Quotient and Kantorovich Metric via Observation-Algebra in Lawvere Theory

Hiroshi Ogawa
University of Tokyo (Hasuo lab. M1)

CSCAT 2015 3/14

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Outline

- Coincidence of
  - Quotient Monad via TT-lifting (top top lifting)
  - Quotient Lawvere theory via observational-algebra
- Kantorovich Metric via observational-algebra
- Conclusion / Future work

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TT-lifting briefly

- Originates from the proof of strong norm. of Moggi's comp. metalang. [Lindley, Stark]
- Semantic (categorical) formulation [Katsumata]
  - lifting of a strong monad along a "nice" fibration

$$\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array} \quad \text{w/} \quad \begin{cases} T : \mathbb{C} \to \mathbb{C} \\ R \in |\mathbb{C}| \\ S \in |\mathbb{P}_{TR}| \end{cases}$$

  - logical predicates/relations
  - enumerating the order-enrichment on $\mathcal{K}\ell$(T)

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$\mathbb{P} \downarrow p\ \mathbb{C}$ w/ $\begin{cases} T : \mathbb{C} \to \mathbb{C} \\ R \in |\mathbb{C}| \\ S \in |\mathbb{P}_{TR}| \end{cases}$ → TT-lifting

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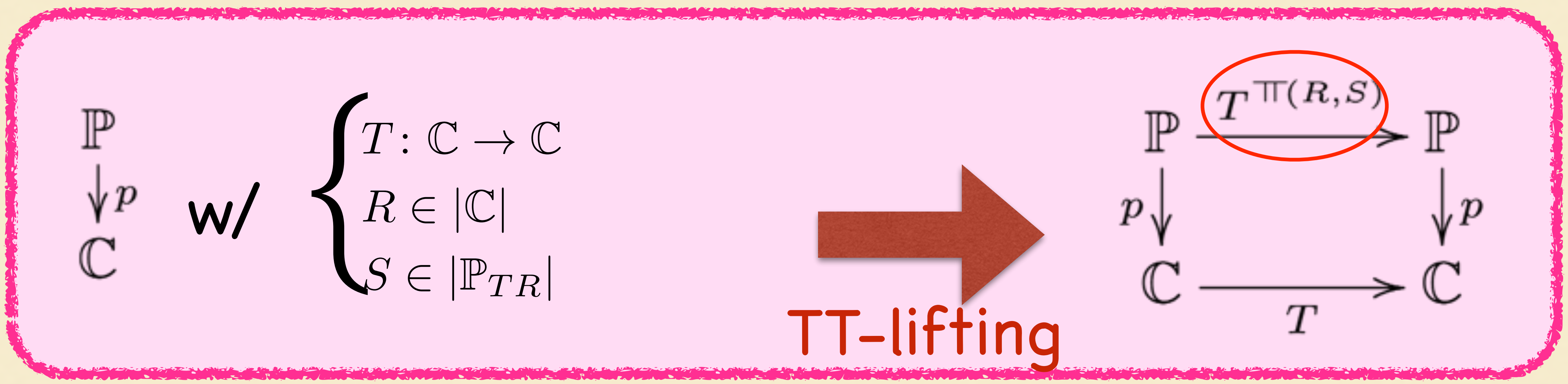

