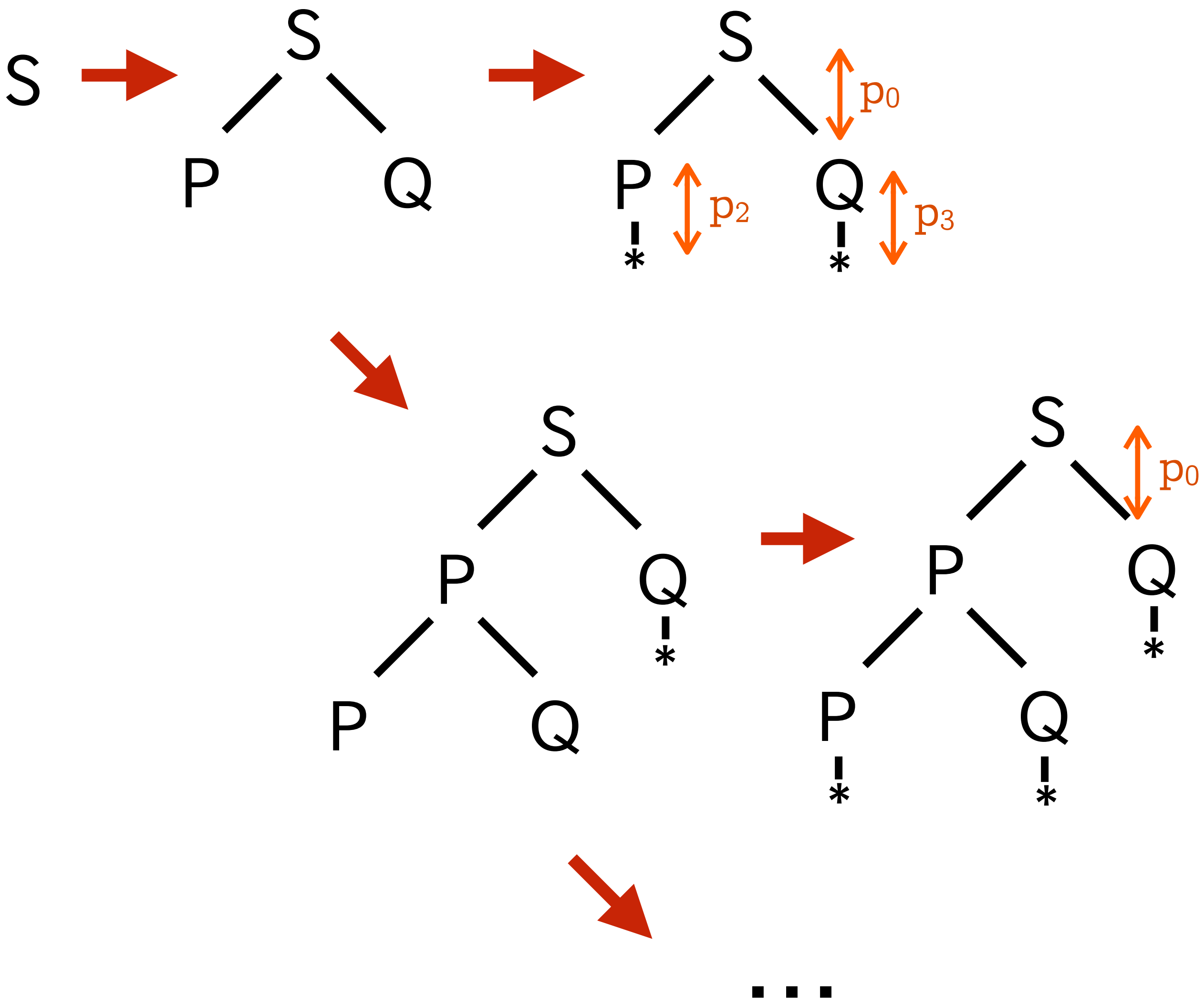

Case Study $\underset{\{S,\, P,\, Q\}}{\underline{X}} \to \mathcal{D}(\underset{\{\, *\, \}}{\underline{1}} + X \times X)$

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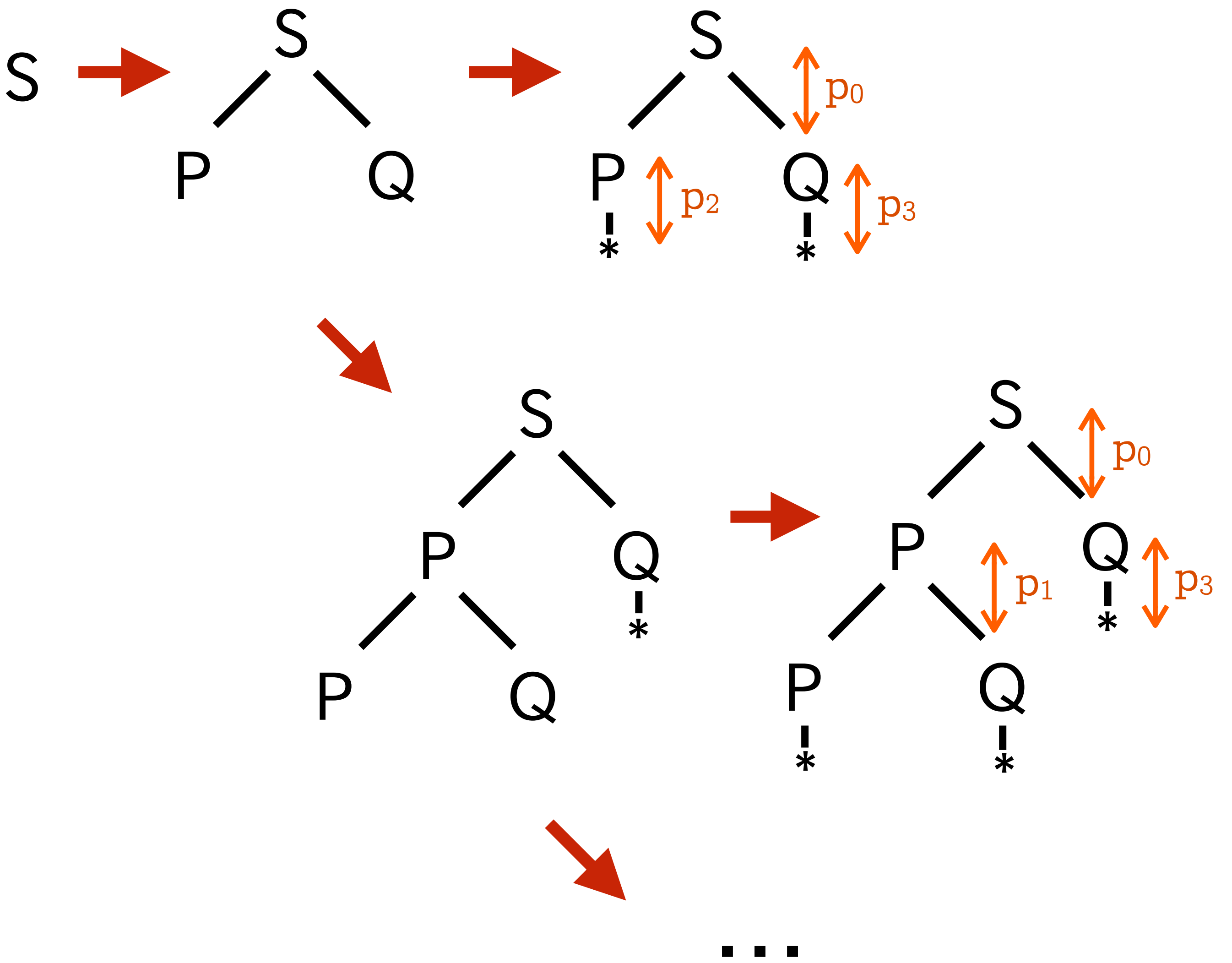

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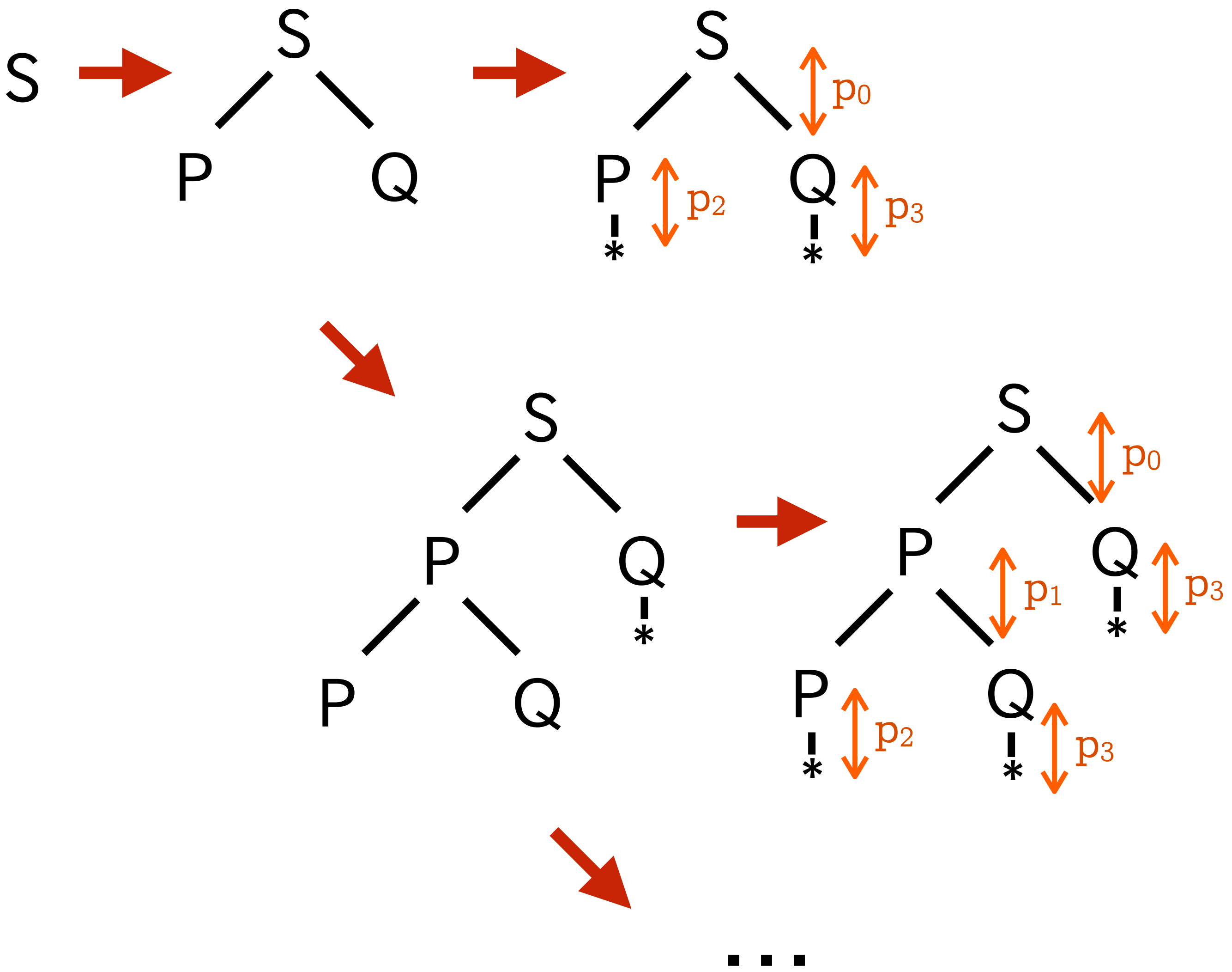

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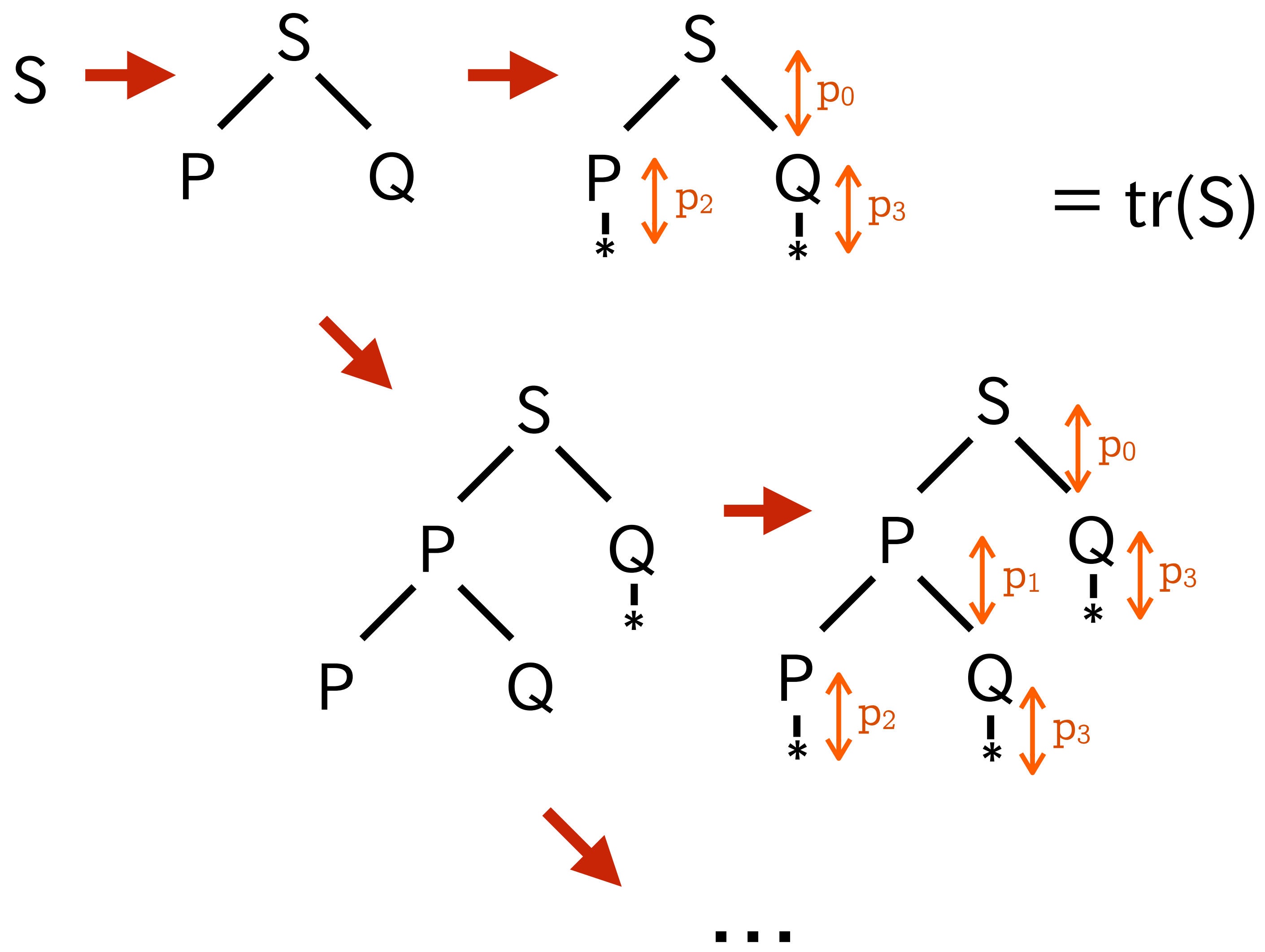

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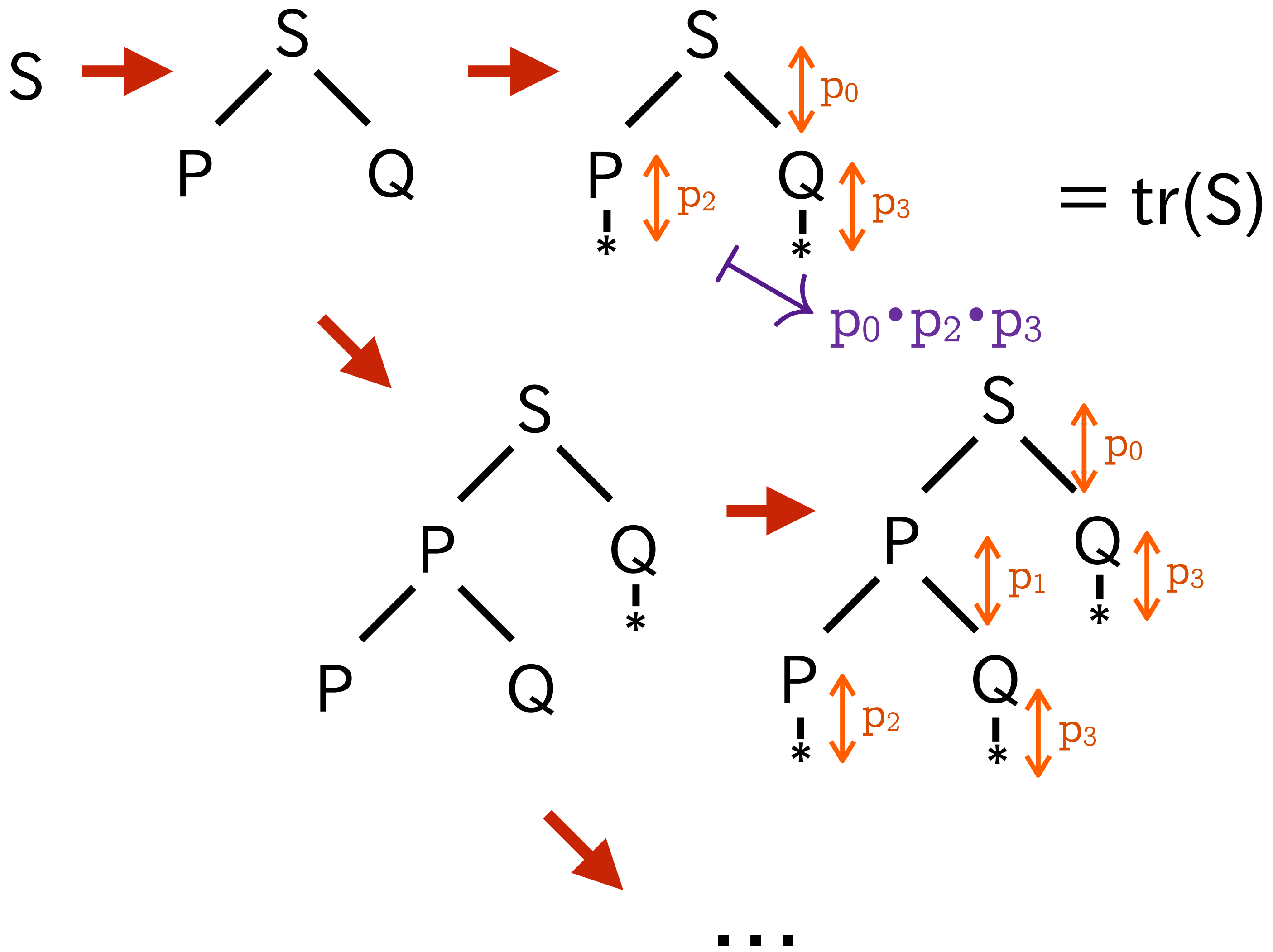

Case Study $\underset{\{S,\,P,\,Q\}}{X} \to \mathcal{D}(\underset{\{\,*\,\}}{1} + X \times X)$

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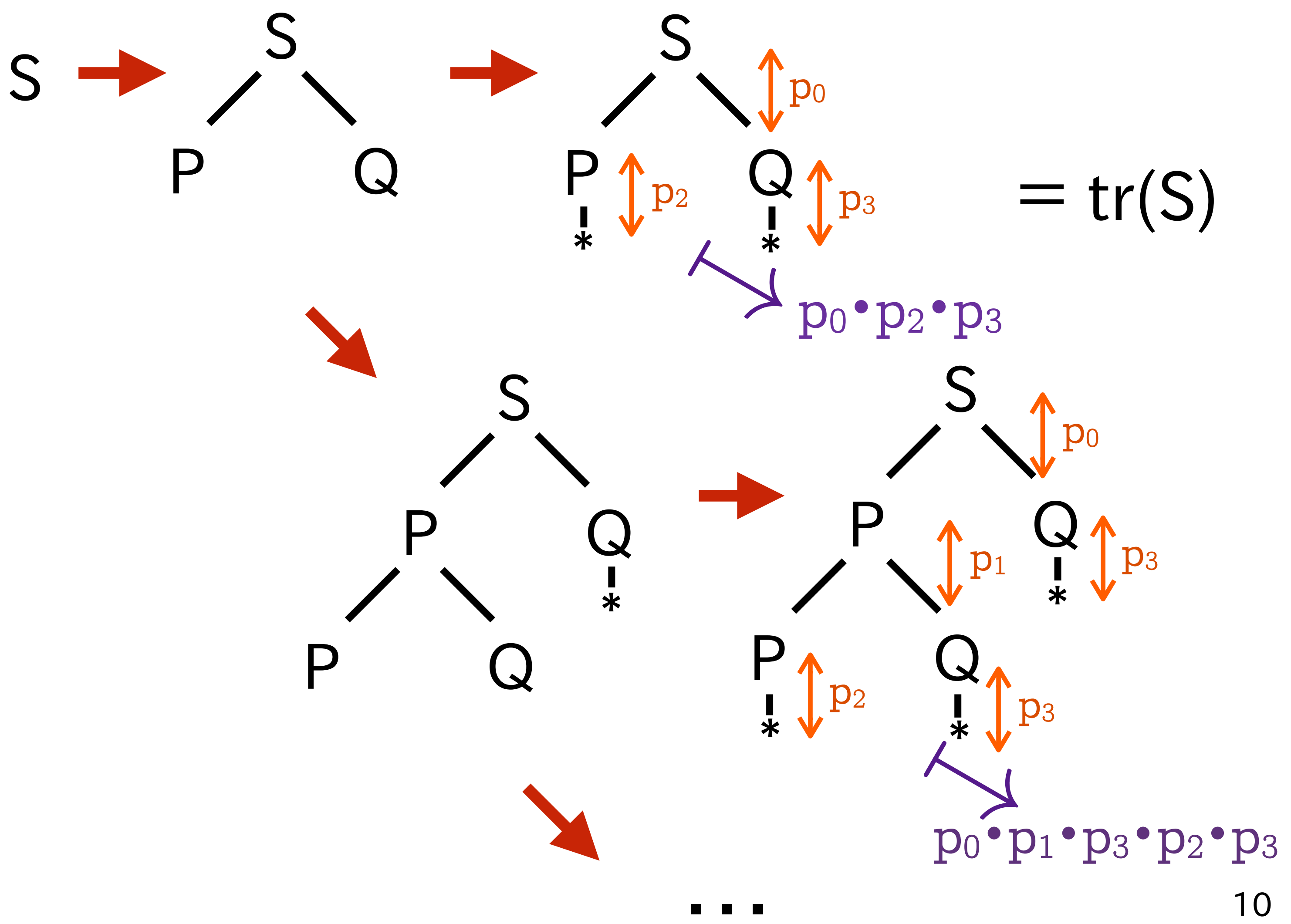

Case Study $\underset{\{S,\ P,\ Q\}}{X} \to \mathcal{D}(\underset{\{\ *\ \}}{1} + X \times X)$

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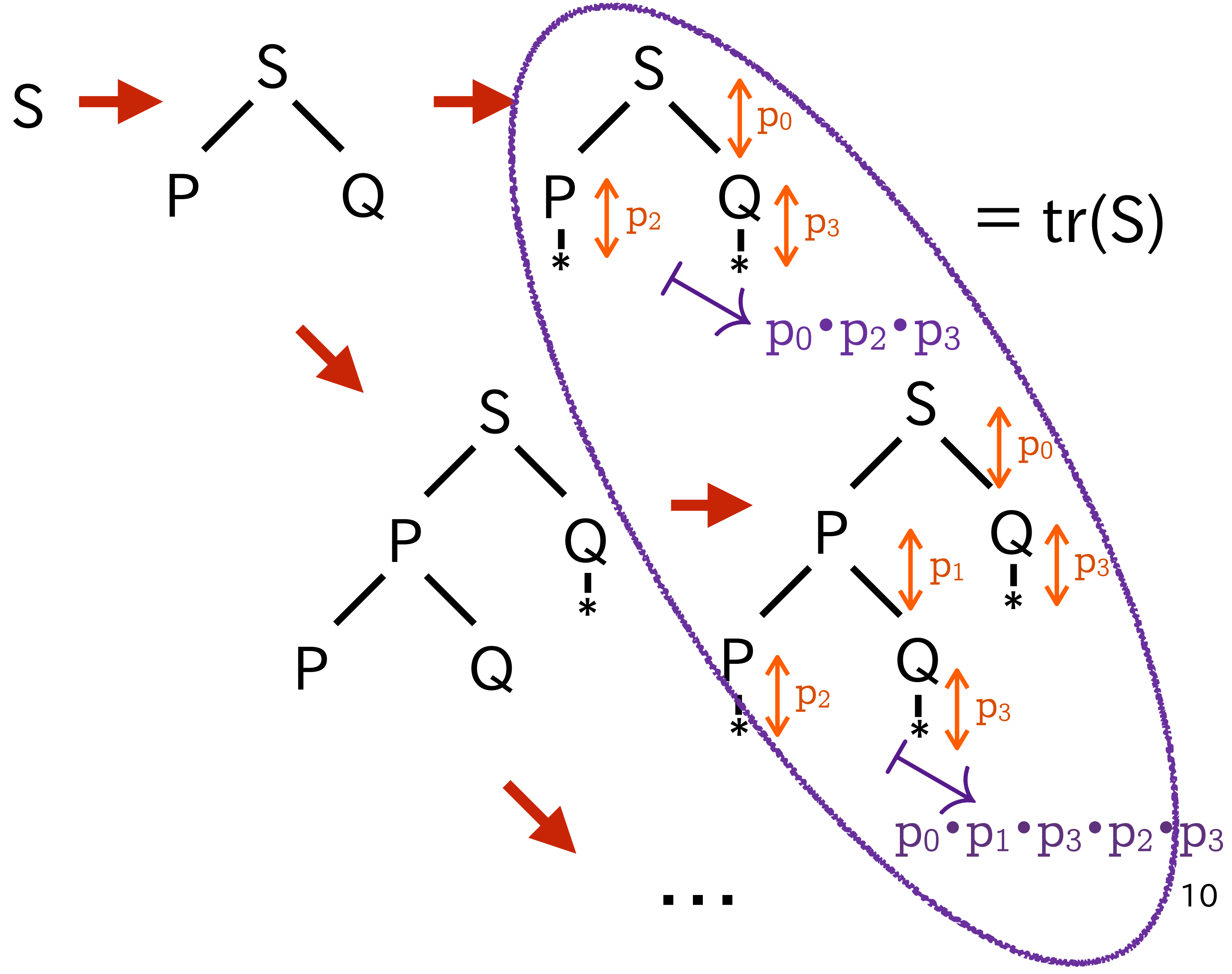


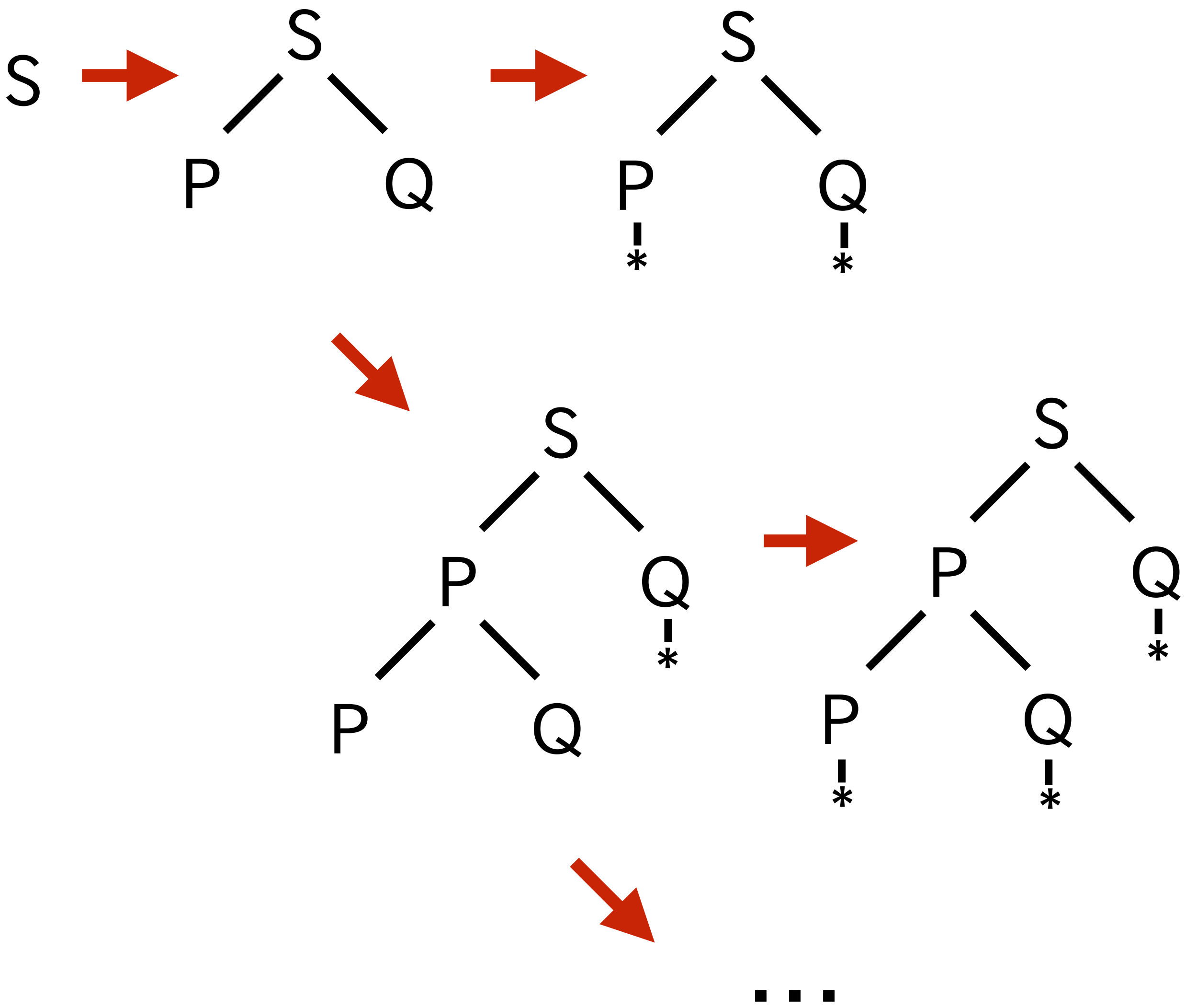
Case Study $\underbrace{X}_{\{S, P, Q\}} \to Q(\underbrace{1}_{\{ * \}} + X \times X)$

$S \longrightarrow PQ$

$P \longrightarrow PQ$

$P \longrightarrow *$

$Q \longrightarrow *$



Case Study $\underset{\{S,\ P,\ Q\}}{\underline{X}} \to Q(\underset{\{\ *\ \}}{\underline{1}} + X \times X)$

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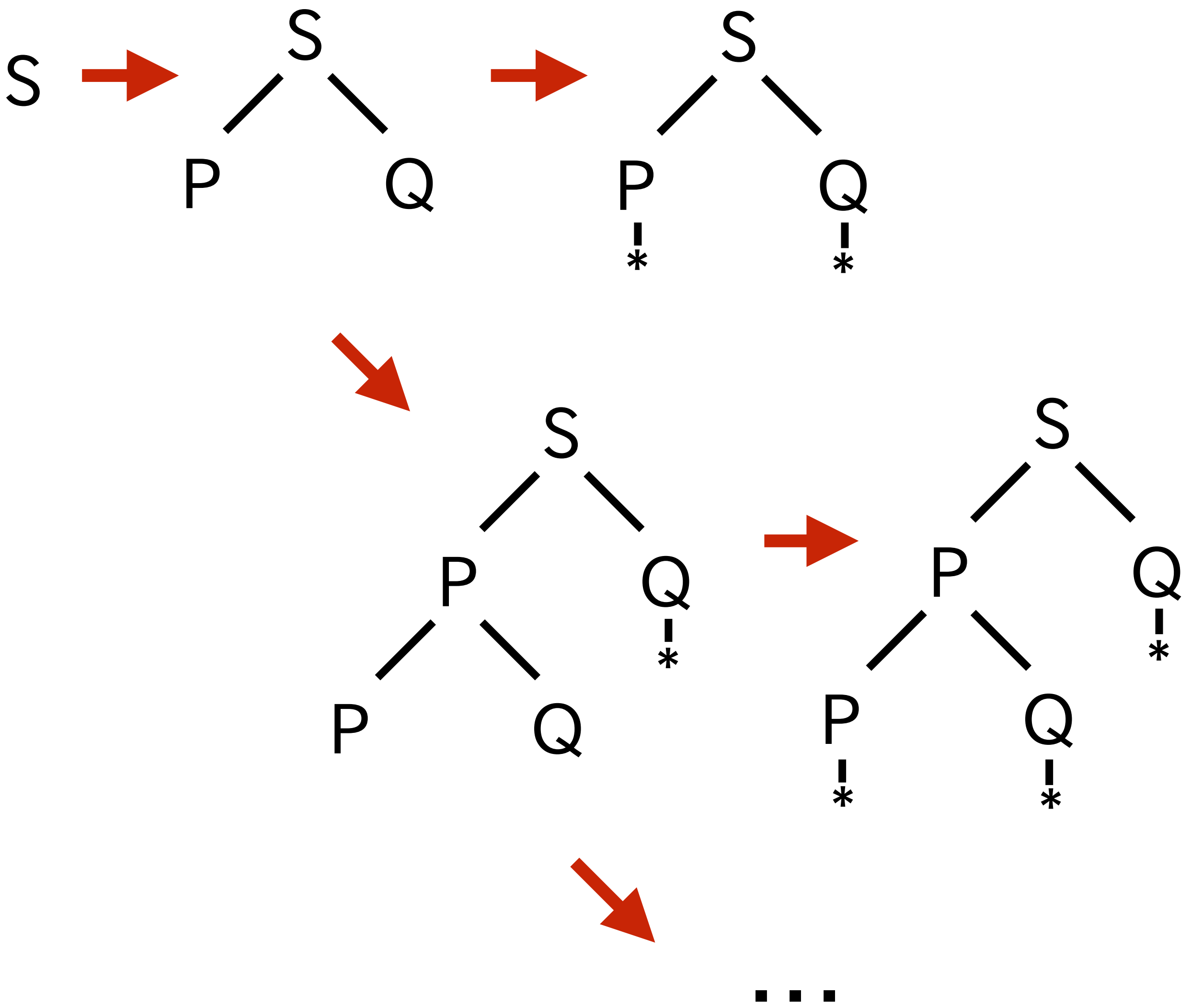

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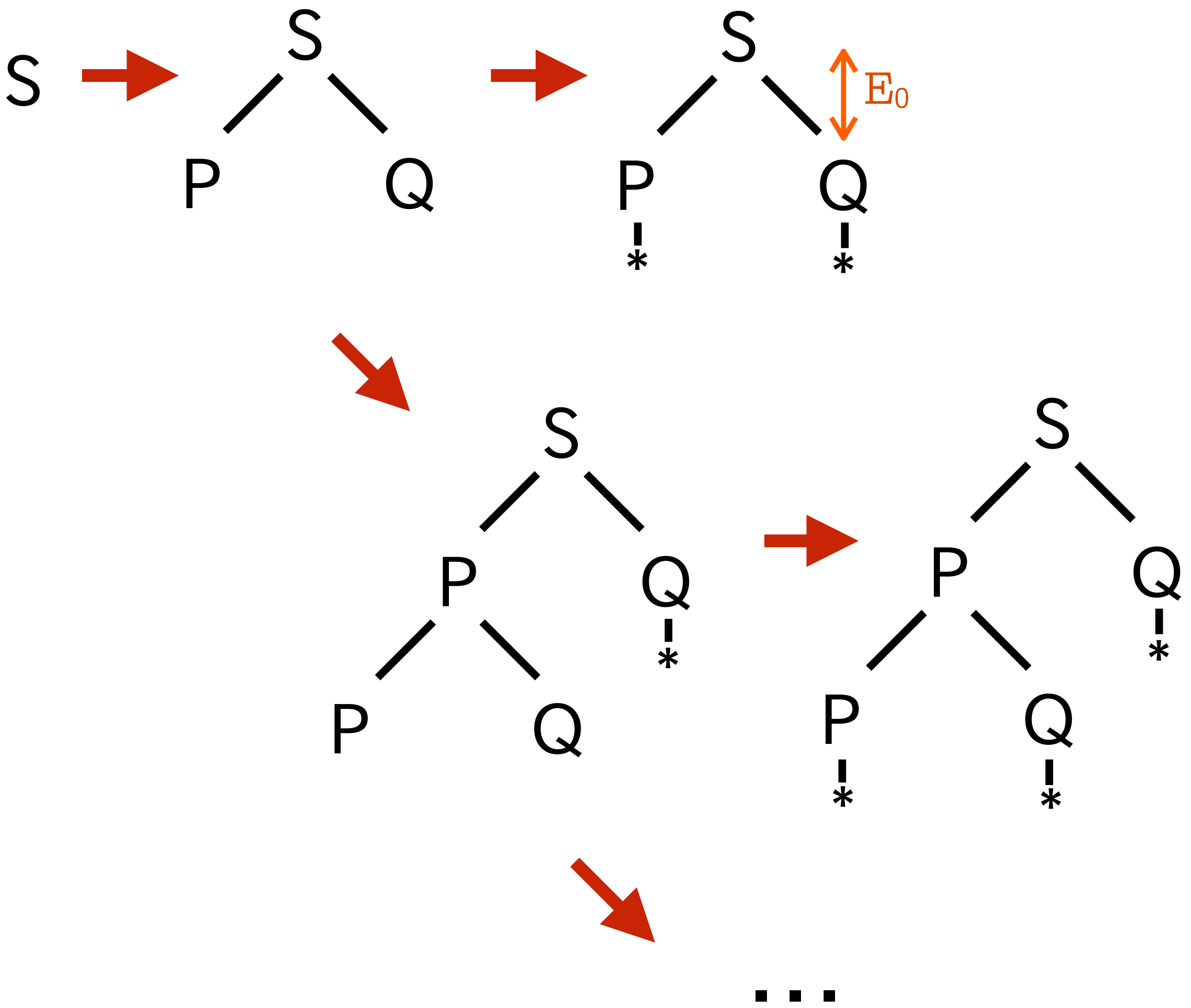

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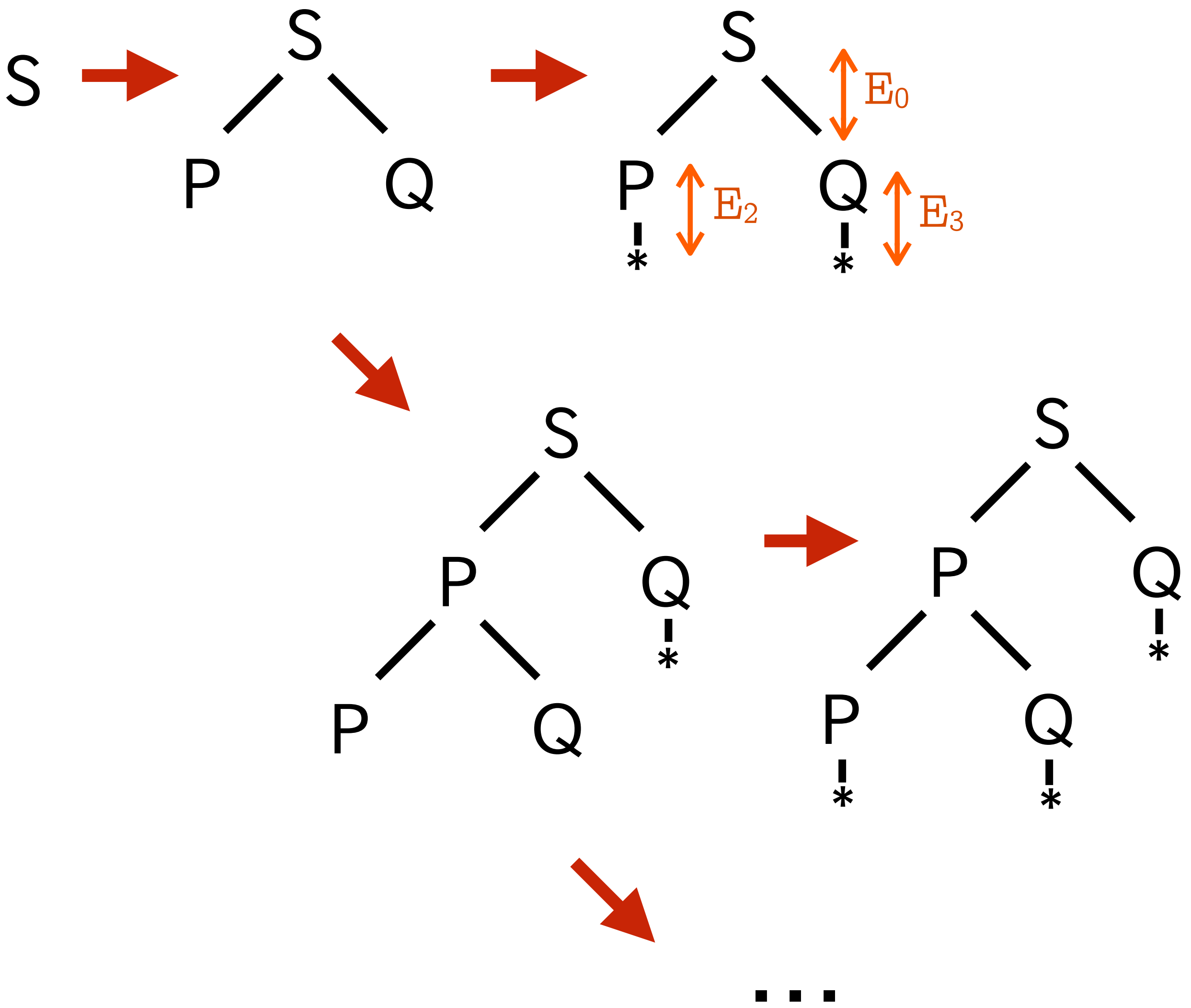

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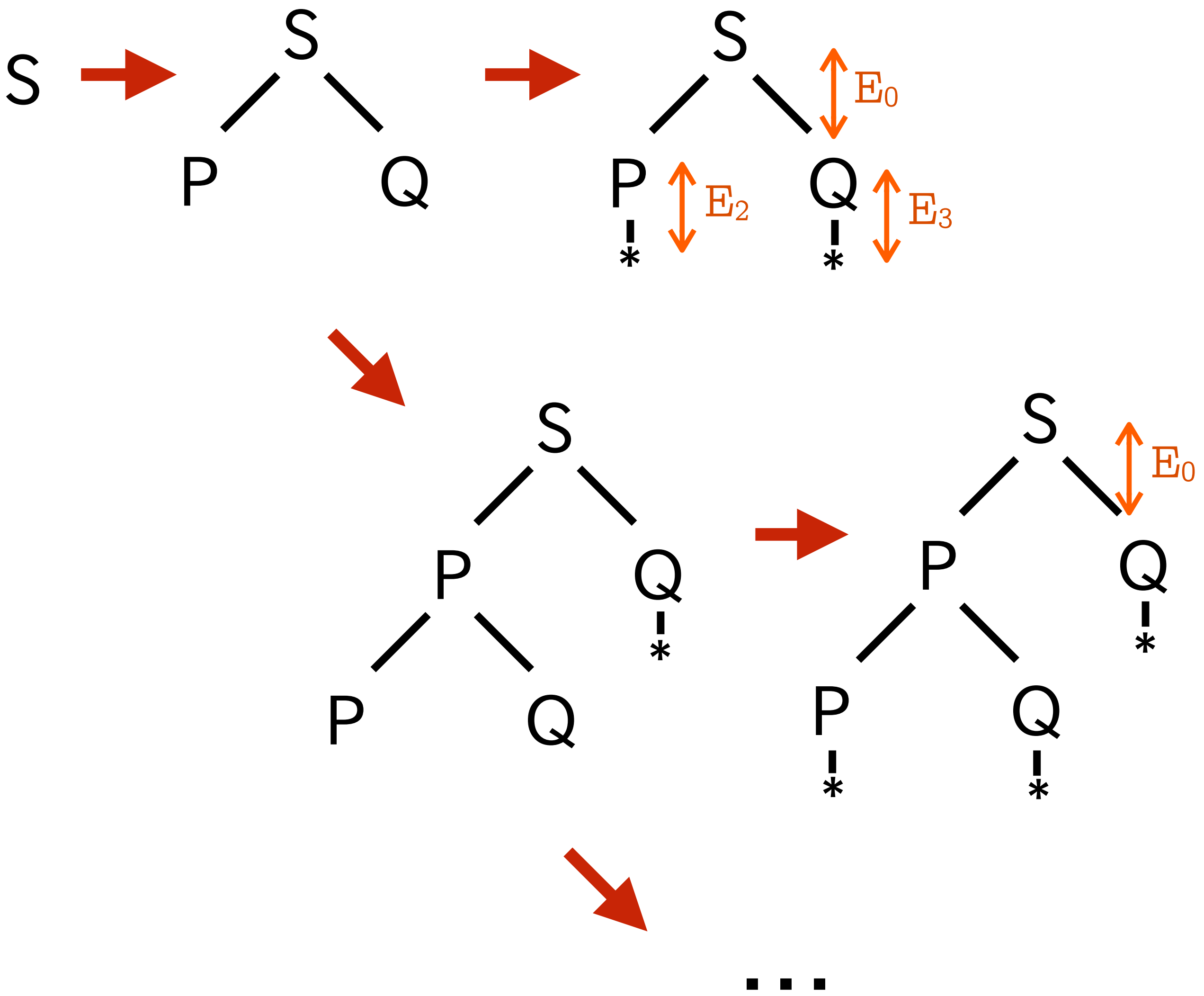

Case Study $\underset{\{S,\ P,\ Q\}}{\underline{X}} \to Q(\underset{\{\ *\ \}}{\underline{1}} + X \times X)$

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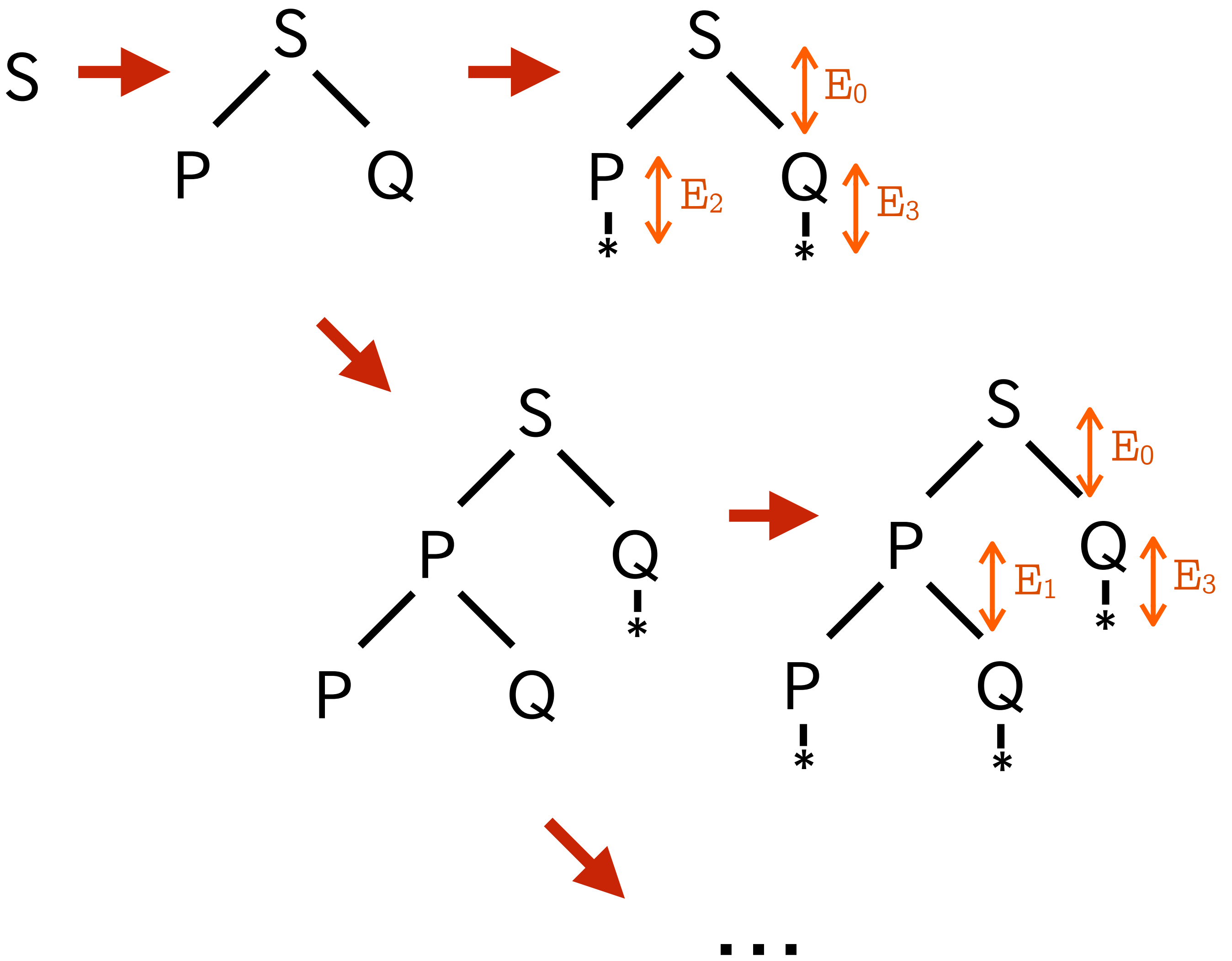

Case Study $\underset{\{S,\, P,\, Q\}}{\underline{X}} \to Q(\underset{\{\ *\ \}}{\underline{1}} + X \times X)$