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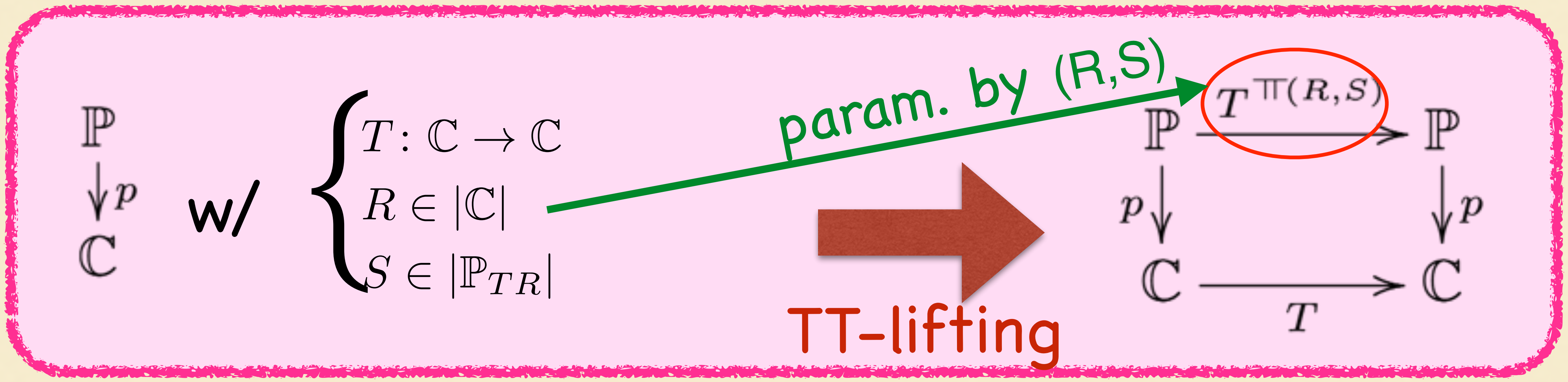

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TT-lifting w/ equiv.rel

- we applied TT-lifting to $\begin{array}{c}\mathbf{Rel}\\ \downarrow p\\ \mathbf{Sets}\end{array}$.

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Def.

Given a monad T: Sets → Sets

a $\mathcal{EM}$-alg. α: TA → A w/ cong.equiv.rel $\sim_A \subseteq A \times A$,

we define $\approx^{(\alpha, \sim_A)}$: Sets → EqRel by:

$$t \approx_X t' \iff \forall f : X \to A.\ \alpha \bullet Tf\,(t) \sim_A \alpha \bullet Tf\,(t')$$

where

$$\begin{array}{c} X \xrightarrow{\quad f \quad} A \\ \Big\downarrow \\ TX \xrightarrow{\ Tf\ } TA \xrightarrow{\ \alpha\ } A \end{array} \qquad \underline{\text{Sets}}$$

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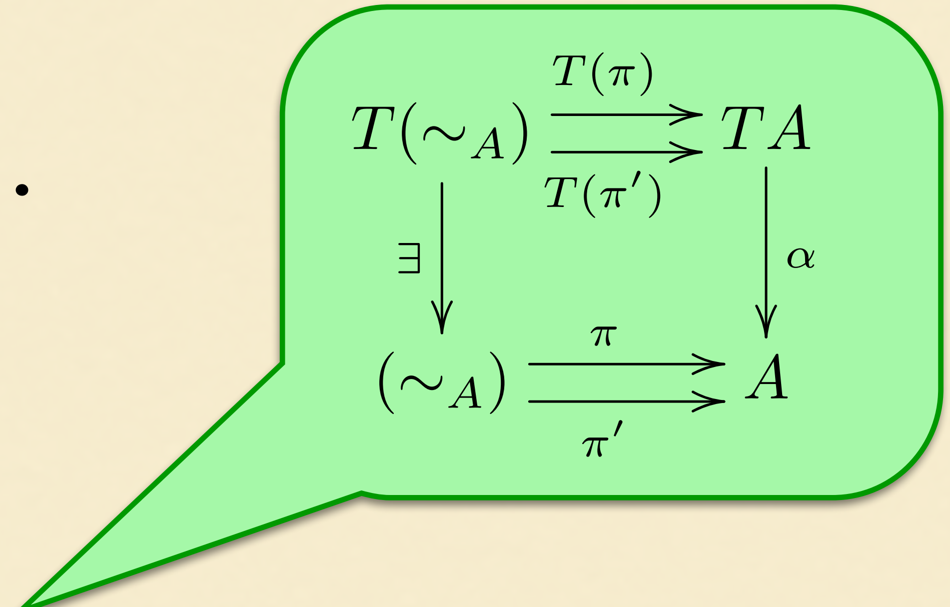