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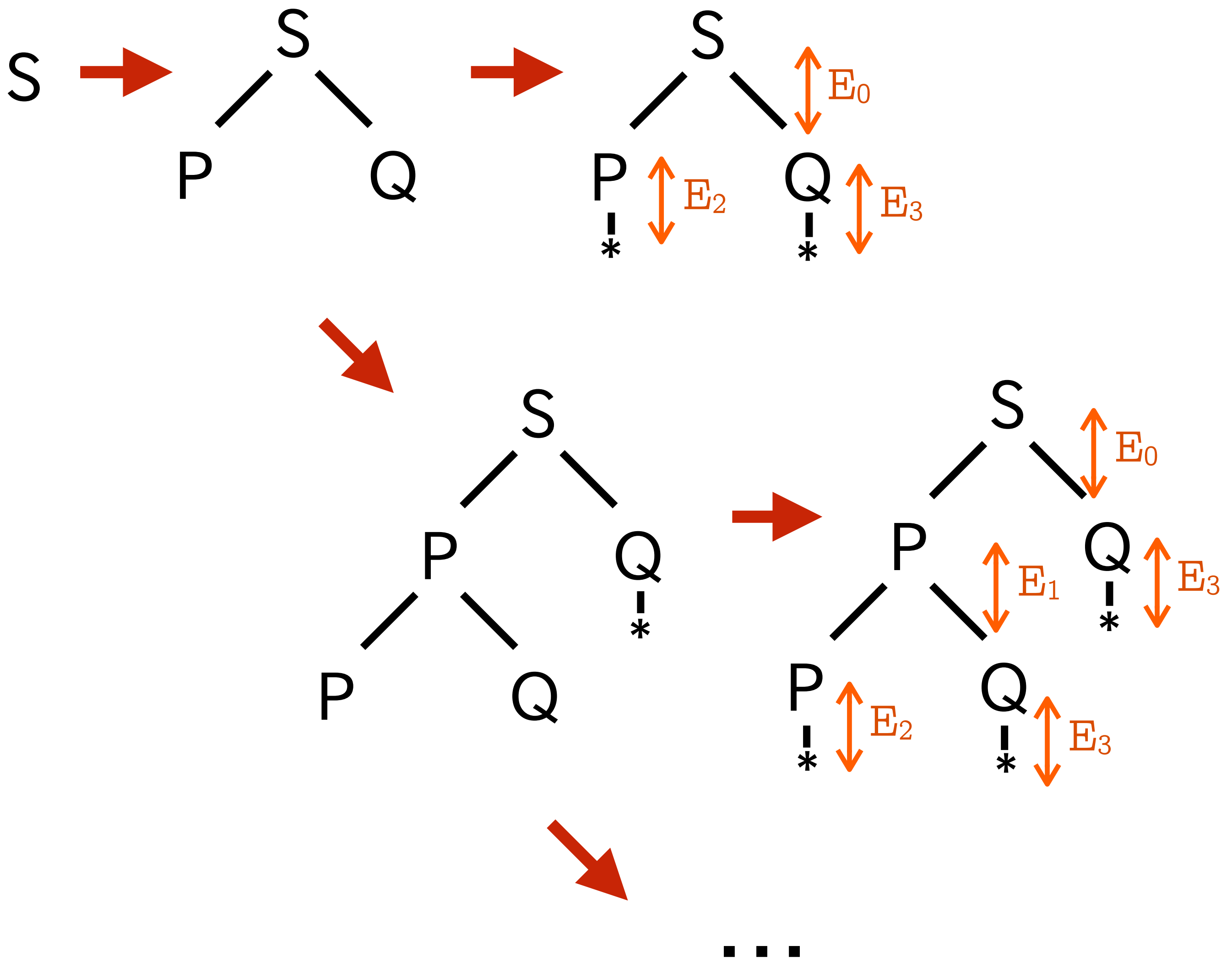

Case Study $\underset{\{S,\ P,\ Q\}}{\underline{X}} \to Q(\underset{\{\ *\ \}}{\underline{1}} + X \times X)$

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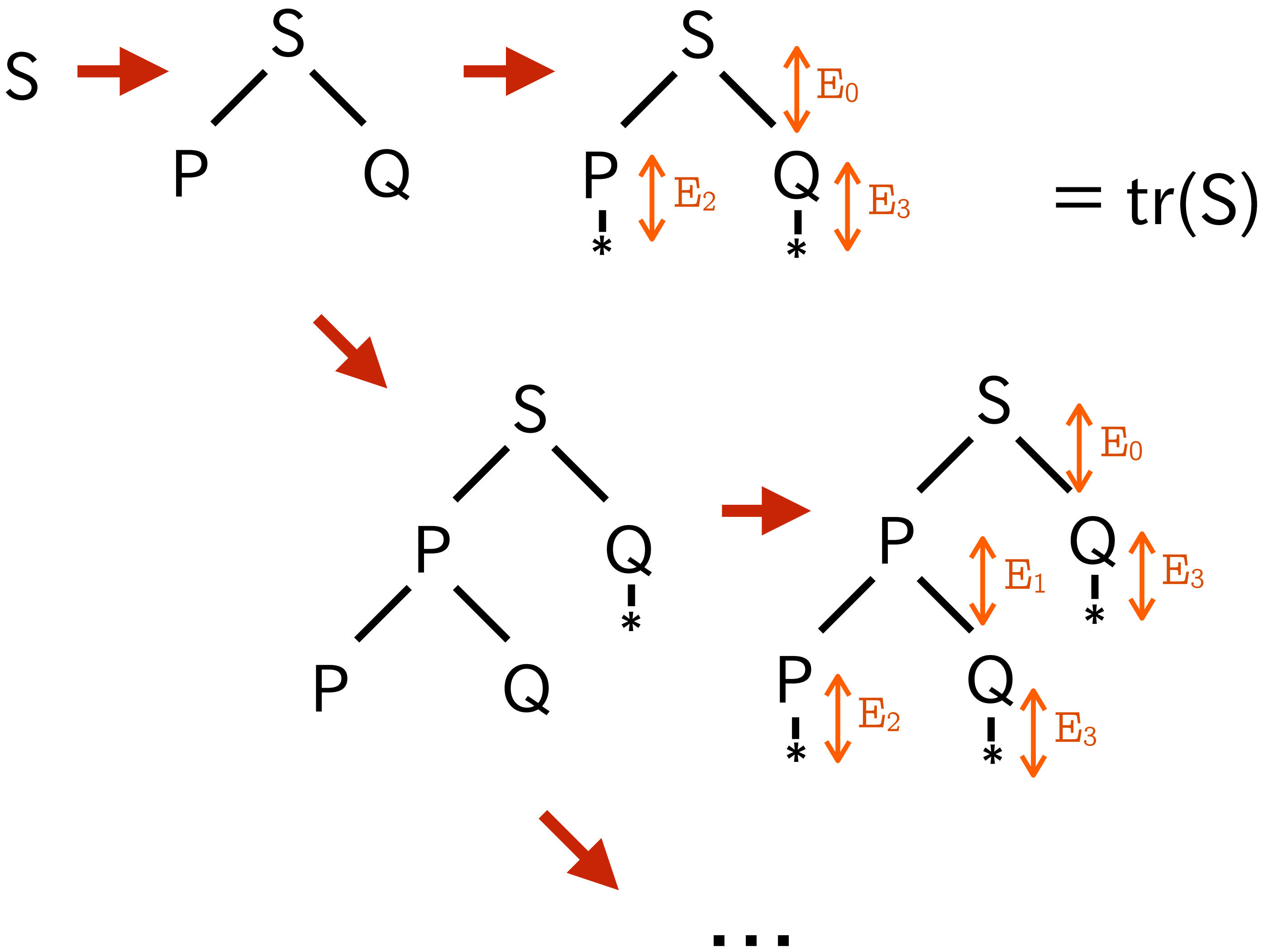

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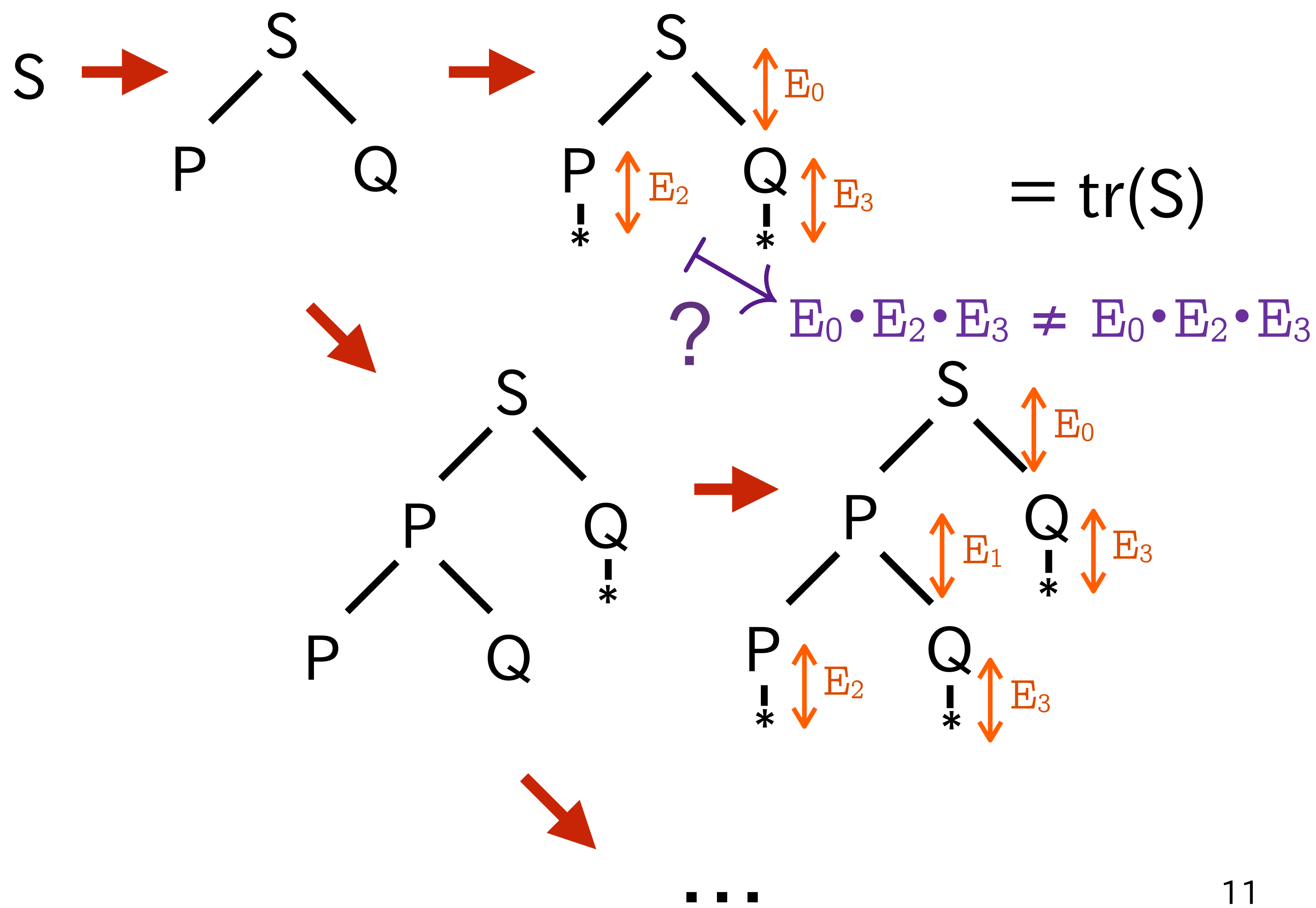

Case Study $X \to Q(1 + X \times X)$

$\{S, P, Q\}$ $\{ * \}$

$S \xrightarrow{E_0} PQ$

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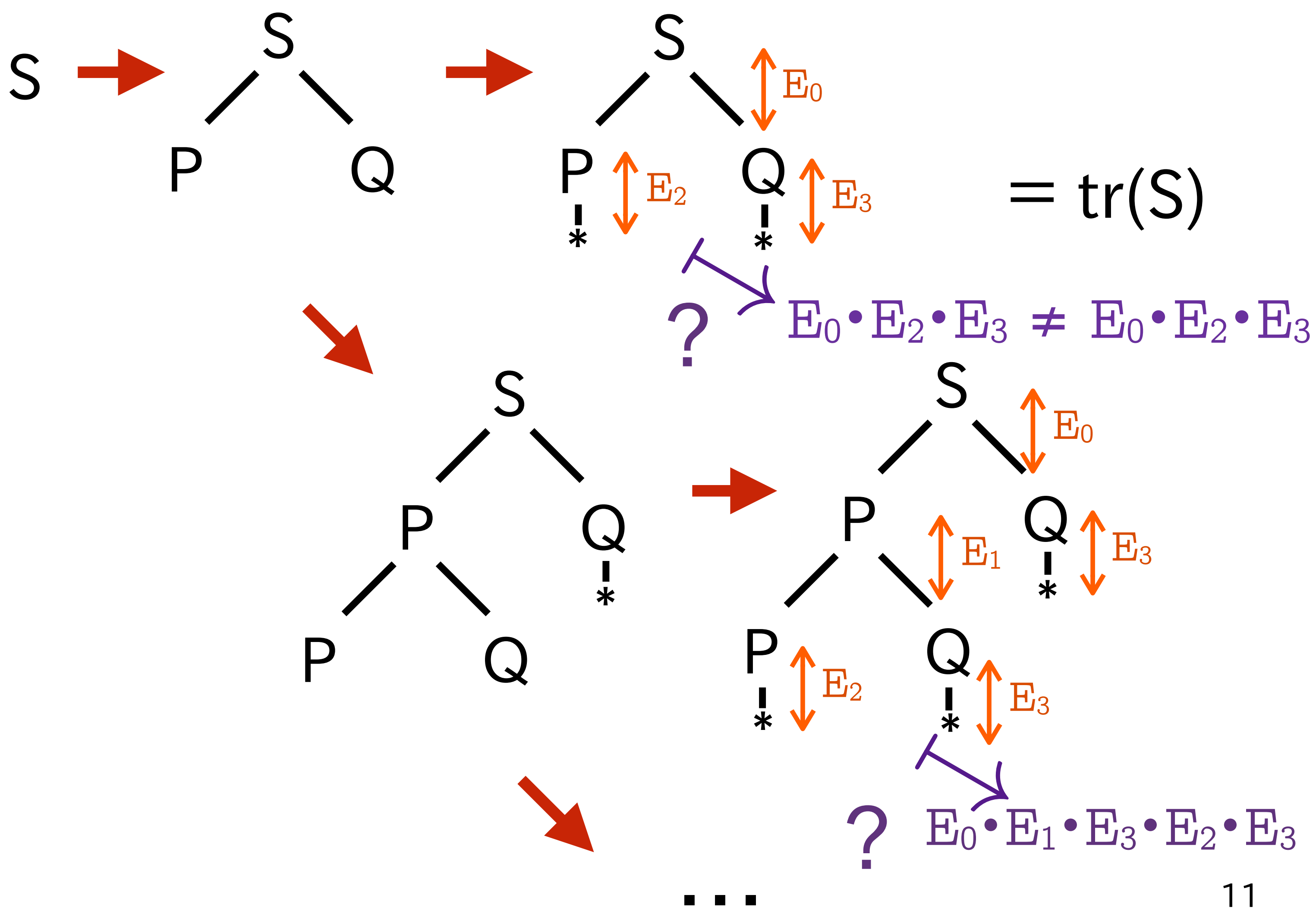

$= \mathrm{tr}(S)$

? $E_0 \bullet E_2 \bullet E_3 \neq E_0 \bullet E_2 \bullet E_3$

? $E_0 \bullet E_1 \bullet E_3 \bullet E_2 \bullet E_3$

Case Study $\underbrace{X}_{\{S,\ P,\ Q\}} \to Q(\underbrace{1}_{\{\ *\ \}} + X \times X)$

$$S \xrightarrow{E_0} PQ$$

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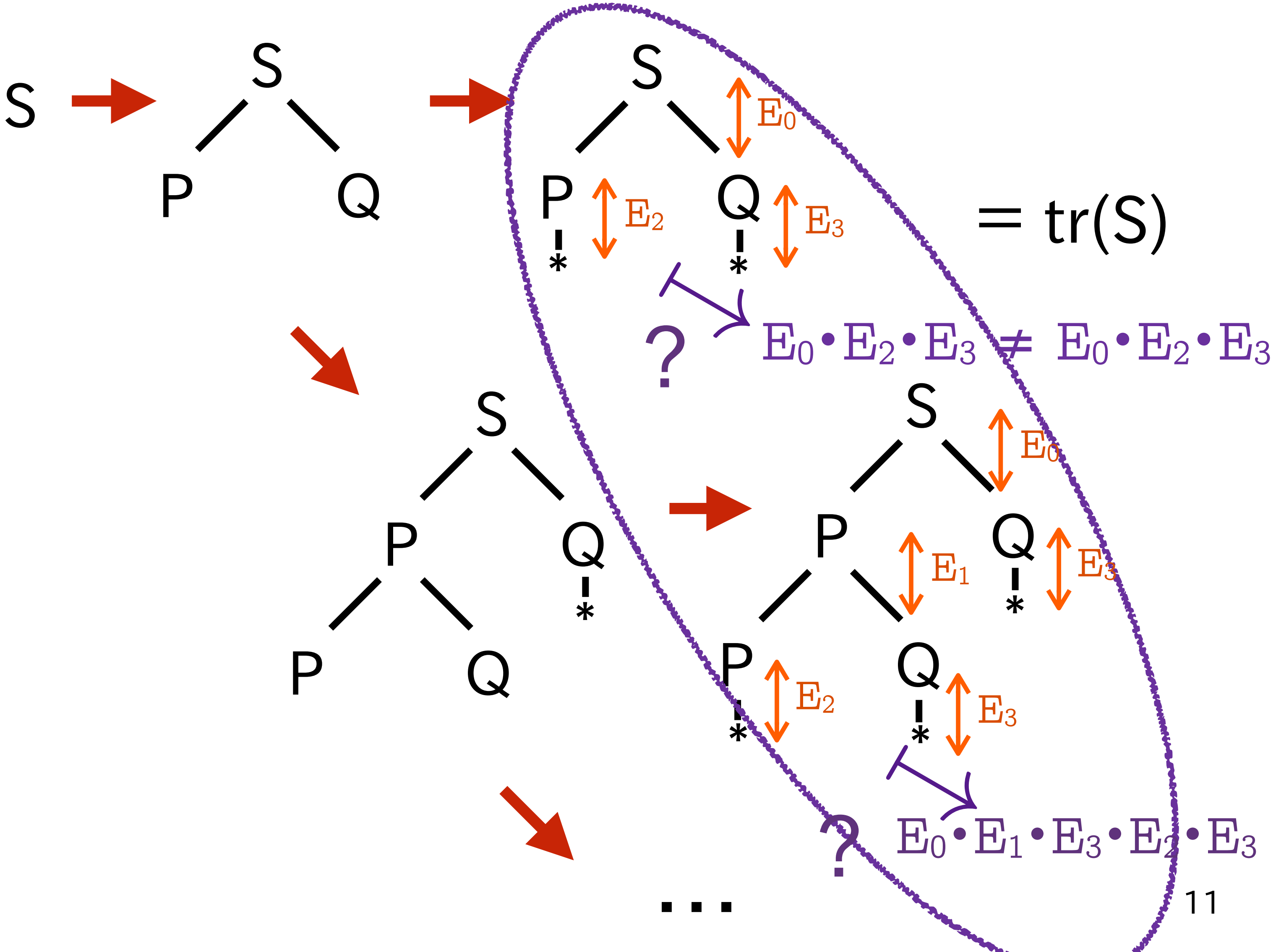

Case Study $\underset{\{S,\ P,\ Q\}}{\underline{X}} \to Q(\underset{\{\ *\ \}}{\underline{1}} + X \times X)$

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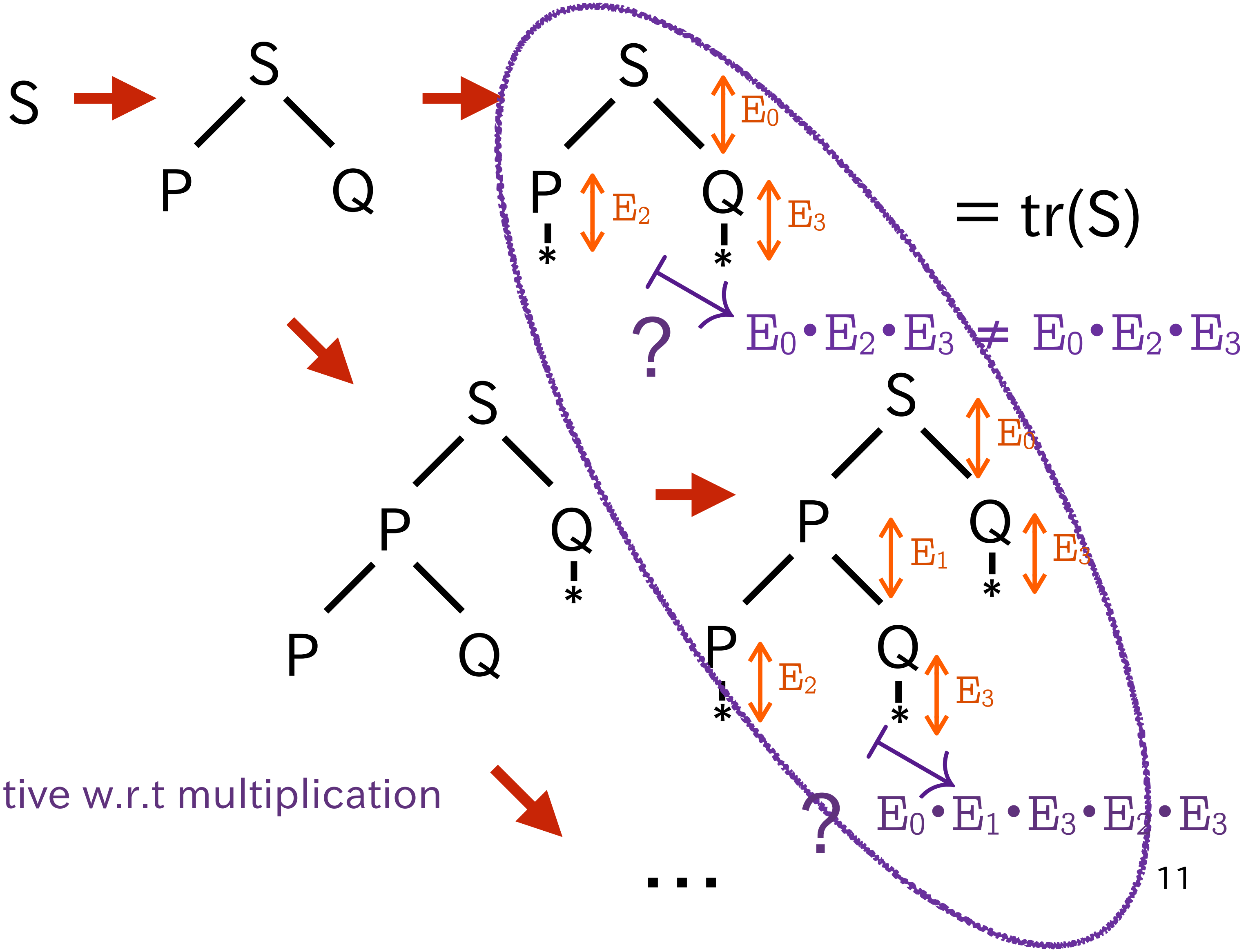

- QO is not commutative w.r.t multiplication

Contribution

- apply existing coalgebra theory to monad Q

	monad D	monad Q
trace sem.	D is commutative	Q is not commutative
bisim., behav. eq.	[0,1] is refinable	QO is not refinable
coalgebraic modal logic	[0,1] is cancellative	QO is cancellative

trace situation for
$$\begin{tikzcd} QFX \\ X \arrow[u] \end{tikzcd}$$
$$F ::= \mathrm{id} \mid \Sigma \mid \Sigma \times F' \mid \coprod F_i$$

Q-bisim.
$\neq$
Q-behav. eq

expressive modal logic
for Q-coalgebra

Contribution

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expressive modal logic for Q-coalgebra