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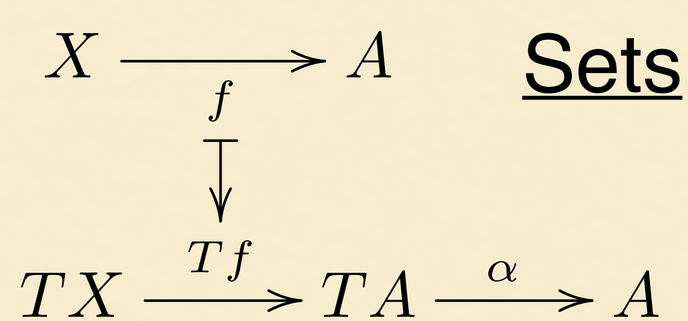

TT-lifting w/ equiv.rel

- we applied TT-lifting to $\begin{array}{c}\mathbf{Rel}\\ \downarrow p\\ \mathbf{Sets}\end{array}$.

$$\begin{array}{ccc} T(\sim_A) & \overset{T(\pi)}{\underset{T(\pi')}{\rightrightarrows}} & TA \\ \exists\downarrow & & \downarrow\alpha \\ (\sim_A) & \overset{\pi}{\underset{\pi'}{\rightrightarrows}} & A \end{array}$$

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Given a monad T: Sets → Sets

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where $\begin{array}{c} X \xrightarrow{f} A \\ \downarrow \\ TX \xrightarrow{Tf} TA \xrightarrow{\alpha} A \end{array}$ <u>Sets</u>

Quotient Monad via TT-lifting

- from the previous $\approx^{(\alpha, \sim_A)}$: Sets → EqRel,

$$X \mapsto \left(\approx_X \rightarrowtail TX \times TX \right)$$

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Prop.

We define $T/_{(\alpha,\sim_A)}$: Sets → Sets by $T/_{(\alpha,\sim_A)}(X) := TX/{\approx_X}$,

as in:

$$\underline{\text{Sets}} \longrightarrow \underline{\text{Sets}}$$

$$X \longmapsto \approx_X \rightrightarrows TX \overset{q_X}{\twoheadrightarrow} T/_{(\alpha,\sim_A)}(X)$$

Then,

- $T/_{(\alpha,\sim_A)}$ is a monad,
- $(q_X)_X$ forms a monad map $q : T \Rightarrow T/_{(\alpha,\sim_A)}$.

proof.

Check Hino-san's condition (especially in Sets).

- substitutivity, congruencity.

Ex. Quotient monad in Sets

List ⇒ (fin.supp.) Multiset ⇒ (finite) Powerset

$X^* \twoheadrightarrow \mathcal{M}_{\mathrm{N}}(X) \twoheadrightarrow \mathcal{P}(X)$

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List $\Rightarrow$ (fin.supp.) Multiset $\Rightarrow$ (finite) Powerset

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$x \cdot y \cdot x$

$x \cdot x \cdot y$

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$x \cdot y \cdot x \mapsto 2x + y$

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- $T = (-)^*$, *the parameters for TT-lifting*
- $\alpha = \mu_2 : (2^*)^* \to 2^*$ w/ $\sim_{2^*} = \langle a \cdot b = b \cdot a \rangle$

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the parameters for TT-lifting

- $T = \mathcal{M}_{\mathbb{N}}$,
- $\alpha = \mu_1 : \mathcal{M}_{\mathbb{N}}(\mathcal{M}_{\mathbb{N}}(1)) \to \mathcal{M}_{\mathbb{N}}(1)$ w/ $\sim_{\mathcal{M}(1)} = \langle a + a = a \rangle$

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- Algebraic (Lawvere theoretic) view

opr.

eq.

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"adding certain equations!"

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- Categorical approach to Universal Algebra [Lawvere] (e.g. Monoid, Group, Vect.Space)
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$\mathcal{M}on\mathcal{T}h$

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Mod(GrpTh, Sets)

G ⇒ G'

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MonTh → GrpTh

the category of law. ths.

Mod(GrpTh, Sets)

G ⇒ G'

the category of (GrpTh, Sets)-models
(≃ the category of groups)

Lawvere Theory briefly

- Correspondense w/ finitary monads on Sets

<u>Thm.</u> [Barr, Wells]

There exists an equivalence betw. $\mathrm{FinMon}(\mathsf{Sets})$ and Th, as in:

$$\begin{array}{ccc} \underline{\mathrm{FinMon}(\mathsf{Sets})} & \xrightarrow{\simeq} & \underline{\mathrm{Th}} \\ \mathsf{T} & \longmapsto & \mathcal{T}_{\mathsf{T}} := \mathcal{K}\ell(\mathsf{T})_{\mathrm{N}}{}^{\mathsf{op}} \end{array}$$

<u>Thm.</u> [Barr, Wells]

There exists an equivalence as:

$$\begin{array}{ccc} \underline{\mathcal{EM}(\mathsf{T})} & \xrightarrow{\simeq} & \underline{\mathsf{Mod}(\mathcal{T}_{\mathsf{T}}, \mathsf{Sets})} \\ (\alpha\colon \mathsf{TA} \to \mathsf{A}) & \stackrel{\mathsf{M}}{\longmapsto} & \mathsf{M}\alpha \end{array}$$

Lawvere Theory briefly

Thm. [Borceux]

The cat. of models Mod($\mathcal{T}$, Sets) admits quotient.

Lawvere Theory briefly

Thm. [Borceux]

The cat. of models $\mathrm{Mod}(\mathcal{T}, \mathrm{Sets})$ admits quotient.

- Thus, we get a quotient model $Q_{(\alpha, \sim_A)}$ from the assumptions of TT-lifting as in:

$$\begin{array}{ccc} \mathcal{EM}(\mathrm{T}) & \longrightarrow & \mathrm{Mod}(\mathcal{T}_{\mathrm{T}}, \mathrm{Sets}) \\ \sim_A \rightrightarrows \alpha & \longmapsto & M(\sim_A) \rightrightarrows M\alpha \twoheadrightarrow Q_{(\alpha, \sim_A)} \end{array}$$

$$\begin{array}{ccc} T(\sim_A) & \overset{T(\pi)}{\underset{T(\pi')}{\rightrightarrows}} & TA \\ \exists \downarrow & & \downarrow \alpha \\ (\sim_A) & \overset{\pi}{\underset{\pi'}{\rightrightarrows}} & A \end{array}$$

Quotient Lawvere theory via observation-algebra [Power]

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Prop.

Given $\mathcal{T}$: Law. Th,

N : ($\mathcal{T}$, Sets)-model (thus, N : $\mathcal{T} \to$ Sets),

we define a cat. $\mathcal{T}$/N as follows:

kernel pair Ker(N) $\rightrightarrows \mathcal{T} \xrightarrow{N}$ Sets,

coequalizer Ker(N) $\rightrightarrows \mathcal{T} \twoheadrightarrow \mathcal{T}$/N.

Then, $\mathcal{T}$/N is a Law. th.