Comparing Coalgebraic Equivalences

(Aczel-Mendler) F-bisimilarity

$$\begin{tikzcd}
& FR \arrow[dl, "F\pi"'] \arrow[dr, "F\pi'"] & \\
FX & & FY \\
& R \arrow[uu, "e"] \arrow[dl, "\pi"] \arrow[dr, "\pi'"'] & \\
X \arrow[uu, "c"] & & Y \arrow[uu, "d"']
\end{tikzcd}$$

$$(x, y) \in R$$

F-behavioral equivalence

$$\begin{tikzcd}
FX \arrow[dr, "Fh"] & & FY \arrow[dl, "Fk"'] \\
& FZ & \\
X \arrow[uu, "c"] \arrow[dr, "h"'] & & Y \arrow[uu, "d"'] \arrow[dl, "k"] \\
& Z \arrow[uu, "e"] &
\end{tikzcd}$$

$$h(x) = k(y)$$

Comparing Coalgebraic Equivalences

(Aczel-Mendler) F-bisimilarity

$$\begin{tikzcd}
 & FR \arrow[dl, "F\pi"'] \arrow[dr, "F\pi'"] & \\
FX & & FY \\
 & R \arrow[uu, "e"] \arrow[dl, "\pi"] \arrow[dr, "\pi'"'] & \\
X \arrow[uu, "c"] & & Y \arrow[uu, "d"']
\end{tikzcd}$$

$$(x, y) \in R$$

span

F-behavioral equivalence

$$\begin{tikzcd}
FX \arrow[dr, "Fh"] & & FY \arrow[dl, "Fk"'] \\
 & FZ & \\
X \arrow[uu, "c"] \arrow[dr, "h"'] & & Y \arrow[uu, "d"'] \arrow[dl, "k"] \\
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\end{tikzcd}$$

$$h(x) = k(y)$$

Comparing Coalgebraic Equivalences

(Aczel-Mendler) F-bisimilarity

$$\begin{tikzcd}
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\end{tikzcd}$$

$$(x, y) \in R$$

span

F-behavioral equivalence

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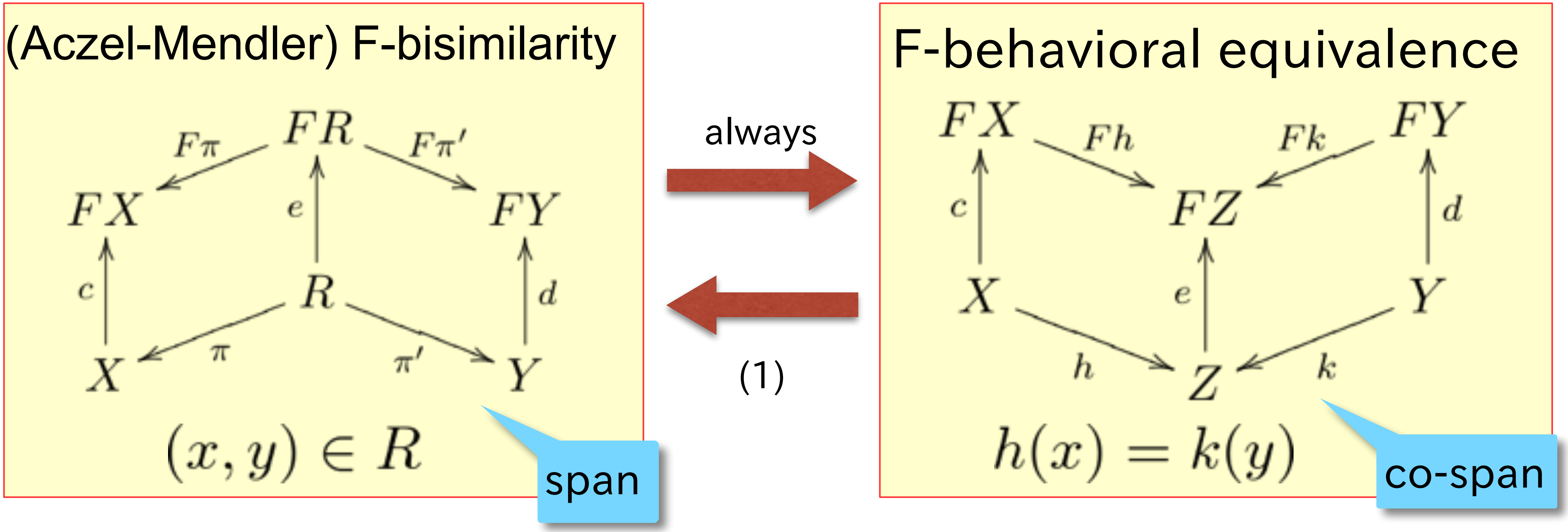
Comparing Coalgebraic Equivalences



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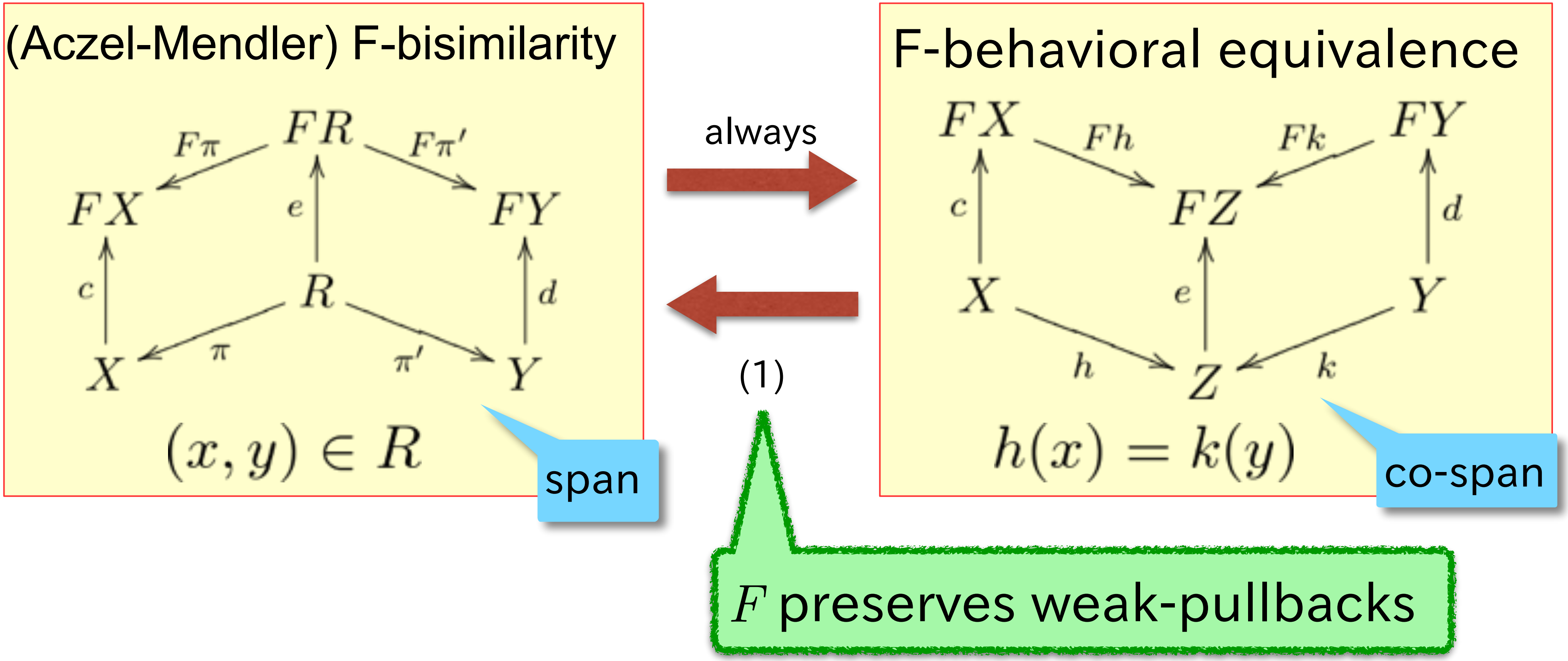
co-span

F preserves weak-pullbacks

Comparing Coalgebraic Equivalences



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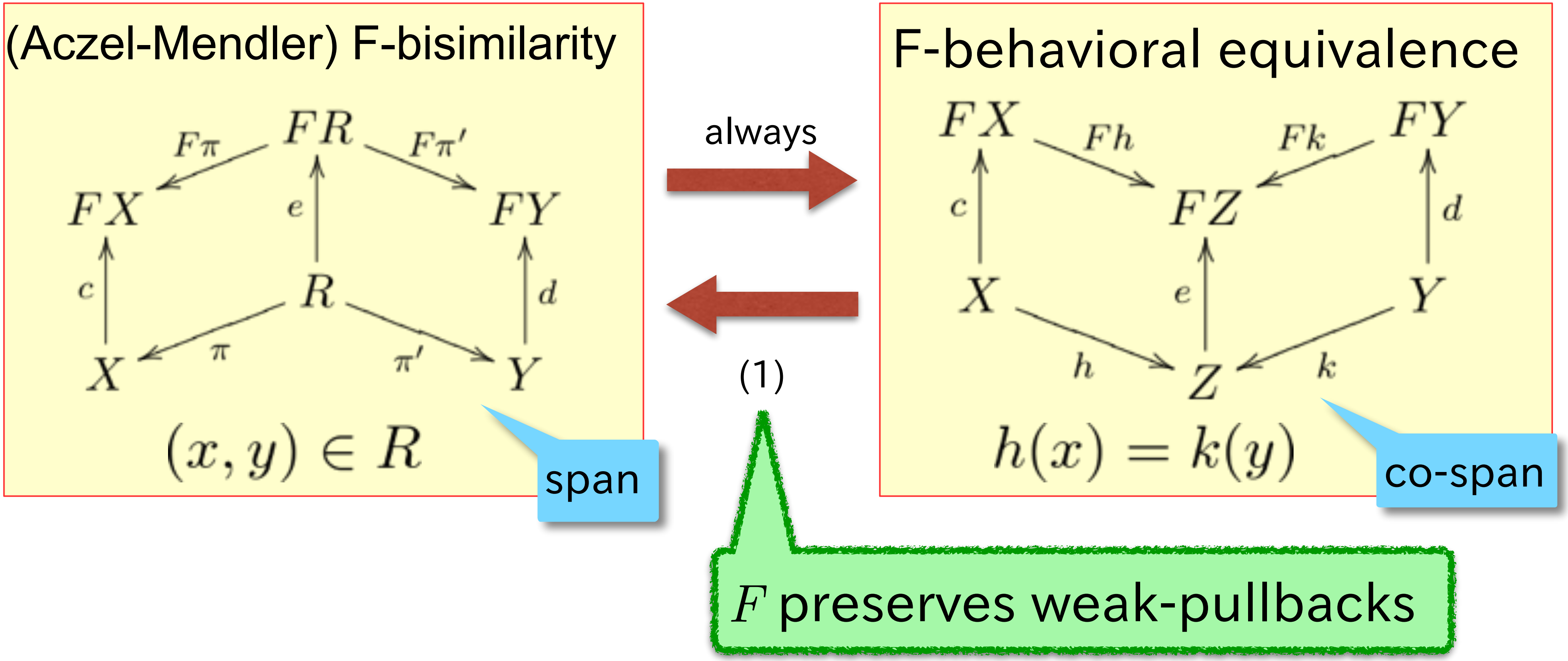
Comparing Coalgebraic Equivalences



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$\Longrightarrow$

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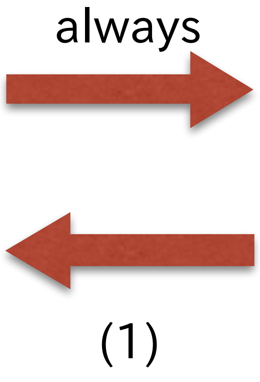

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In case of Multiset Functor [Gumm, Schroder 2001]

$\mathcal{M}_M$ preserves weak-pullbacks $\Longleftrightarrow$ $\left\{ \right.$

Monoid : $(M, +, 0)$

$\mathcal{M}_M(X) = \{\phi : X \to M \mid supp(\phi) : \text{finite}\}$

In case of Multiset Functor [Gumm, Schroder 2001]

$\mathcal{M}_M$ preserves weak-pullbacks ⟷

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$$r_1 + r_2 = c_1 + c_2$$

$$\Downarrow$$

$$\exists (m_{i,j}).$$

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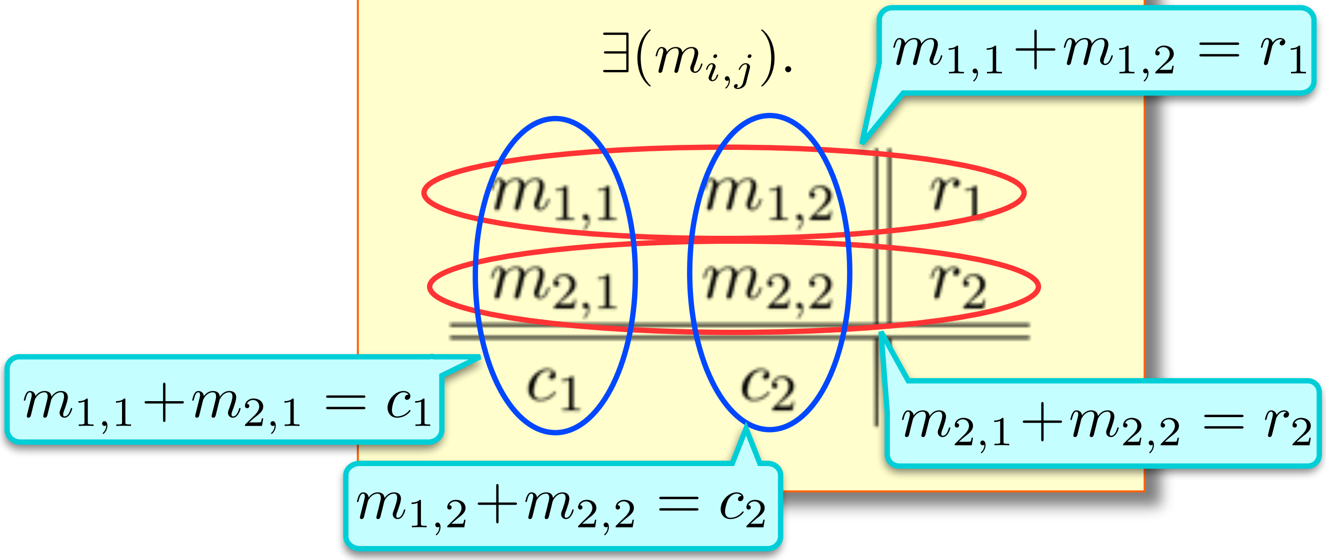

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Example : [0,1]

$$0.3+0.6 = 0.2+0.7\,(=0.9)$$

$$\begin{array}{cc||c} 0.1 & 0.2 & 0.3 \\ 0.1 & 0.5 & 0.6 \\ \hline\hline 0.2 & 0.7 & \end{array}$$

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⇩

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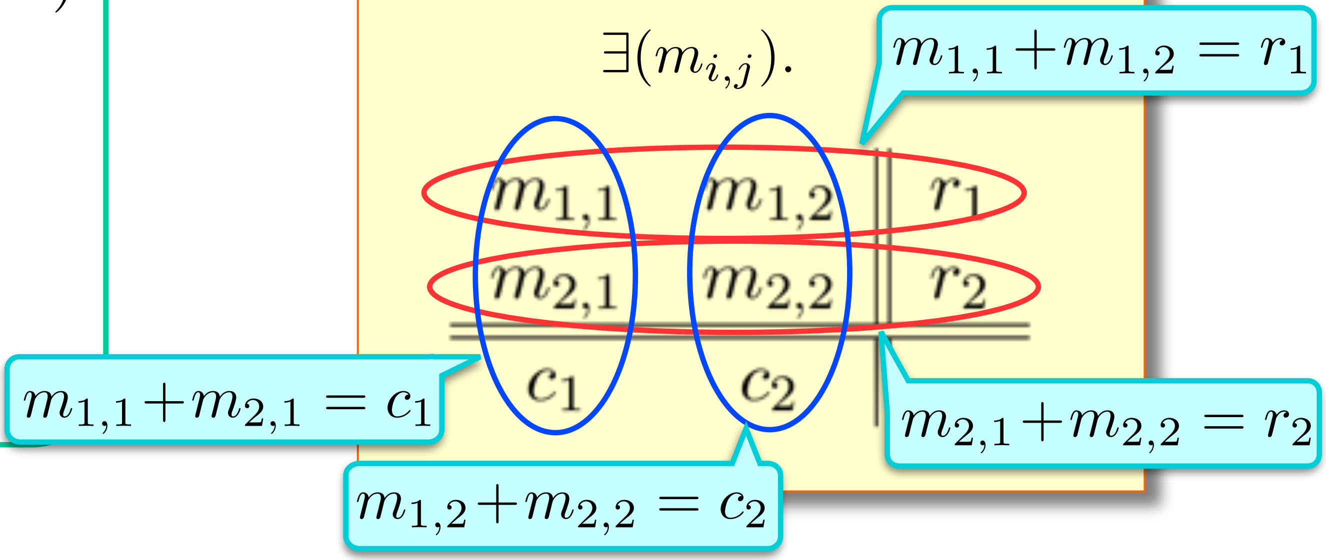

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Thm. QO is not refinable.

Counter Example:

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$$\langle 0| \_ |0\rangle + \langle 1| \_ |1\rangle = \langle +| \_ |+\rangle + \langle -| \_ |-\rangle$$

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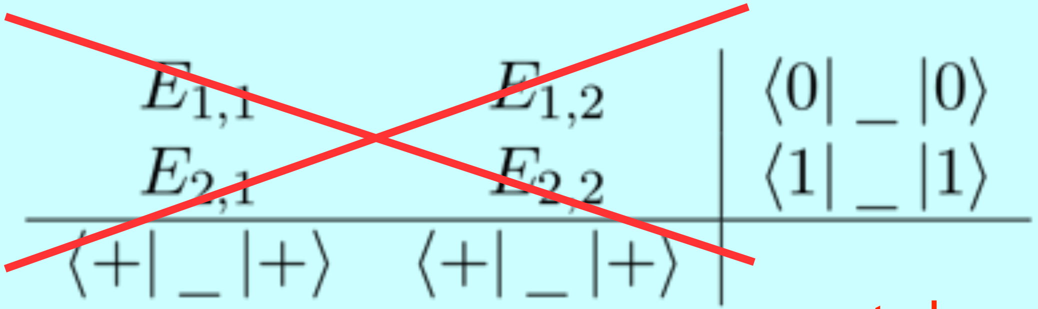

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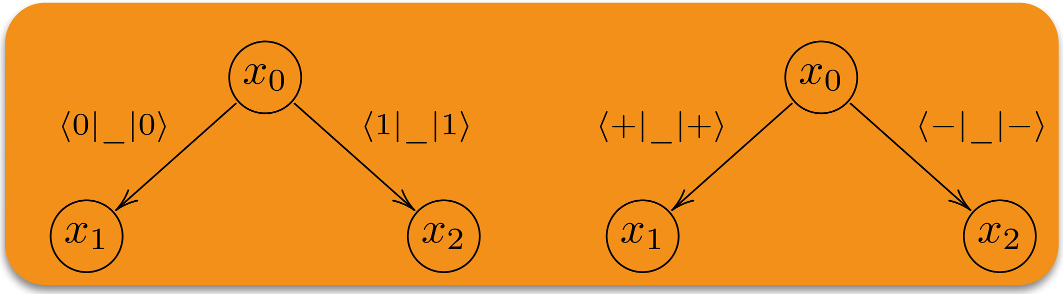

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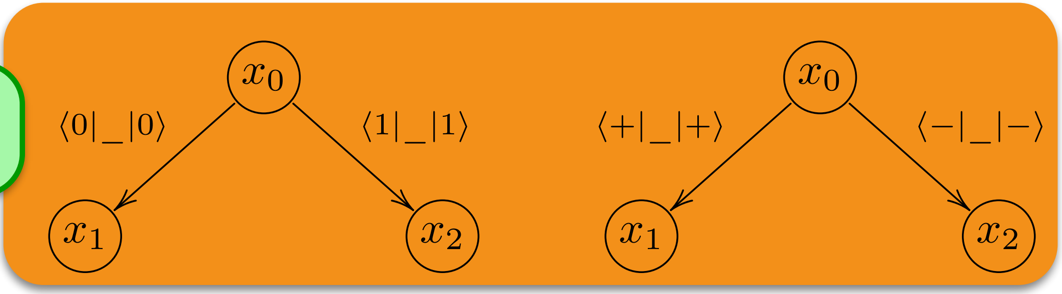

Contribution

- apply existing coalgebra theory to monad Q

	monad D	monad Q
trace sem.	D is commutative	Q is not commutative
bisim., behav. eq.	[0,1] is refinable	QO is not refinable
coalgebraic modal logic	[0,1] is cancellative	QO is cancellative

trace situation for
$$\begin{tikzcd} QFX \\ X \arrow[u] \end{tikzcd}$$
$$F ::= \mathrm{id} \mid \Sigma \mid \Sigma \times F' \mid \coprod F_i$$

Q-bisim.

$\neq$

Q-behav. eq

expressive modal logic
for Q-coalgebra

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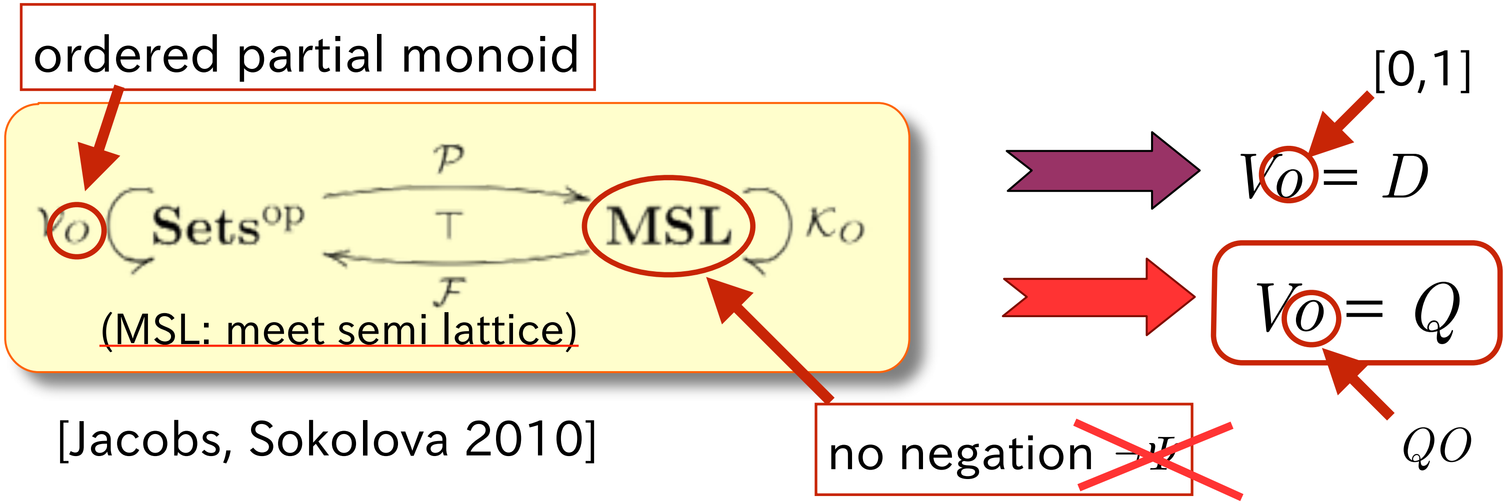
Coalgebraic Modal Logic

- framework for considering modal logic fits to coalgebra



ordered partial monoid

$$\begin{tikzcd} \mathbf{Sets}^{\mathrm{op}} \arrow[loop left, "\mathcal{V}_O"] \arrow[r, bend left, "\mathcal{P}"] \arrow[r, phantom, "\top"] & \mathbf{MSL} \arrow[l, bend left, "\mathcal{F}"] \arrow[loop right, "\mathcal{K}_O"] \end{tikzcd}$$

(MSL: meet semi lattice)

[Jacobs, Sokolova 2010]

no negation ~~$\neg\psi$~~

[0,1]

$\mathcal{V}_O = D$

$\mathcal{V}_O = Q$

QO

- Thm. Logic is expressive if O is canncellative.

$$x + z \leq y + z \implies x \leq y$$

$\psi ::= \top \mid \psi_1 \wedge \psi_2 \mid \Box_E \psi$

$x \models_c \Box_E \psi \iff \sum_{x' \models_c \psi} c(x)(x') \sqsupseteq E$

$E \in QO$

$\{\psi \in L \mid x \models \psi\} = \{\psi \in L \mid y \models \psi\}$

⇧ ⬇ expressive!

x and *y* are behavioral equivalent

Conclusions and Future Work

- apply existing coalgebra theory to monad Q

	monad D	monad Q
trace sem.	D is commutative	Q is not commutative
bisim., behav. eq.	[0,1] is refinable	QO is not refinable
coalgebraic modal logic	[0,1] is cancellative	QO is cancellative

trace situation for

$$\begin{tikzcd} QFX \\ X \arrow[u] \end{tikzcd}$$

$$F ::= \mathrm{id} \mid \Sigma \mid \Sigma \times F' \mid \coprod F_i$$

Q-bisim.

$\neq$

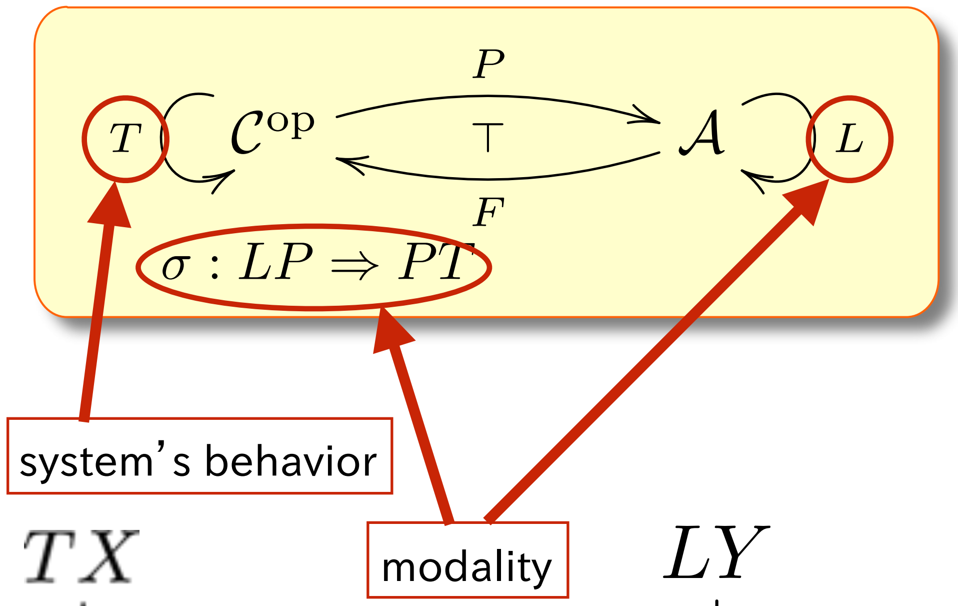
Q-behav. eq

expressive modal logic for Q-coalgebra

- Future Work
  - bialgebra and structural operational semantics [Turi, Plotkin]

→ quantum process calculus

Coalgebraic Modal Logic



$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[loop left, "T"] \arrow[r, bend left, "P"] \arrow[r, phantom, "\top"] & \mathcal{A} \arrow[l, bend left, "F"] \arrow[loop right, "L"]
\end{tikzcd}$$

$\sigma : LP \Rightarrow PT$

system's behavior

modality

$$\begin{tikzcd}
TX & & LY \arrow[d] \\
X \arrow[u] & & Y
\end{tikzcd}$$

概要

目的: 余代数を量子システムへ応用

余代数の理論	量子システム
trace semantics, fwd/bwd simulation	量子プロトコルの検証
bisimilarity, behavioral equivalence	bisimilarity ≠ behavioral equivalence
coalgebraic modal logic	量子的振る舞いを 表現するmodal logic (correct by construction)

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$$\begin{tikzcd}
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trace semantics, fwd/bwd simulation	量子プロトコルの検証
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余代数とは

- 様々な種類の状態遷移系を統一的に表現
  - 共通する概念を抽象的に議論
  - ファンクタ F : パラメータ

$$\begin{tikzcd} FX \\ X \arrow[u] \end{tikzcd}$$

F				

余代数とは

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F	P			

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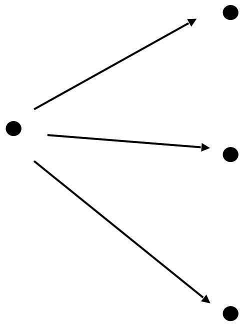
F	P			
	 非決定的			

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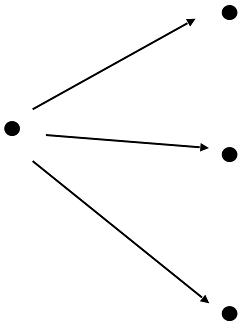
F	P	D		
	 非決定的			

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	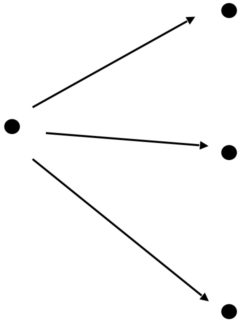	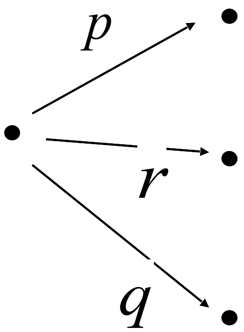 $p+q+r\leq 1$		
	非決定的	確率的		

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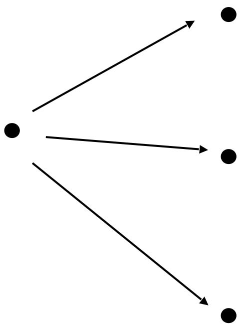
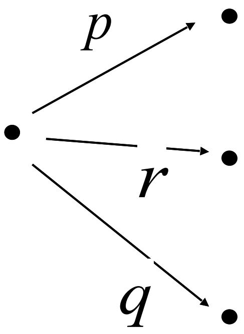
$$\begin{tikzcd} FX \\ X \arrow[u] \end{tikzcd}$$

F	P	D	M(マルチセット)	
		 $p+q+r\leq 1$		
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	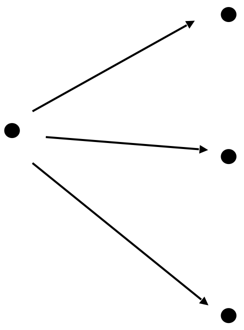	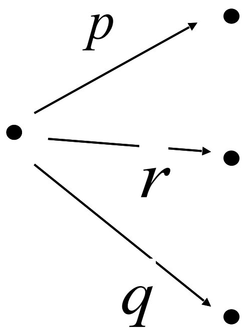	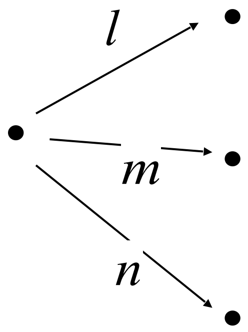
		$p+q+r\leq 1$	$l,m,n\in\mathcal{M}$ モノイド M $\mathcal{M}$ (e.g. 自然数、実数)
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余代数とは

- 様々な種類の**状態遷移系**を統一的に表現
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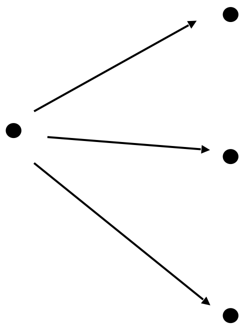
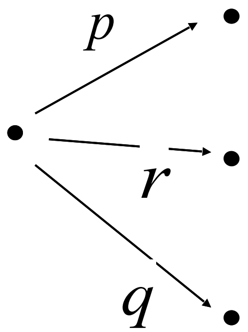
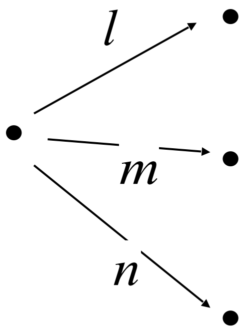
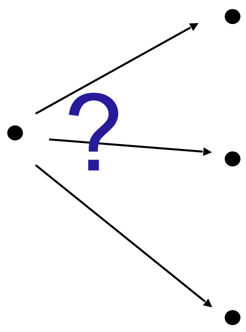
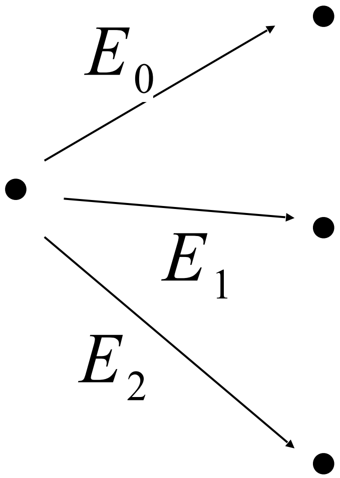
F	P	D	M(マルチセット)	?
		 $p+q+r\leq 1$	 $l,m,n\in\mathcal{M}$	
	非決定的	確率的	モノイド M $\mathcal{M}$ (e.g. 自然数、実数)	量子的遷移?

量子分岐モナドQ

[Hasuo&Hoshino, LICS'11]

- quantum operationで重み付けされた遷移
  - 量子状態の準備
  - ユニタリ変換
  - 量子測定

cf. qMC [Ying et al. 2013]



$$QX = \{\phi : X \to \prod_{m,n} QO_{m,n} \mid \text{trace condition (1)}\}$$

$$(1) : \forall m. \forall \rho \in DM_m. \sum_{x \in X} \sum_{n \in N} (\mathrm{tr}(\phi(x))_{m,n}(\rho)) \le 1$$

$$\forall \rho \in DM. \sum_i E_i(\rho) \le 1$$

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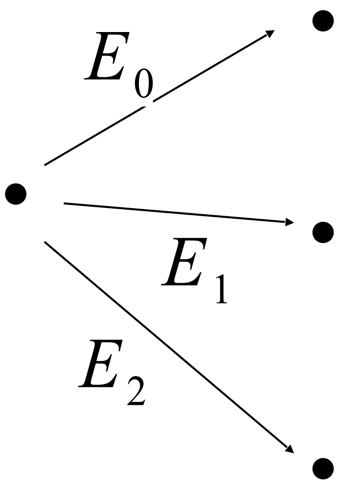
量子計算における
3つの基本操作

cf. qMC [Ying et al. 2013]



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Ogawa (Tokyo)

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$\underline{\mathrm{fwd}}$

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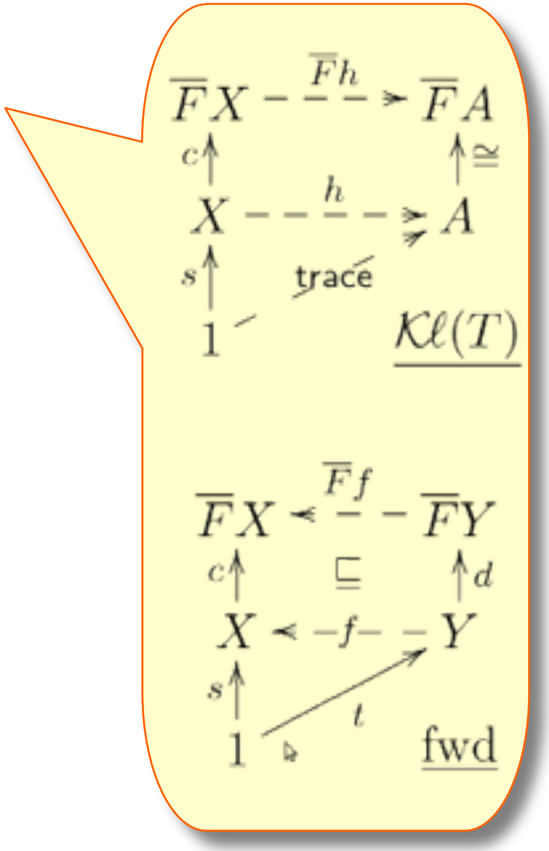

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$T = D$

確率的ネットワークプロトコル
(Crowds protocol)
の匿名性の検証
[Hasuo, Kawabe & Sakurada 2010]

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$T = D$

$T = Q$

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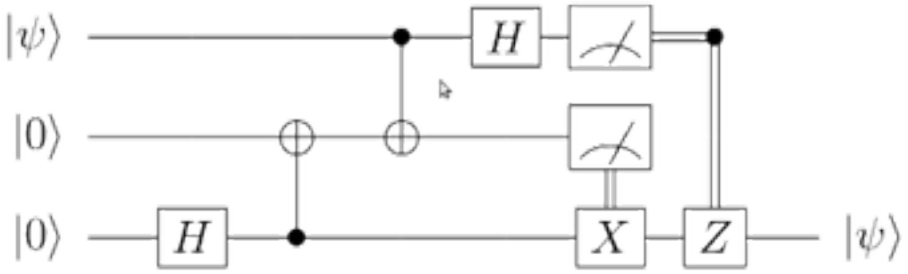
量子プロトコルの
確率的振る舞いに関する
検証

量子システム(プロトコル)

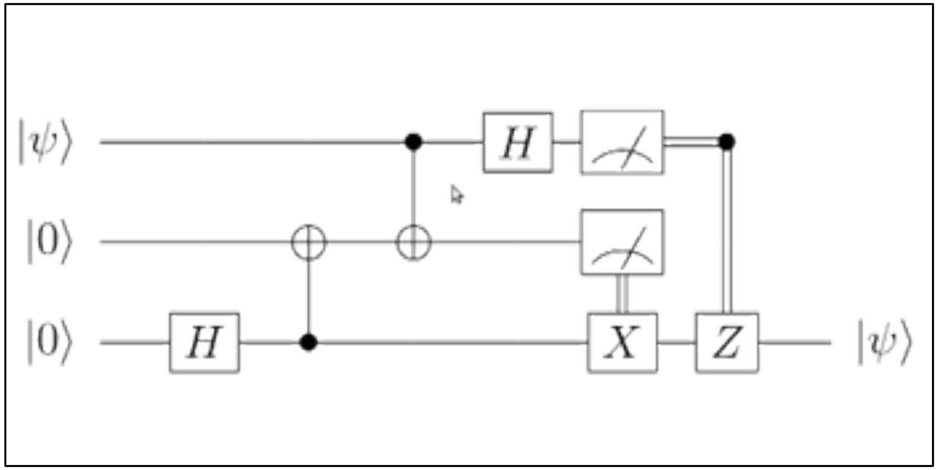
量子的遷移系でモデル

量子システム(プロトコル)



量子的遷移系でモデル

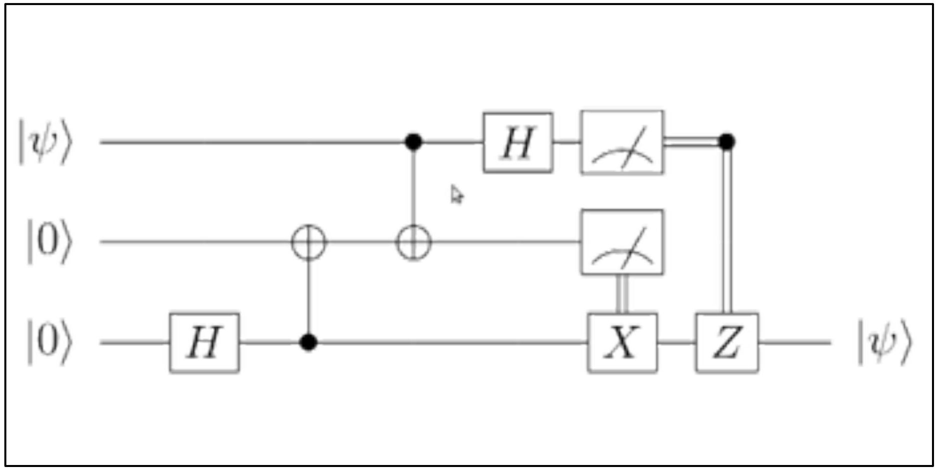
量子システム(プロトコル)



$\Rightarrow$

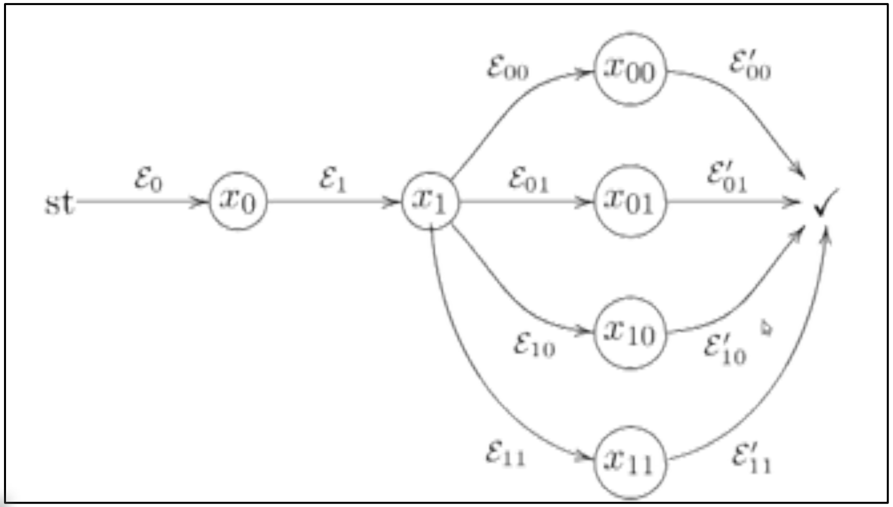
量子的遷移系でモデル

量子システム(プロトコル)



⇒

量子的遷移系でモデル



$\underline{\mathcal{K}\ell(\mathcal{Q})}$

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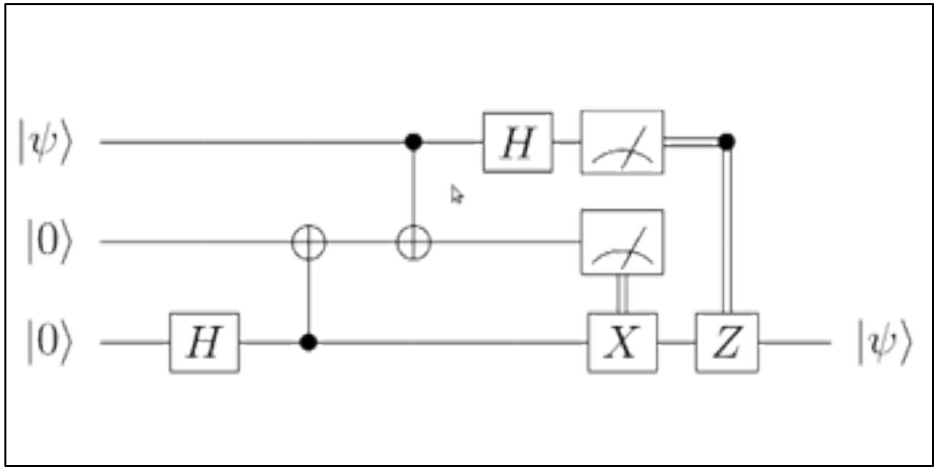
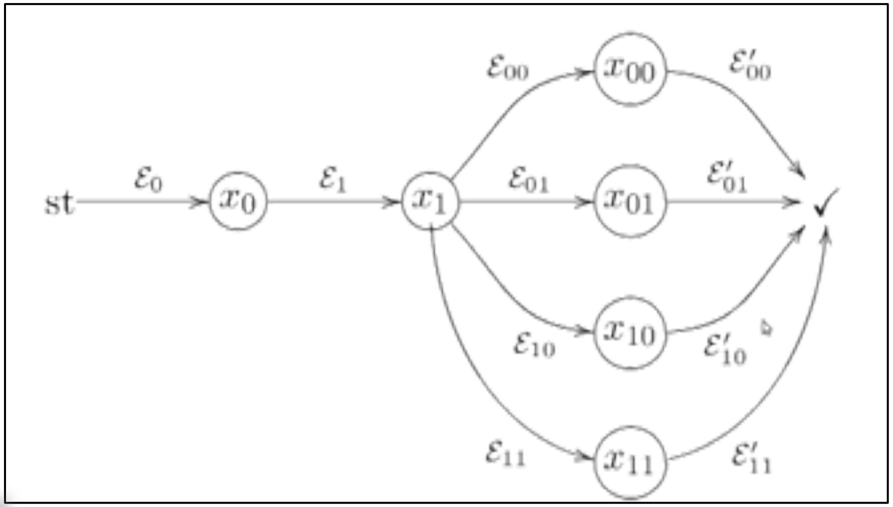
量子システム(プロトコル)

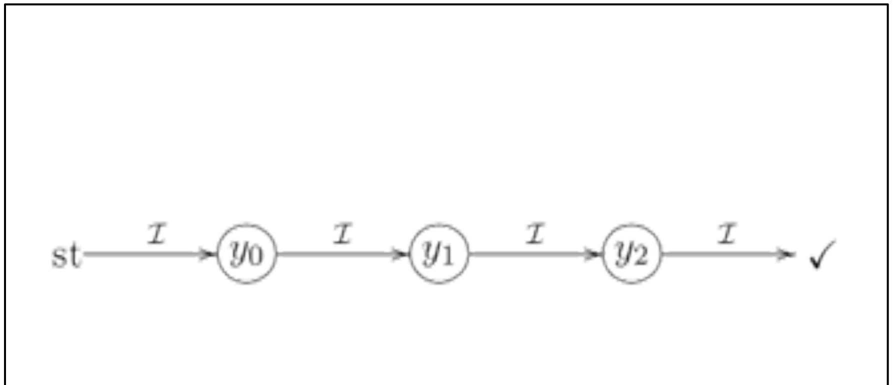


⇒

量子的遷移系でモデル

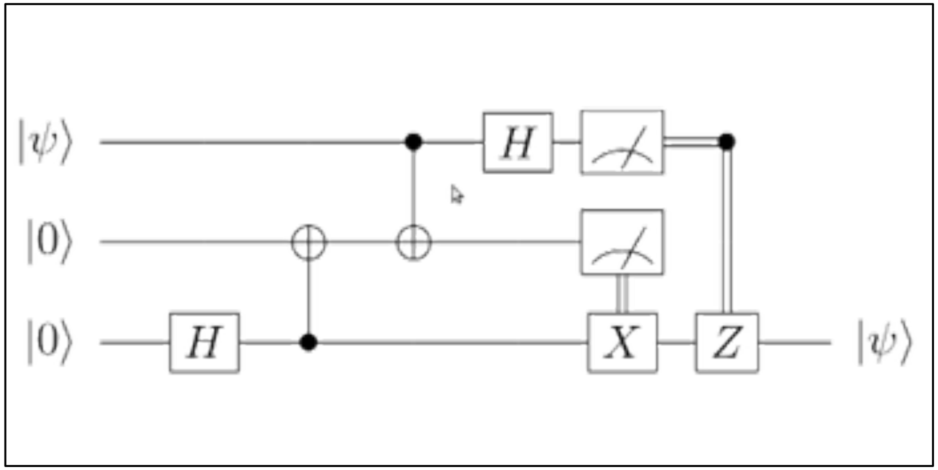


$$\underline{\mathcal{K}\ell(\mathcal{Q})}\qquad \begin{tikzcd} \overline{F}X \\ X \arrow[u, "c"] \\ 1 \arrow[u, "s"] \end{tikzcd}$$



specification

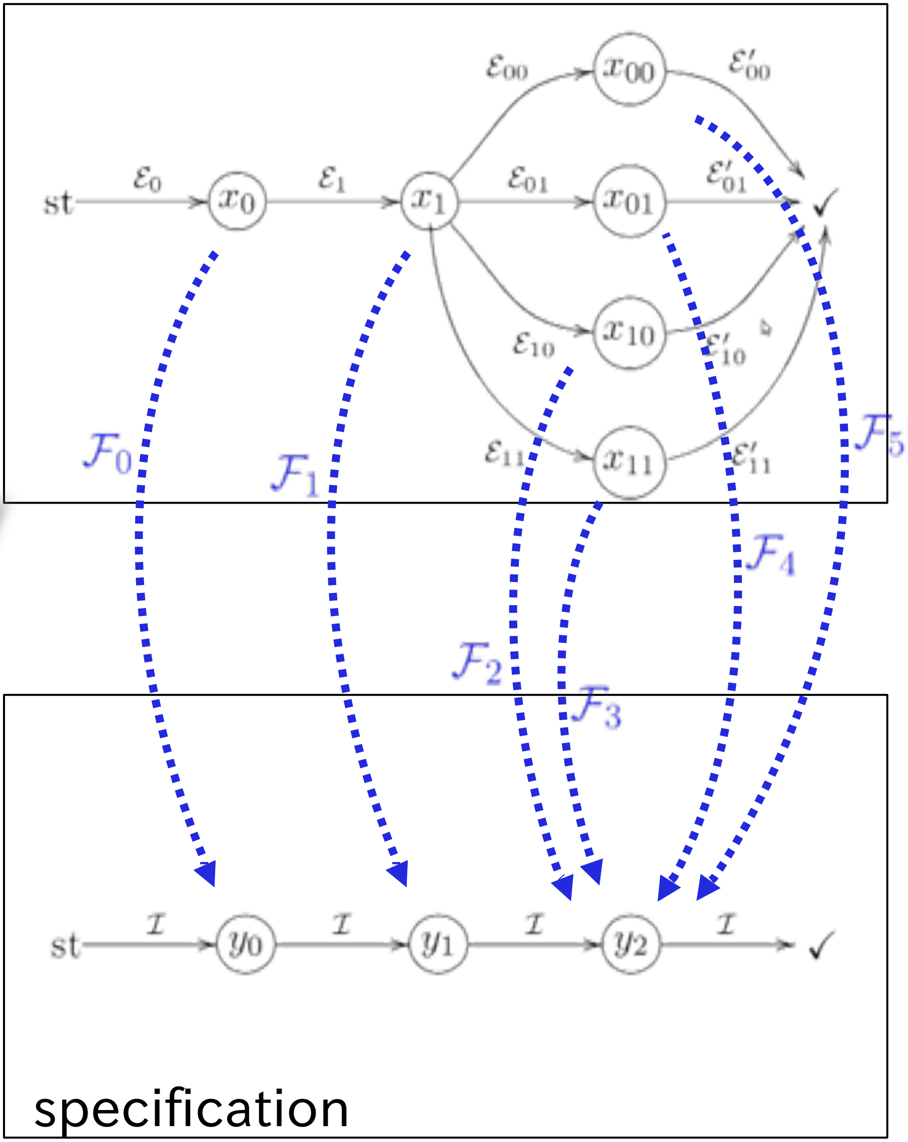
量子システム(プロトコル)