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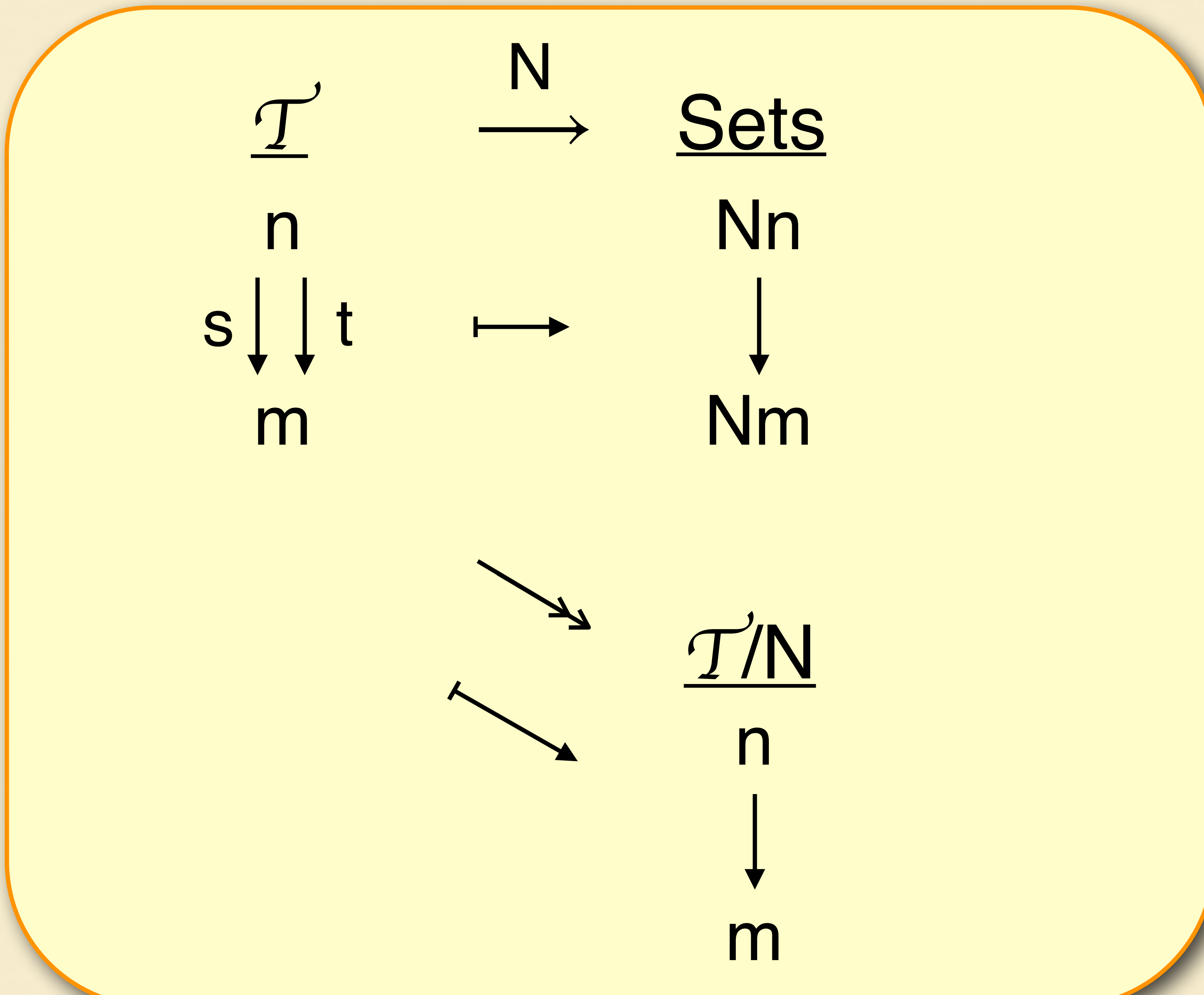
Then, $\mathcal{T}/N$ is a Law. th.

Why "observation" ?

- If two terms s, t are "observed" similarly in the model N, then $\mathcal{T}/N$ includes an additional eq. s = t.



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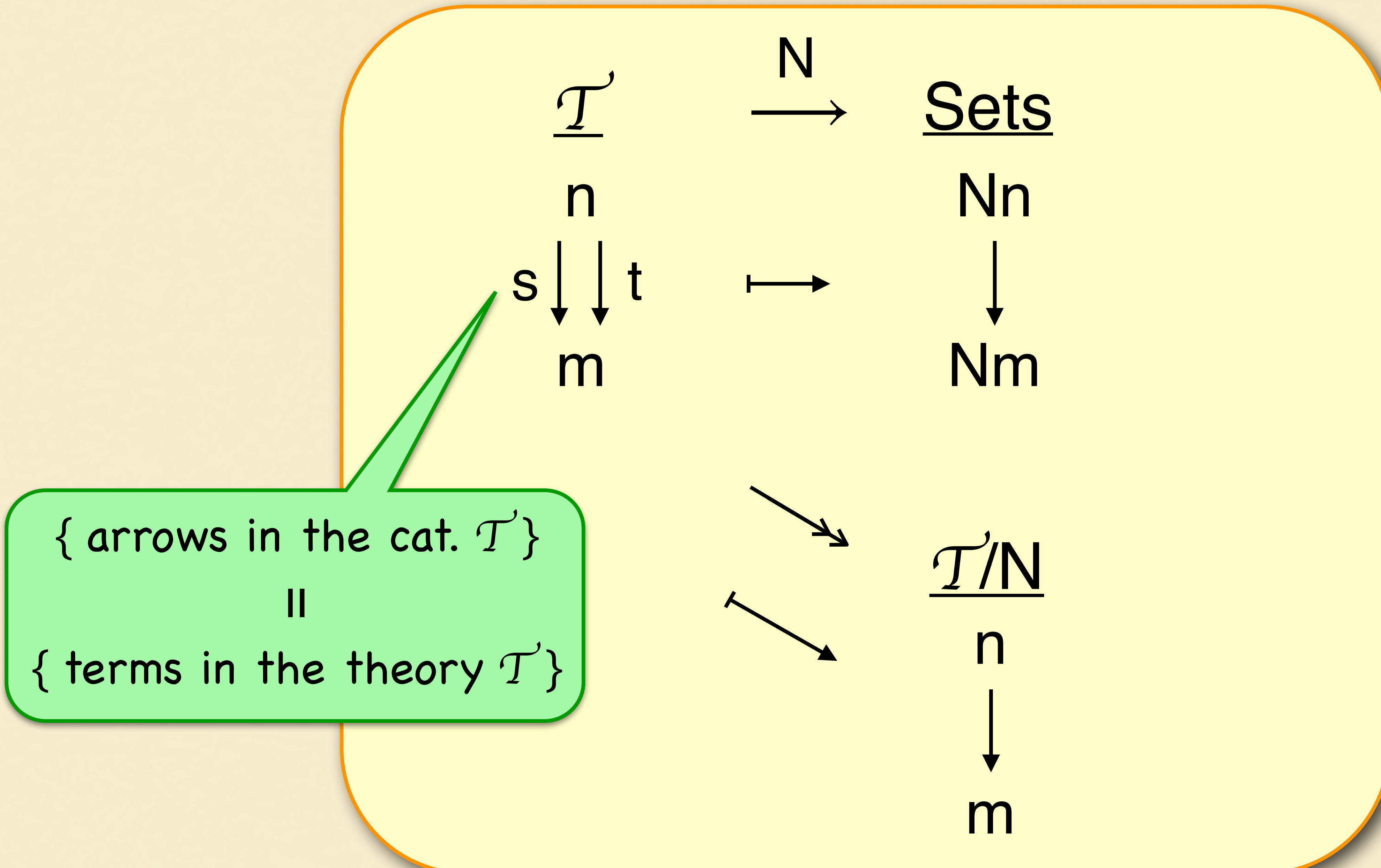