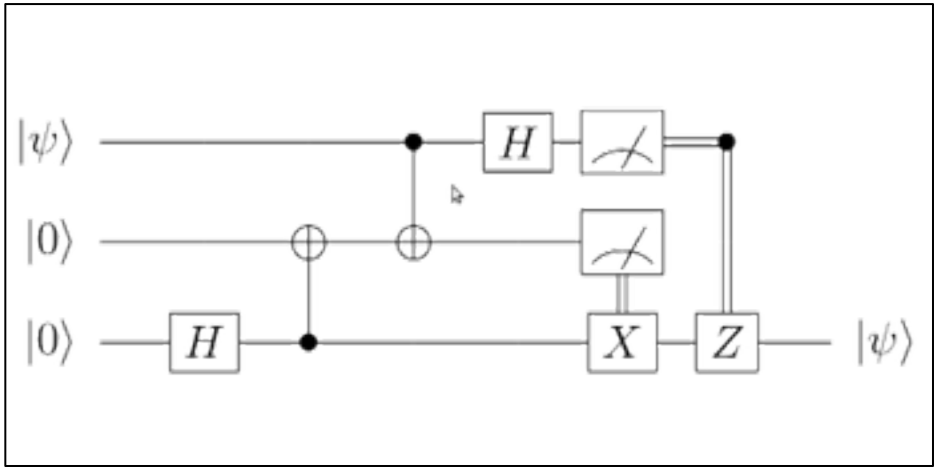
⇒

量子的遷移系でモデル

$$\begin{tikzcd} & \overline{F}X \\ \underline{\mathcal{K}\ell(\mathcal{Q})} & X \arrow[u, "c"] \\ & 1 \arrow[u, "s"] \end{tikzcd}$$



量子システム(プロトコル)

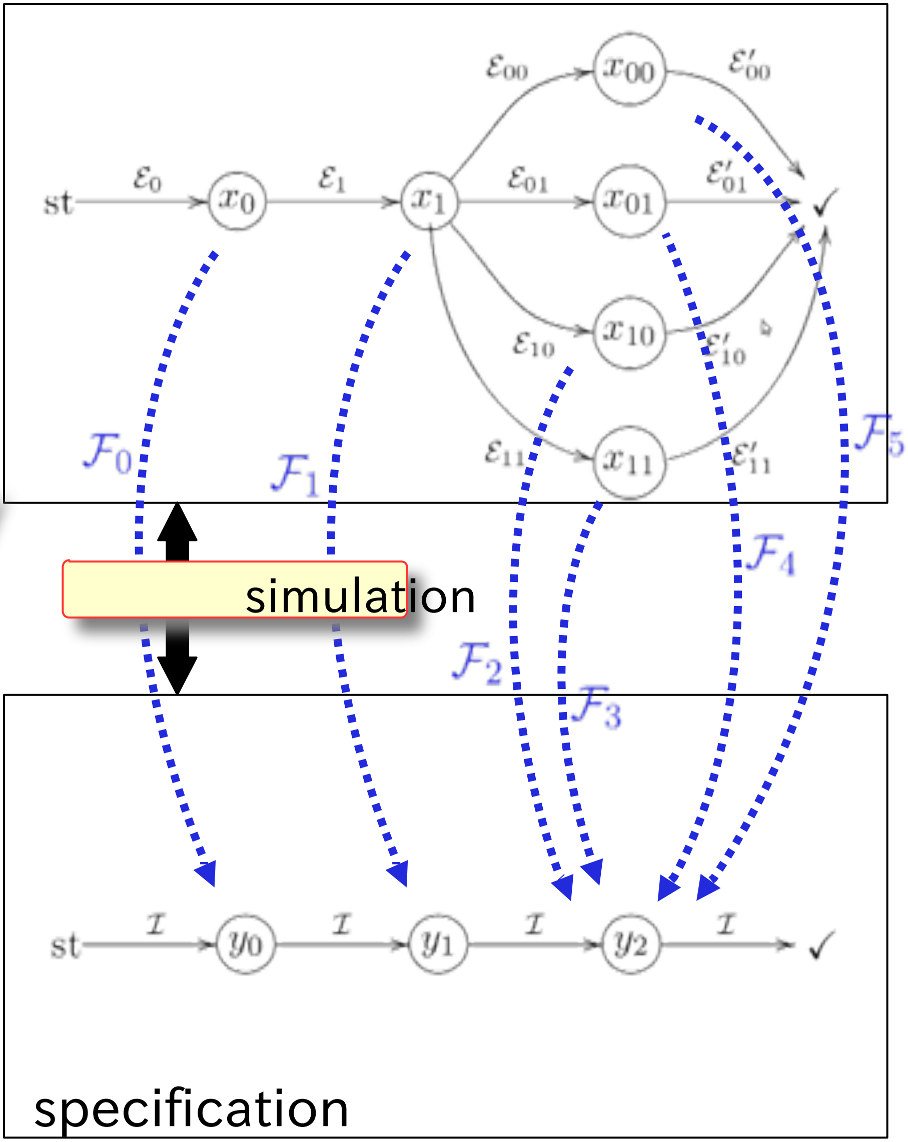


⇒

量子的遷移系でモデル

$\underline{\mathcal{K}\ell(\mathcal{Q})}$

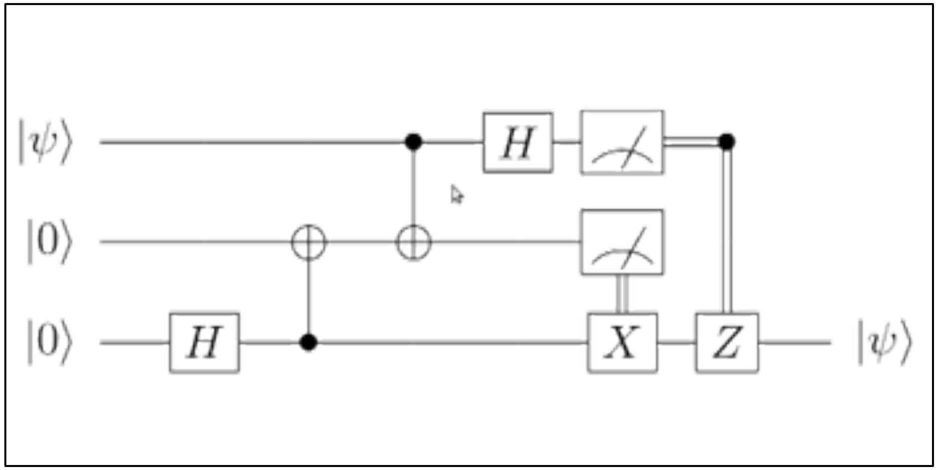
$$\begin{tikzcd}
\overline{F}X \\
X \arrow[u, "c"] \\
1 \arrow[u, "s"]
\end{tikzcd}$$



simulation

specification

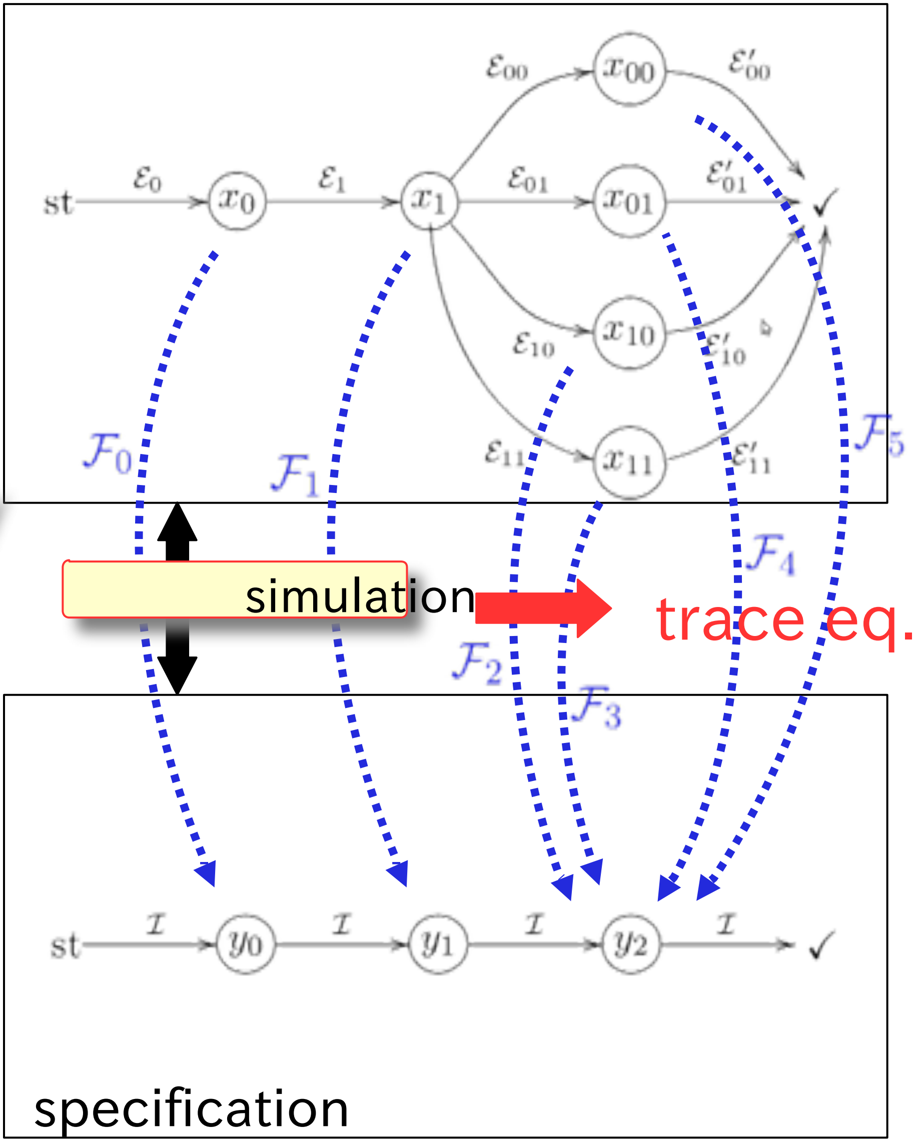
量子システム(プロトコル)



⇒

量子的遷移系でモデル

$$\begin{tikzcd} & \overline{F}X \\ \underline{\mathcal{K}\ell(\mathcal{Q})} & X \arrow[u, "c"] \\ & 1 \arrow[u, "s"] \end{tikzcd}$$

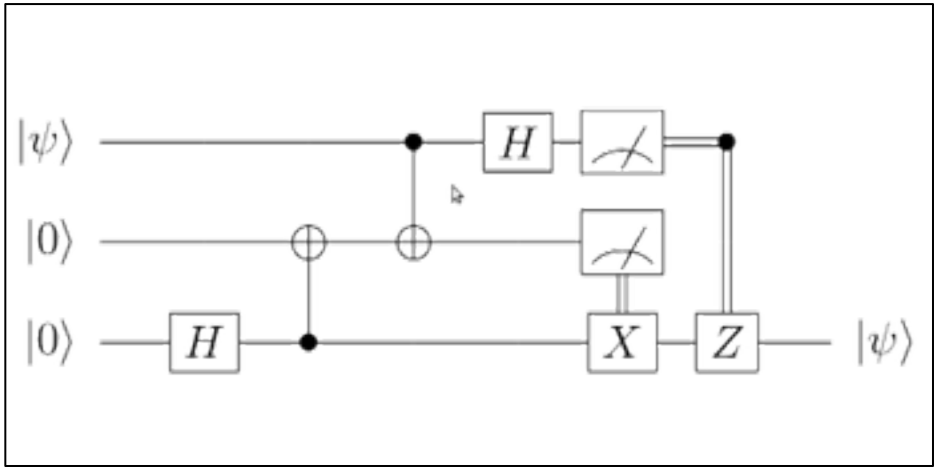


simulation

trace eq.

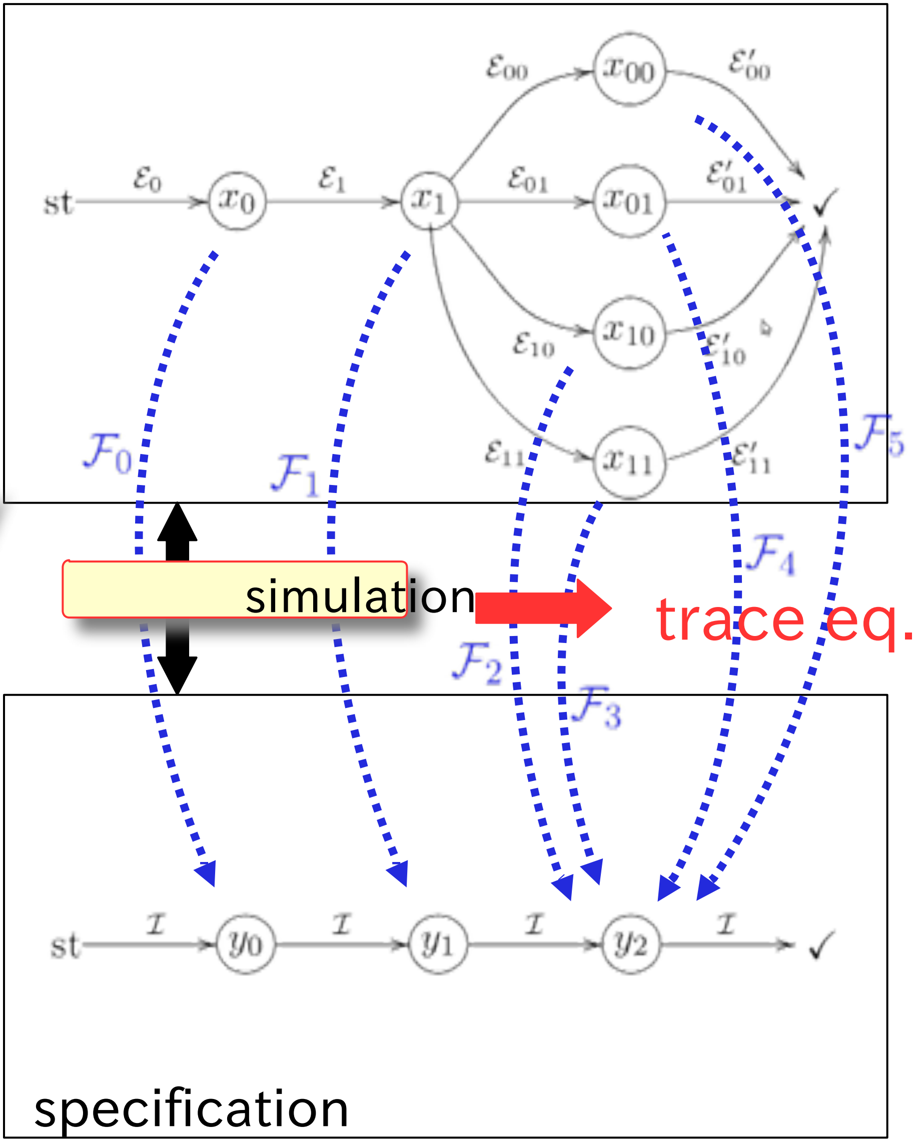
specification

量子システム(プロトコル)



⇒

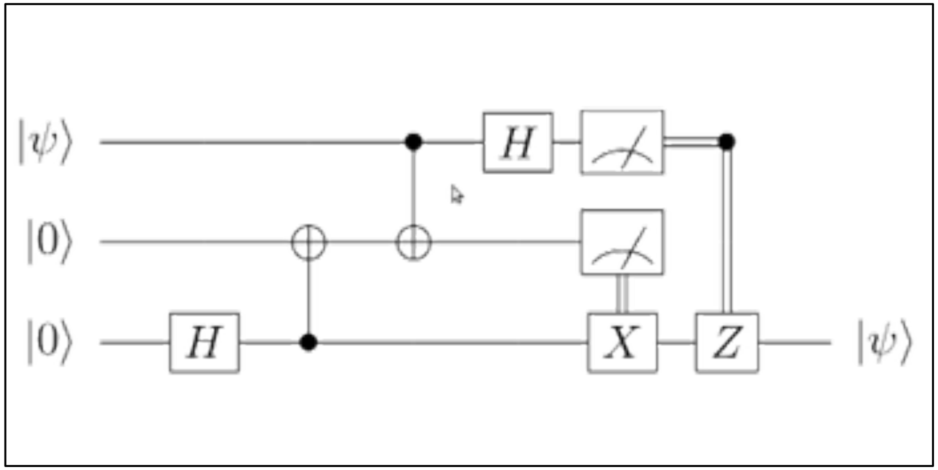
量子的遷移系でモデル



$$\underline{\mathcal{K}\ell(\mathcal{Q})}\qquad \begin{tikzcd} \overline{F}X \\ X \arrow[u, "c"] \\ 1 \arrow[u, "s"] \end{tikzcd}$$

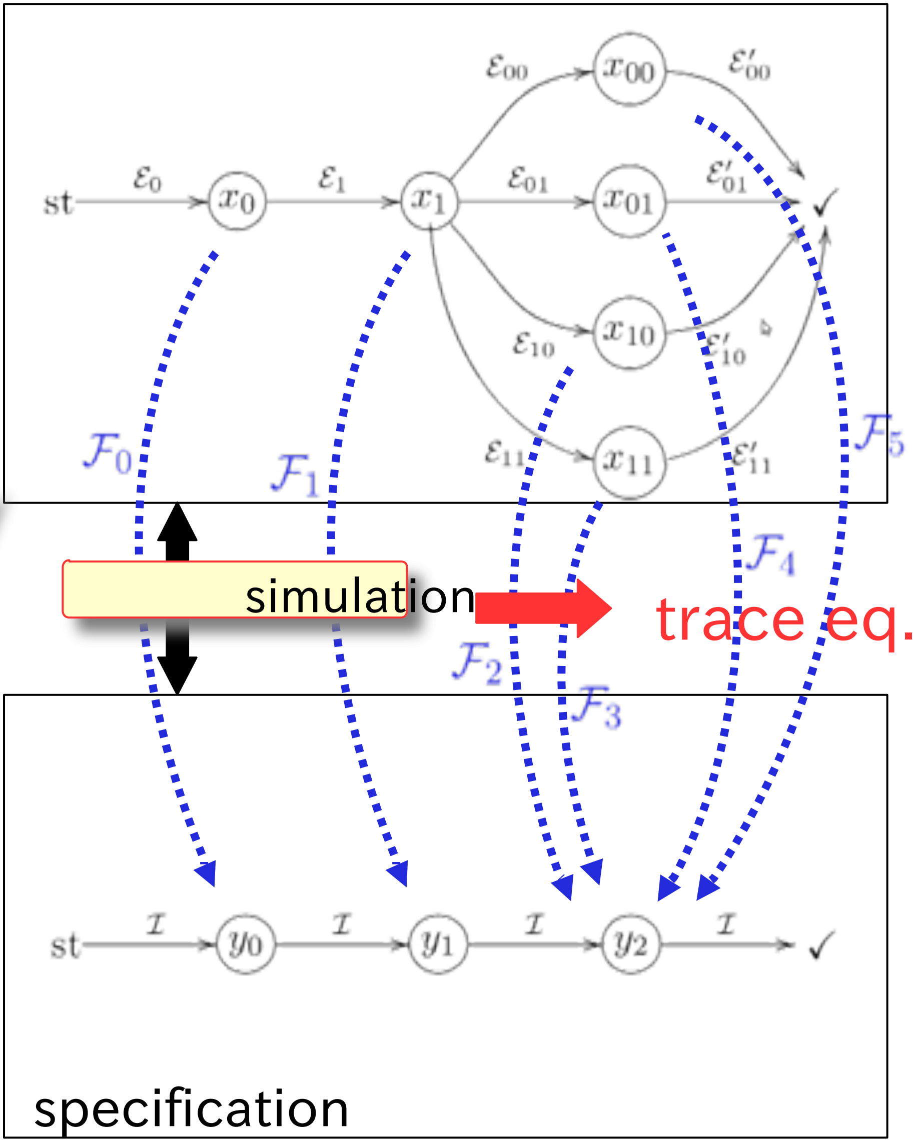
- 量子テレポーテーション
- スーパーデンス・コーディング

量子システム(プロトコル)



⇒

量子的遷移系でモデル



$$\begin{tikzcd} & \overline{F}X \\ \mathcal{K}\ell(\mathcal{Q}) & X \arrow[u, "c"] \\ & 1 \arrow[u, "s"] \end{tikzcd}$$

- 量子テレポーテーション
- スーパーデンス・コーディング

simulationを用いて検証

概要

目的: 余代数を量子システムへ応用

$$\begin{tikzcd}
\overline{F}X \arrow[rr, dashed, "\overline{F}(h)"] & & \overline{F}A \\
X \arrow[u, "c"] \arrow[rr, dashed, "h"] & & A \arrow[u, "\cong"'] \\
1 \arrow[u, "s"] \arrow[urr, dashed, "\mathsf{trace}"'] & & \underline{\mathcal{K}\ell(T)}
\end{tikzcd}$$

$$\begin{tikzcd}
FX & FR \arrow[l, "F\pi"'] \arrow[r, "F\pi'"] & FY \\
X \arrow[u, "c"] & R \arrow[u, "e"] \arrow[l, "\pi"] \arrow[r, "\pi'"'] & Y \arrow[u, "d"']
\end{tikzcd}$$

$$\begin{tikzcd}
FX \arrow[r, "Fh"] & FZ & FY \arrow[l, "Fk"'] \\
X \arrow[u, "c"] \arrow[r, "h"'] & Z \arrow[u, "e"] & Y \arrow[u, "d"'] \arrow[l, "k"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[loop left, "T"] \arrow[rr, bend left, "P"] \arrow[rr, phantom, "\top"] & & \mathcal{A} \arrow[loop right, "L"] \arrow[ll, bend left, "F"]
\end{tikzcd}$$

余代数の理論	量子システム
trace semantics, fwd/bwd simulation	量子プロトコルの検証
bisimilarity, behavioral equivalence	bisimilarity $\neq$ behavioral equivalence
coalgebraic modal logic	量子的振る舞いを 表現するmodal logic (correct by construction)

概要

目的: 余代数を量子システムへ応用

$$\begin{tikzcd}
\overline{F}X \arrow[r, dashed, "\overline{F}(h)"] & \overline{F}A \\
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\end{tikzcd}$$

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FX & FR \arrow[l, "F\pi"'] \arrow[r, "F\pi'"] & FY \\
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FX \arrow[r, "Fh"] & FZ & FY \arrow[l, "Fk"'] \\
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\end{tikzcd}$$

余代数の理論	量子システム
trace semantics, fwd/bwd simulation	量子プロトコルの検証
bisimilarity, behavioral equivalence	bisimilarity ≠ behavioral equivalence ★
coalgebraic modal logic	量子的振る舞いを 表現するmodal logic (correct by construction)

システムの等価性

(Aczel-Mendler) bisimilarity

$$\begin{tikzcd}
& FR \arrow[dl, "F\pi"'] \arrow[dr, "F\pi'"] & \\
FX & & FY \\
& R \arrow[uu, "e"] \arrow[dl, "\pi"] \arrow[dr, "\pi'"'] & \\
X \arrow[uu, "c"] & & Y \arrow[uu, "d"']
\end{tikzcd}$$

$$(x, y) \in R$$

behavioral equivalence

$$\begin{tikzcd}
FX \arrow[dr, "Fh"] & & FY \arrow[dl, "Fk"'] \\
& FZ & \\
X \arrow[uu, "c"] \arrow[dr, "h"'] & & Y \arrow[uu, "d"'] \arrow[dl, "k"] \\
& Z \arrow[uu, "e"] &
\end{tikzcd}$$

$$h(x) = k(y)$$

システムの等価性

(Aczel-Mendler) bisimilarity

$$\begin{tikzcd}
 & FR \arrow[dl, "F\pi"'] \arrow[dr, "F\pi'"] & \\
FX & & FY \\
 & R \arrow[uu, "e"] \arrow[dl, "\pi"] \arrow[dr, "\pi'"'] & \\
X \arrow[uu, "c"] & & Y \arrow[uu, "d"']
\end{tikzcd}$$

$$(x, y) \in R$$

span

behavioral equivalence

$$\begin{tikzcd}
FX \arrow[dr, "Fh"] & & FY \arrow[dl, "Fk"'] \\
 & FZ & \\
X \arrow[uu, "c"] \arrow[dr, "h"'] & & Y \arrow[uu, "d"'] \arrow[dl, "k"] \\
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システムの等価性

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X \arrow[uu, "c"] & & Y \arrow[uu, "d"']
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span

behavioral equivalence

$$\begin{tikzcd}
FX \arrow[dr, "Fh"] & & FY \arrow[dl, "Fk"'] \\
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 & Z \arrow[uu, "e"] &
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$$h(x) = k(y)$$

co-span

システムの等価性

(Aczel-Mendler) bisimilarity

$$\begin{tikzcd}
 & FR \arrow[dl, "F\pi"'] \arrow[dr, "F\pi'"] & \\
FX & & FY \\
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X \arrow[uu, "c"] & & Y \arrow[uu, "d"']
\end{tikzcd}$$

$$(x, y) \in R$$

span

behavioral equivalence

$$\begin{tikzcd}
FX \arrow[dr, "Fh"] & & FY \arrow[dl, "Fk"'] \\
 & FZ & \\
X \arrow[uu, "c"] \arrow[dr, "h"'] & & Y \arrow[uu, "d"'] \arrow[dl, "k"] \\
 & Z \arrow[uu, "e"] &
\end{tikzcd}$$

$$h(x) = k(y)$$

co-span

確率的システムの場合
($F = D$)