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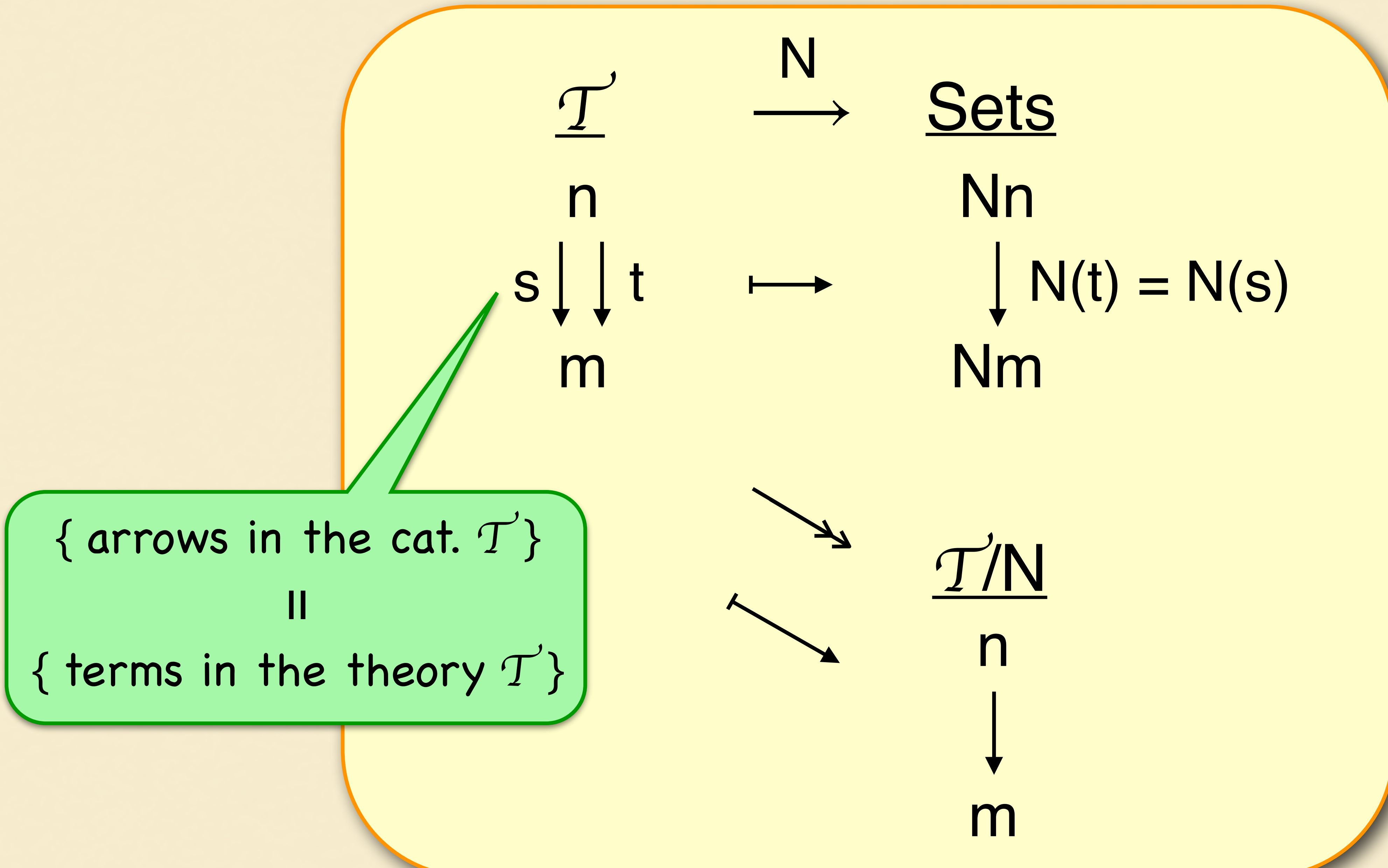