システムの等価性

(Aczel-Mendler) bisimilarity

$$\begin{tikzcd}
 & FR \arrow[ld, "F\pi"'] \arrow[rd, "F\pi'"] & \\
FX & & FY \\
 & R \arrow[uu, "e"] \arrow[ld, "\pi"] \arrow[rd, "\pi'"'] & \\
X \arrow[uu, "c"] & & Y \arrow[uu, "d"']
\end{tikzcd}$$

$$(x, y) \in R$$

span

behavioral equivalence

$$\begin{tikzcd}
FX \arrow[rd, "Fh"] & & FY \arrow[ld, "Fk"'] \\
 & FZ & \\
X \arrow[uu, "c"] \arrow[rd, "h"'] & & Y \arrow[uu, "d"'] \arrow[ld, "k"] \\
 & Z \arrow[uu, "e"] &
\end{tikzcd}$$

$$h(x) = k(y)$$

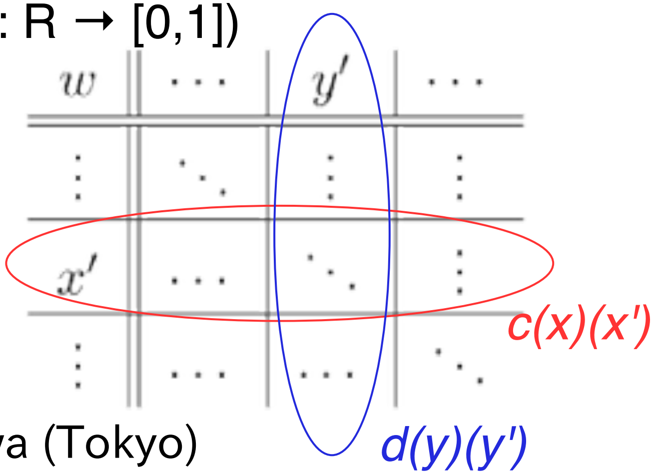
co-span

確率的システムの場合

($F = D$)

weight function based bisimulation

(w: R → [0,1])



システムの等価性

(Aczel-Mendler) bisimilarity

$$\begin{tikzcd}
 & FR \arrow[dl, "F\pi"'] \arrow[dr, "F\pi'"] & \\
FX & & FY \\
 & R \arrow[uu, "e"] \arrow[dl, "\pi"] \arrow[dr, "\pi'"'] & \\
X \arrow[uu, "c"] & & Y \arrow[uu, "d"']
\end{tikzcd}$$

$$(x, y) \in R$$

span

behavioral equivalence

$$\begin{tikzcd}
FX \arrow[dr, "Fh"] & & FY \arrow[dl, "Fk"'] \\
 & FZ & \\
X \arrow[uu, "c"] \arrow[dr, "h"'] & & Y \arrow[uu, "d"'] \arrow[dl, "k"] \\
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\end{tikzcd}$$

$$h(x) = k(y)$$

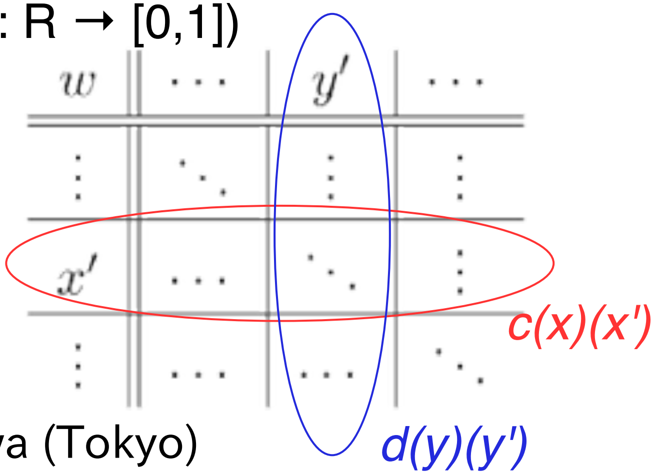
co-span

確率的システムの場合

(F= D)

weight function based bisimulation

(w: R → [0,1])



Ogawa (Tokyo)

equivalence class based bisimulation

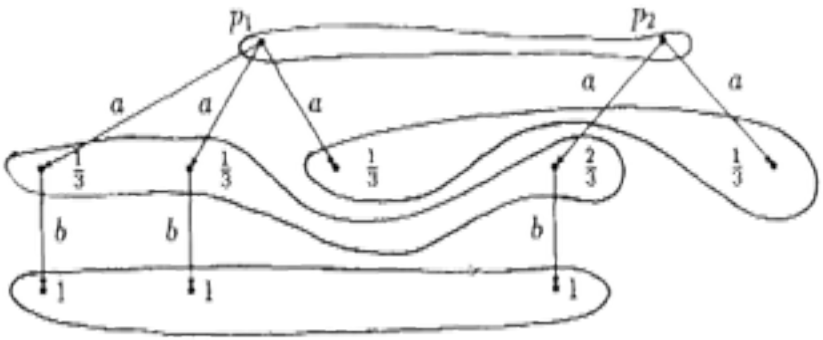


image from [Larsen&Skou]

システムの等価性

(Aczel-Mendler) bisimilarity

$$\begin{tikzcd}
& FR \arrow[dl, "F\pi"'] \arrow[dr, "F\pi'"] & \\
FX & R \arrow[u, "e"] \arrow[dl, "\pi"] \arrow[dr, "\pi'"'] & FY \\
X \arrow[u, "c"] & & Y \arrow[u, "d"']
\end{tikzcd}$$

$$(x, y) \in R$$

span

behavioral equivalence

$$\begin{tikzcd}
FX \arrow[dr, "Fh"] & & FY \arrow[dl, "Fk"'] \\
& FZ & \\
X \arrow[uu, "c"] \arrow[dr, "h"'] & & Y \arrow[uu, "d"'] \arrow[dl, "k"] \\
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\end{tikzcd}$$

$$h(x) = k(y)$$

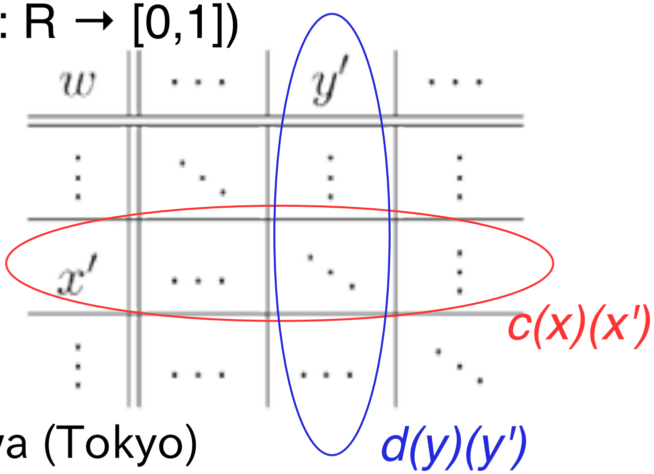
co-span

確率的システムの場合

($F = D$)

weight function based bisimulation

(w: R → [0,1])



Ogawa (Tokyo)

一致する

[Vink & Rutten 1999]

equivalence class based bisimulation

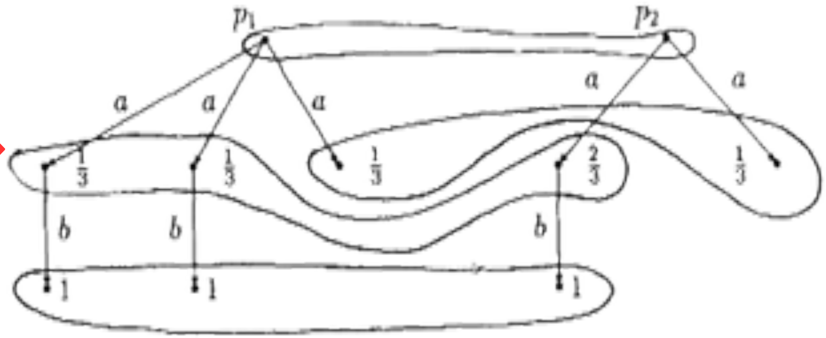


image from [Larsen&Skou]

システムの等価性

(Aczel-Mendler) bisimilarity

$$\begin{tikzcd}
 & FR \arrow[dl, "F\pi"'] \arrow[dr, "F\pi'"] & \\
FX & & FY \\
 & R \arrow[uu, "e"] \arrow[dl, "\pi"] \arrow[dr, "\pi'"'] & \\
X \arrow[uu, "c"] & & Y \arrow[uu, "d"']
\end{tikzcd}$$

$$(x, y) \in R$$

span

behavioral equivalence

$$\begin{tikzcd}
FX \arrow[dr, "Fh"] & & FY \arrow[dl, "Fk"'] \\
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 & Z \arrow[uu, "e"] &
\end{tikzcd}$$

$$h(x) = k(y)$$

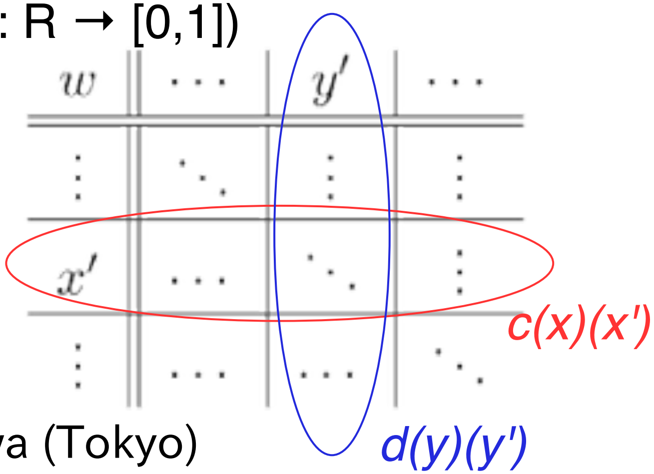
co-span

確率的システムの場合

($F = D$)

weight function based bisimulation

(w: R → [0,1])



Ogawa (Tokyo)

一致する

[Vink & Rutten 1999]

equivalence class based bisimulation

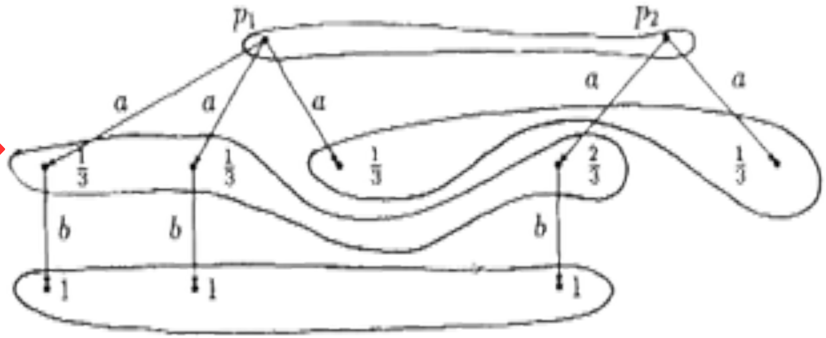


image from [Larsen&Skou]

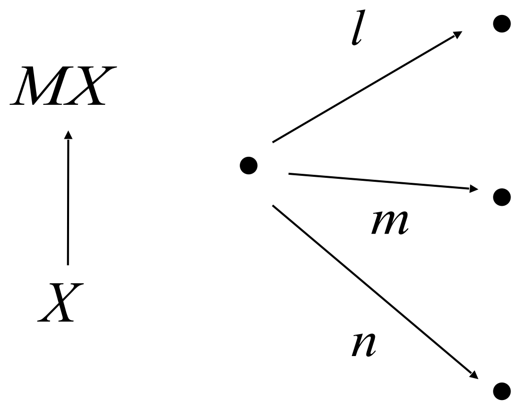
量子的遷移ではどうなるか?

マルチセットファンクタの場合($F = M$)

bisim. $\overset{\text{always}}{\Longrightarrow}$ beh. eq.

bisim. $\Longleftarrow$ beh. eq.

(1) [Gumm & Schröder 2001]



$(\mathcal{M}, +, 0)$:モノイド

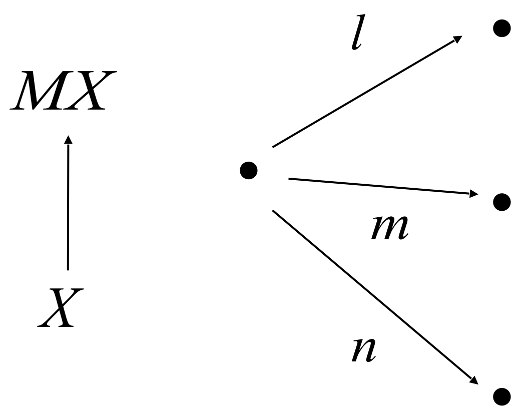
マルチセットファンクタの場合(*F* = *M*)

$$\text{bisim.} \;\overset{\text{always}}{\Longrightarrow}\; \text{beh. eq.}$$

$$\text{bisim.} \;\overset{(1)}{\Longleftarrow}\; \text{beh. eq.}$$

(1) [Gumm & Schröder 2001]

(1)
- $\mathcal{M}$がpositiveである
  $(m+n=0 \Rightarrow m=n=0)$
- $\mathcal{M}$がrefinable である



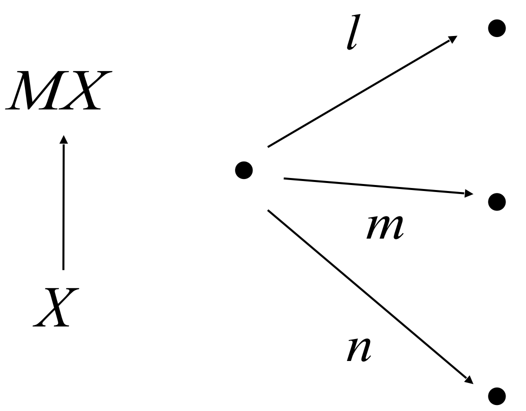
$(\mathcal{M},+,0)$:モノイド

マルチセットファンクタの場合(*F* = *M*)

$$\text{bisim.} \overset{\text{always}}{\Longrightarrow} \text{beh. eq.}$$

$$\Longleftarrow$$

(1) [Gumm & Schröder 2001]

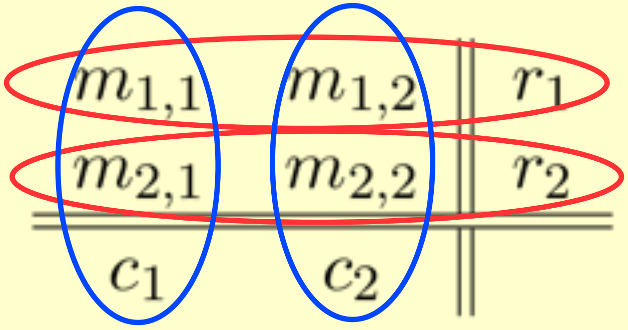


$(\mathcal{M}, +, 0)$:モノイド

(1)
- $\mathcal{M}$がpositiveである $(m+n=0 \Rightarrow m=n=0)$
- $\mathcal{M}$がrefinable である

$$r_1+r_2=c_1+c_2$$

$$\Downarrow$$



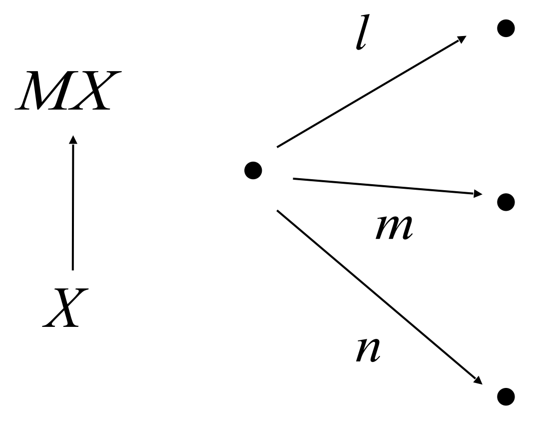
$$\begin{array}{cc||c} m_{1,1} & m_{1,2} & r_1 \\ m_{2,1} & m_{2,2} & r_2 \\ \hline\hline c_1 & c_2 & \end{array}$$

マルチセットファンクタの場合(*F* = *M*)

$$\text{bisim.} \overset{\text{always}}{\Longrightarrow} \text{beh. eq.}$$

$$\text{bisim.} \Longleftarrow \text{beh. eq.}$$

(1) [Gumm & Schröder 2001]

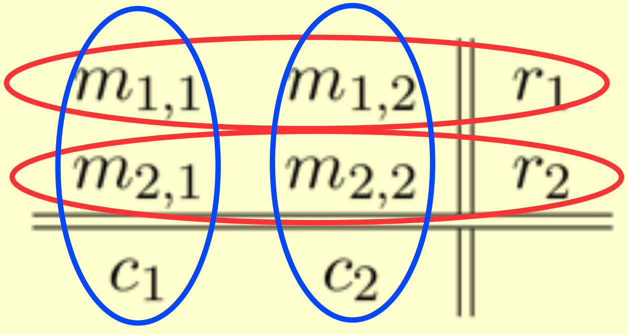


$(\mathcal{M},+,0)$:モノイド

(1)
- $\mathcal{M}$がpositiveである $(m+n=0 \Rightarrow m=n=0)$
- $\mathcal{M}$がrefinable である

$r_1+r_2=c_1+c_2$

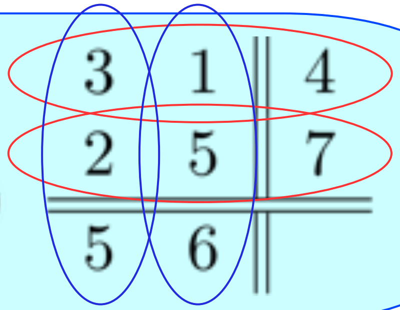
⇩



- 正実数 [0, ∞] (区間 [0, 1])
- 自然数 *N*

例:

$5+6=4+7(=11)$



量子的遷移の場合($F = Q$)

- QO: quantum-operationの集合

定理.　QOはrefinableでない

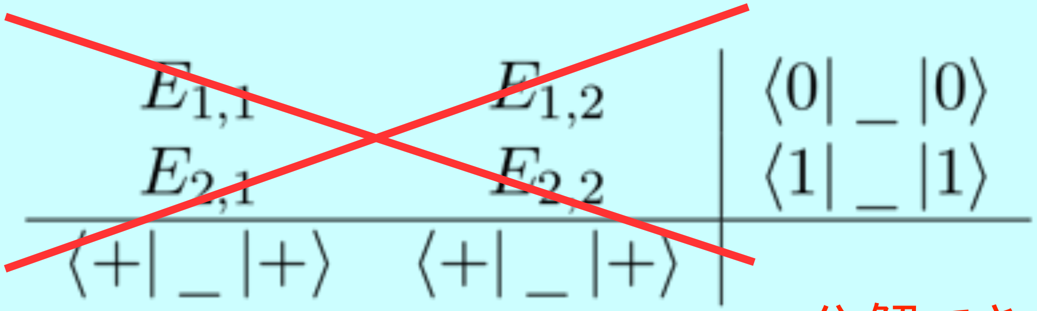
量子的遷移の場合($F = Q$)

- QO: quantum-operationの集合

定理. QOはrefinableでない

反例:

$$\langle 0| \_ |0\rangle + \langle 1| \_ |1\rangle = \langle +| \_ |+\rangle + \langle -| \_ |-\rangle$$



$$\begin{array}{cc|c} E_{1,1} & E_{1,2} & \langle 0| \_ |0\rangle \\ E_{2,1} & E_{2,2} & \langle 1| \_ |1\rangle \\ \hline \langle +| \_ |+\rangle & \langle +| \_ |+\rangle & \end{array}$$

分解できない

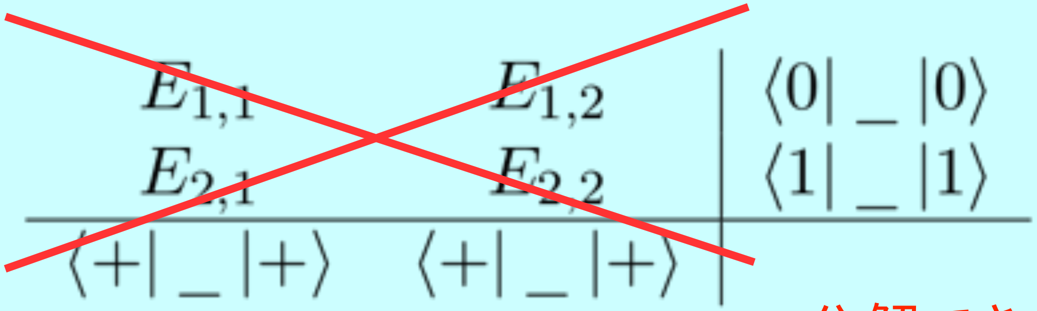
量子的遷移の場合(*F* = *Q*)

- *QO*: quantum-operationの集合

定理.　*QO*はrefinableでない

反例:

$$\langle 0|\_|0\rangle + \langle 1|\_|1\rangle = \langle +|\_|+\rangle + \langle -|\_|-\rangle$$



$$\begin{array}{cc|c} E_{1,1} & E_{1,2} & \langle 0|\_|0\rangle \\ E_{2,1} & E_{2,2} & \langle 1|\_|1\rangle \\ \hline \langle +|\_|+\rangle & \langle +|\_|+\rangle & \end{array}$$

分解できない

量子システムでは bisim. ≠ beh. eq.

概要

目的: 余代数を量子システムへ応用

$$\begin{tikzcd}
\overline{F}X \arrow[r, dashed, "\overline{F}(h)"] & \overline{F}A \\
X \arrow[u, "c"] \arrow[r, dashed, "h"] & A \arrow[u, "\cong"'] \\
1 \arrow[u, "s"] \arrow[ur, dashed, "\text{trace}"'] & \underline{\mathcal{K}\ell(T)}
\end{tikzcd}$$

$$\begin{tikzcd}
& FR \arrow[dl, "F\pi"'] \arrow[dr, "F\pi'"] & \\
FX & & FY \\
& R \arrow[uu, "e"] \arrow[dl, "\pi"] \arrow[dr, "\pi'"'] & \\
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$$\begin{tikzcd}
FX \arrow[r, "Fh"] & FZ & FY \arrow[l, "Fk"'] \\
X \arrow[u, "c"] \arrow[r, "h"'] & Z \arrow[u, "e"] & Y \arrow[u, "d"'] \arrow[l, "k"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[loop left, "T"] \arrow[r, bend left, "P"] \arrow[r, phantom, "\top"] & \mathcal{A} \arrow[loop right, "L"] \arrow[l, bend left, "F"]
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余代数の理論	量子システム
trace semantics, fwd/bwd simulation	量子プロトコルの検証
bisimilarity, behavioral equivalence	bisimilarity ≠ behavioral equivalence
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Coalgebraic Modal Logic

・それぞれのシステムの挙動にフィットし

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- それぞれのシステムの挙動にフィット

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(MSL: meet semi lattice)

[Jacobs & Sokolova 2010]

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$$E \in QO$$

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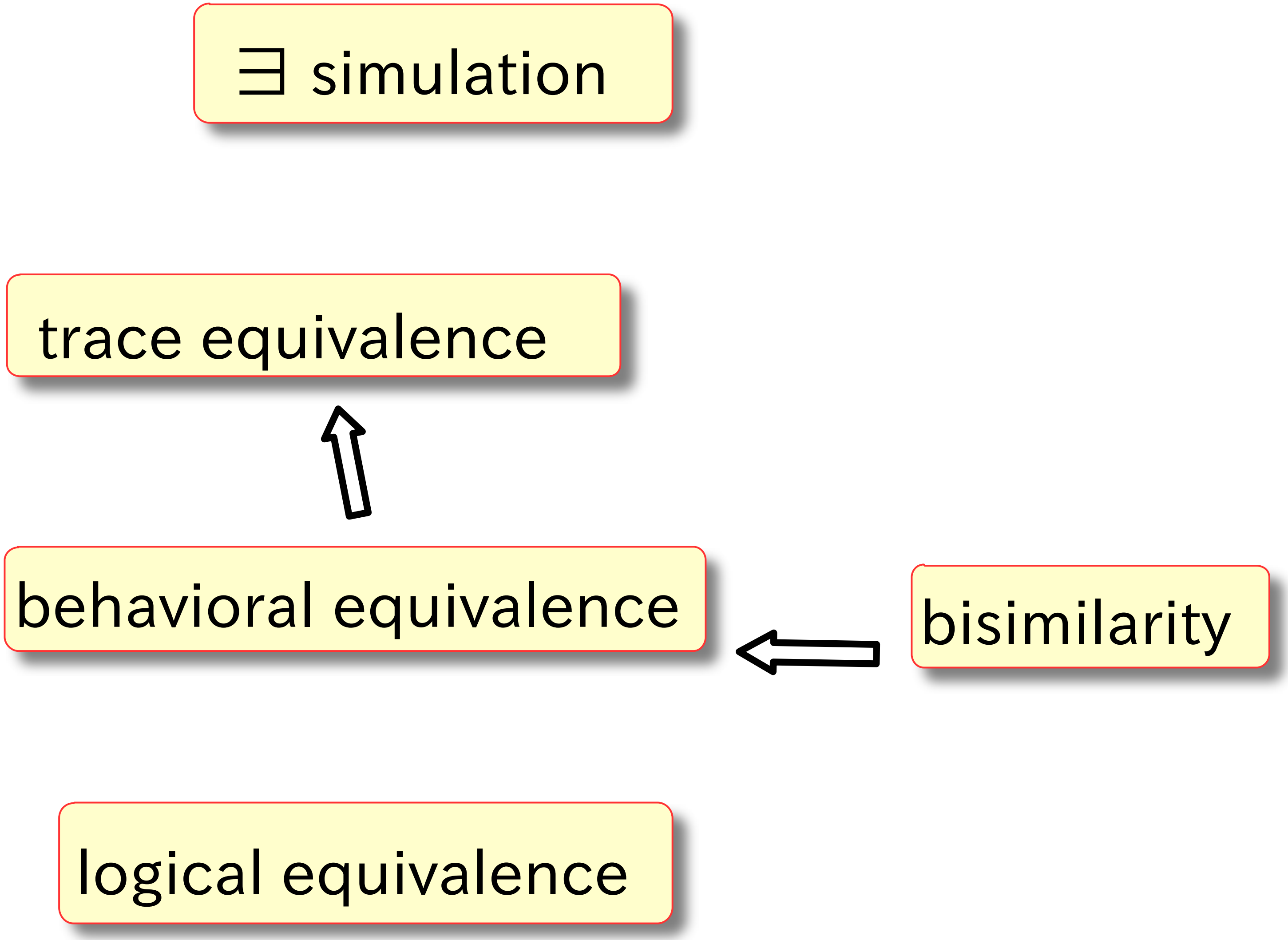
- ここで得たmod

correct by constr

- negationは必要

量子システムに関する等価性のまとめ



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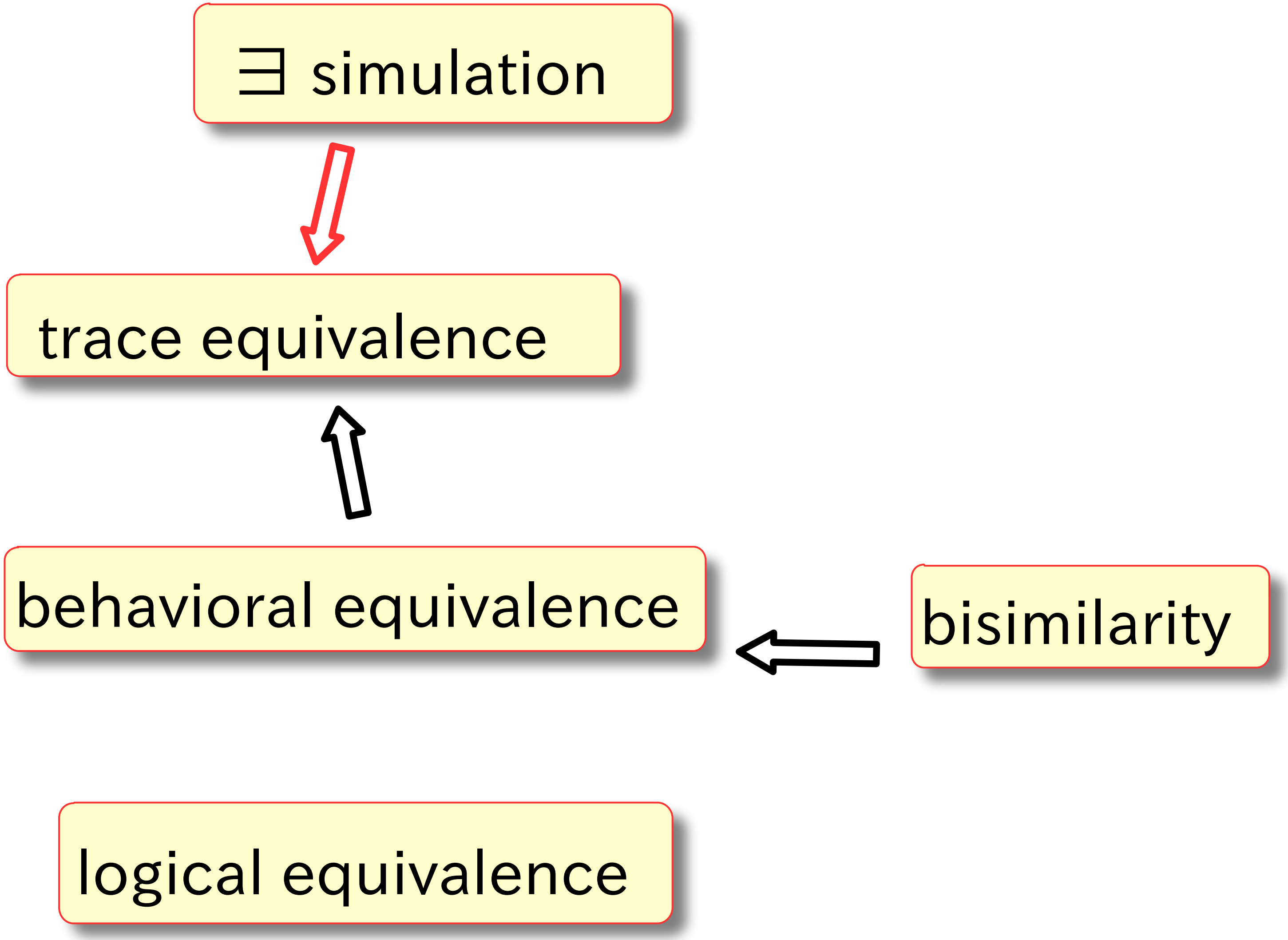

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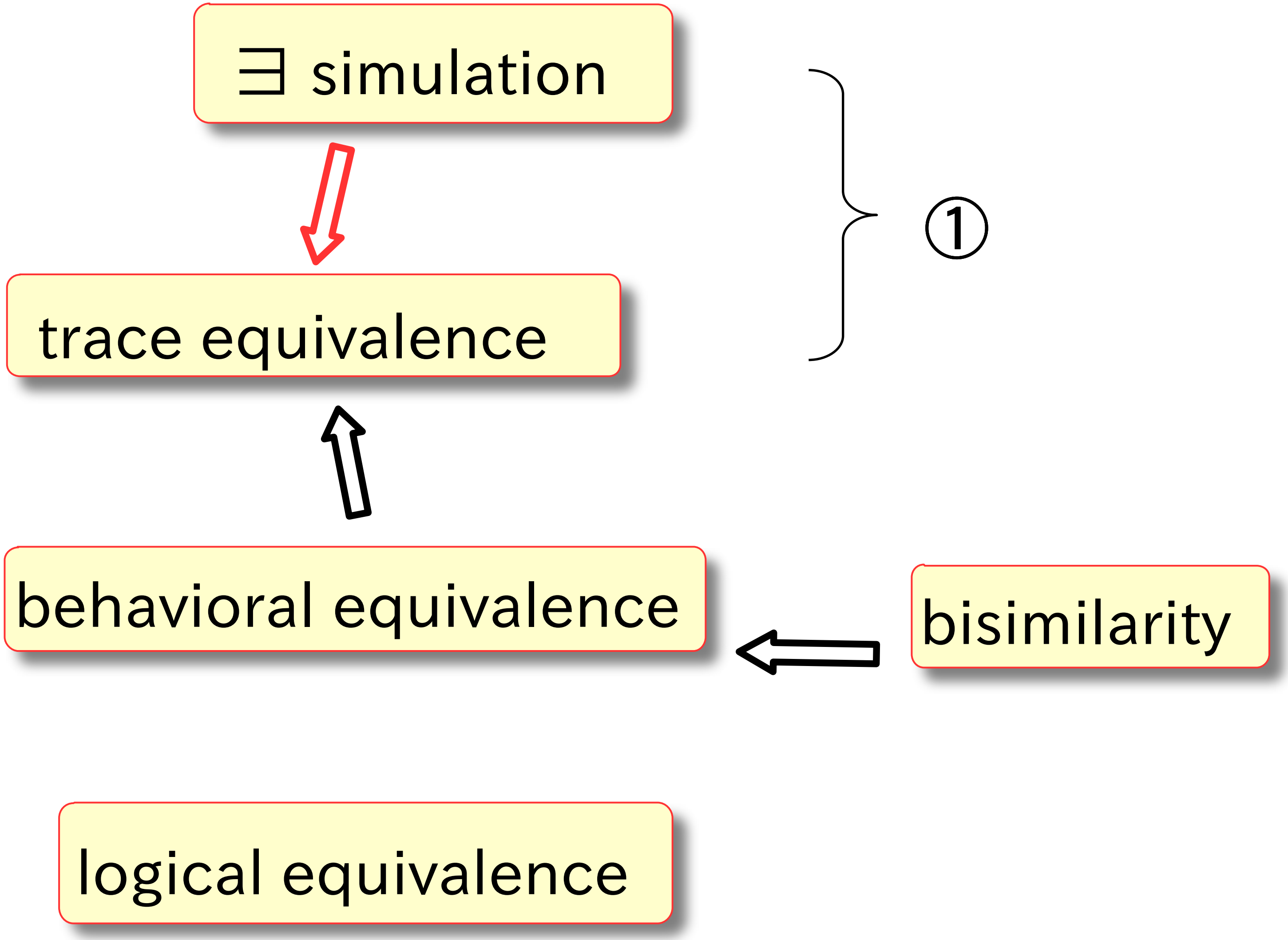

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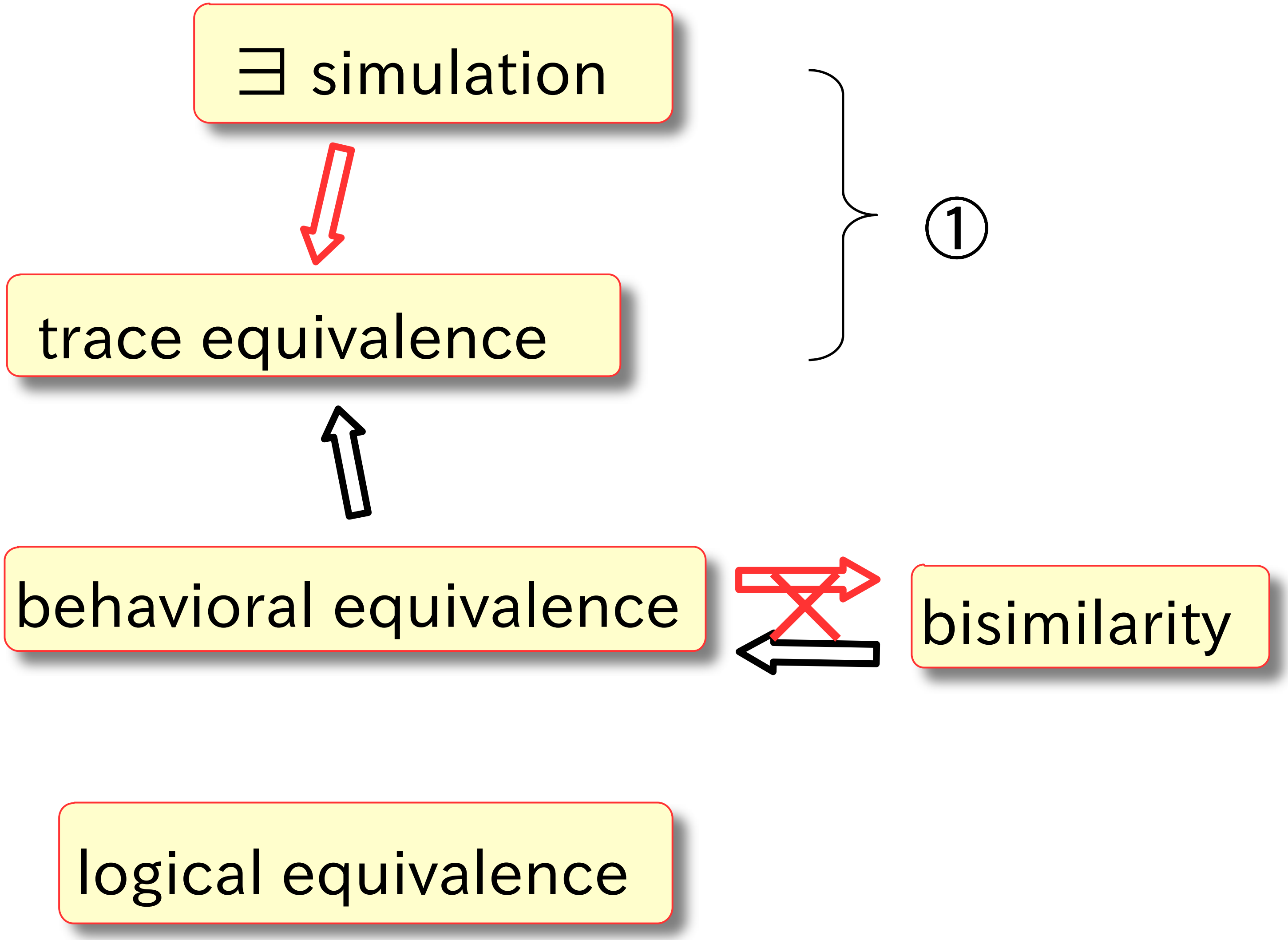

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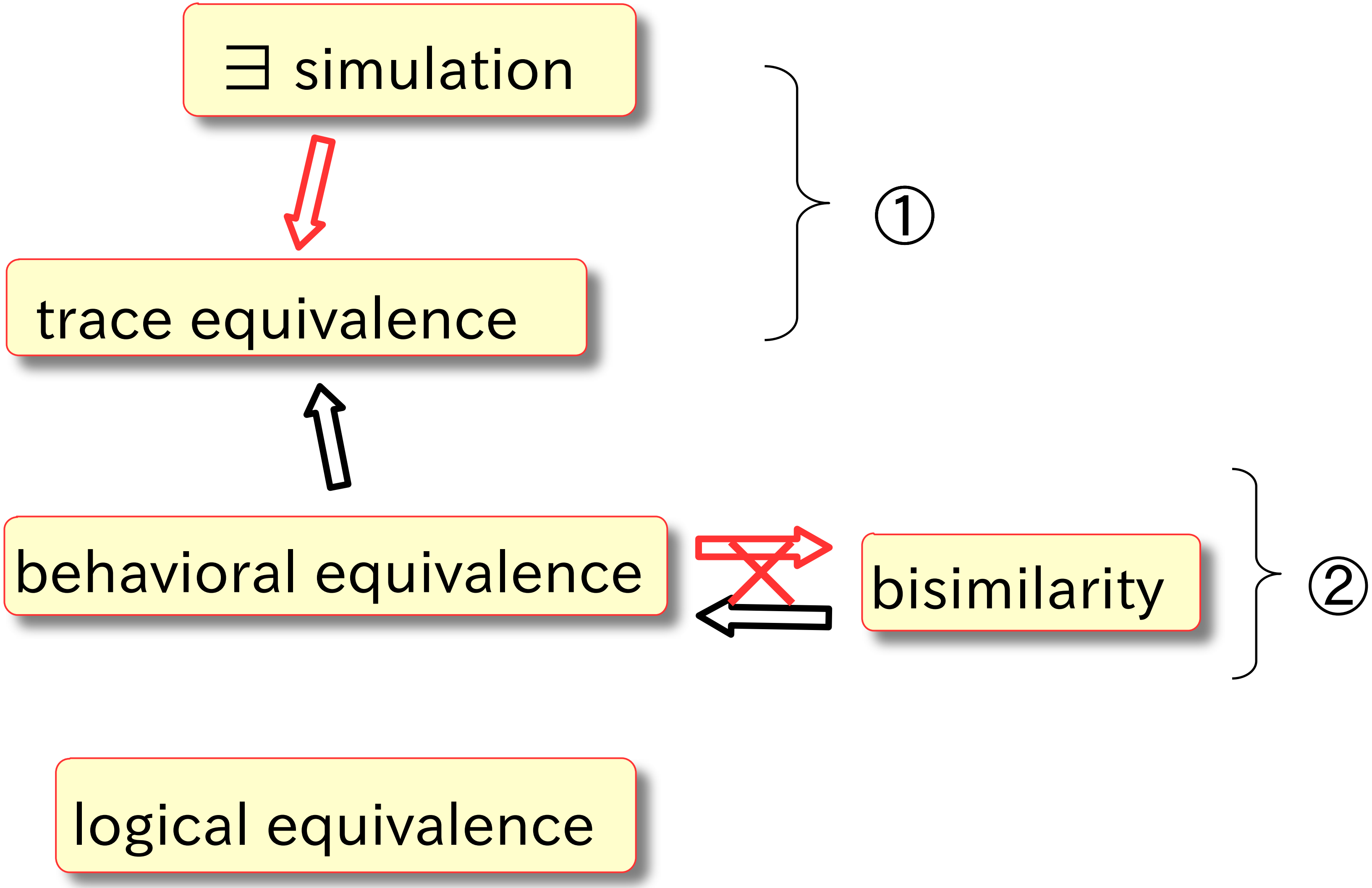

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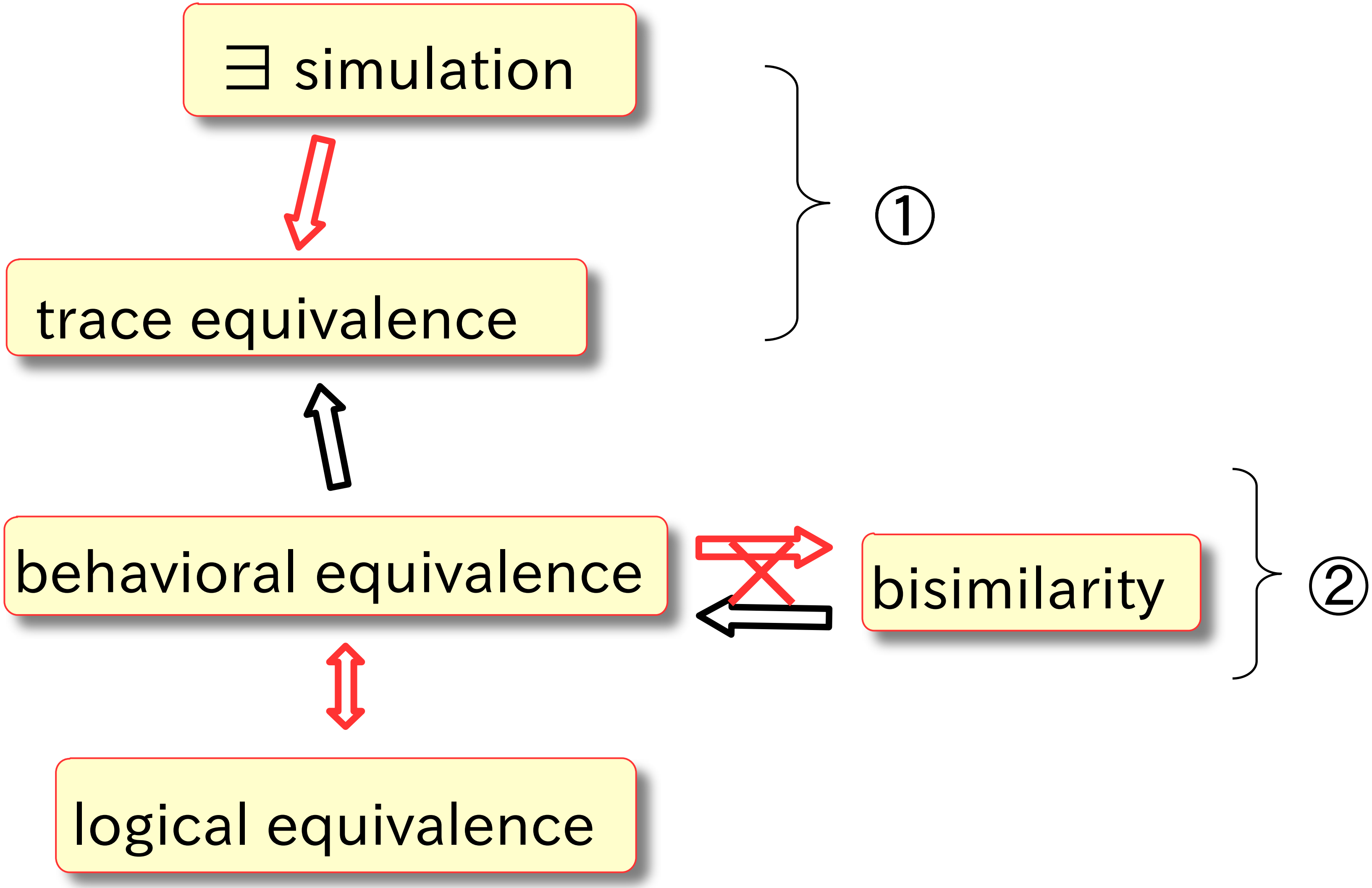

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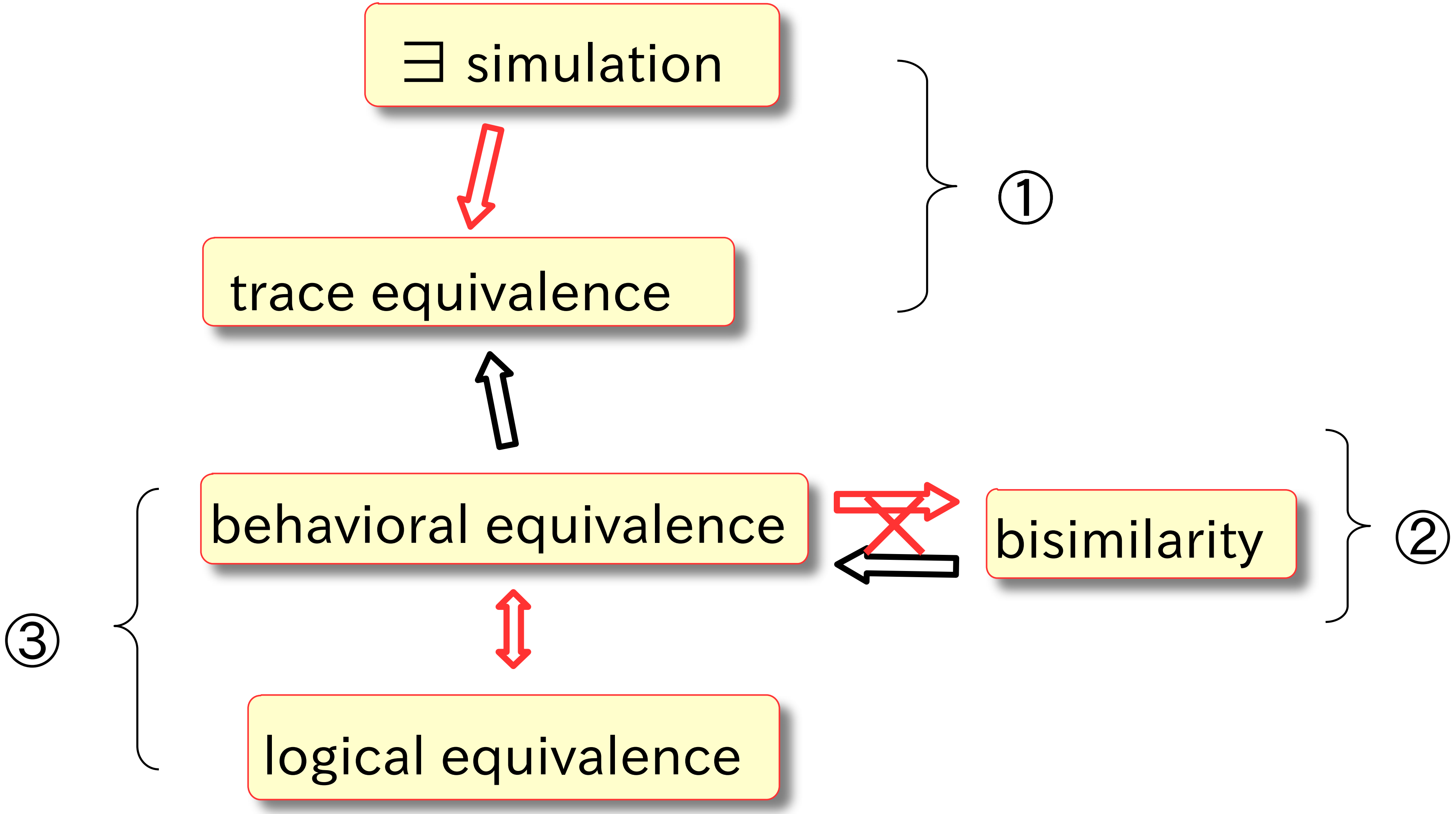

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