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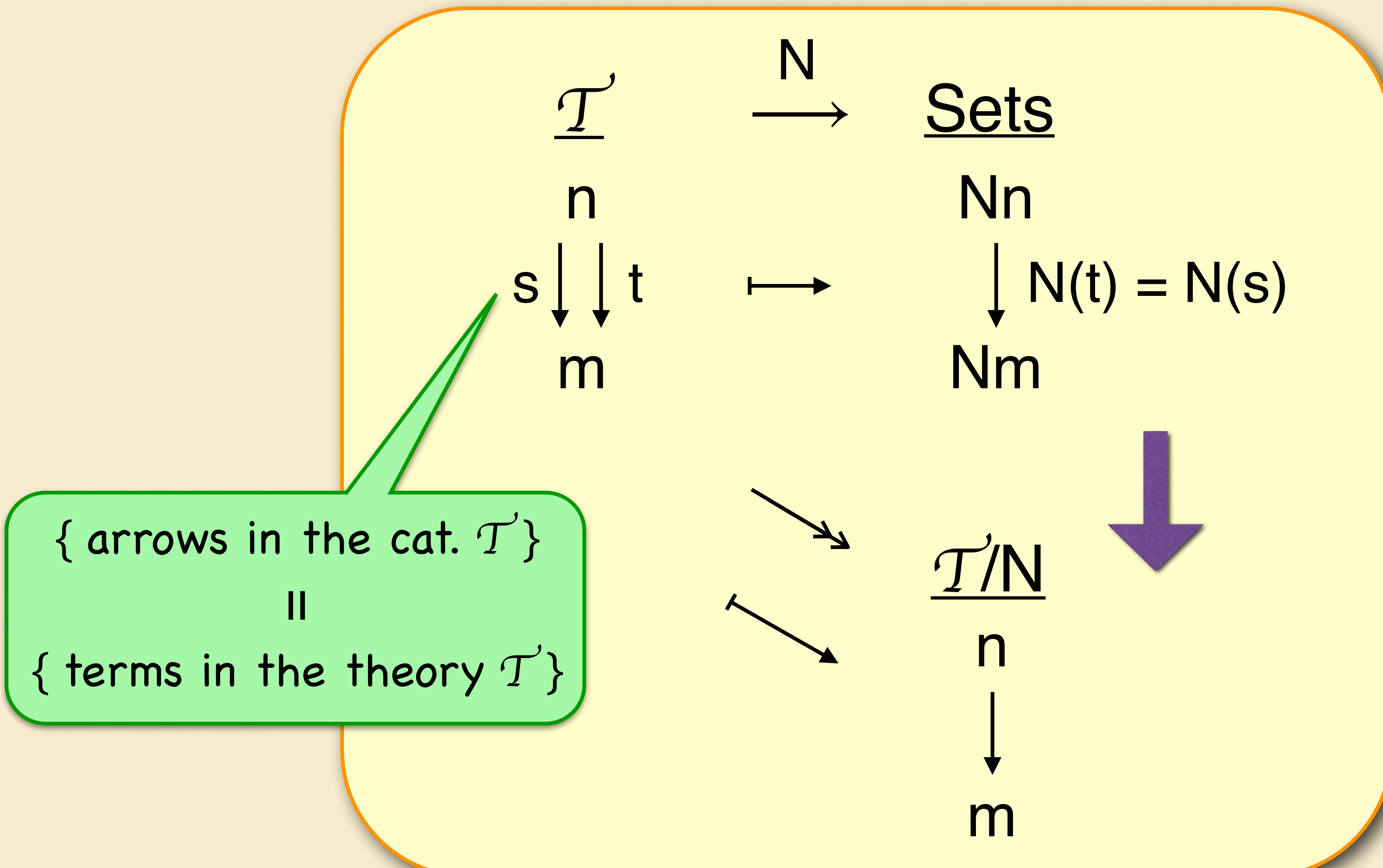

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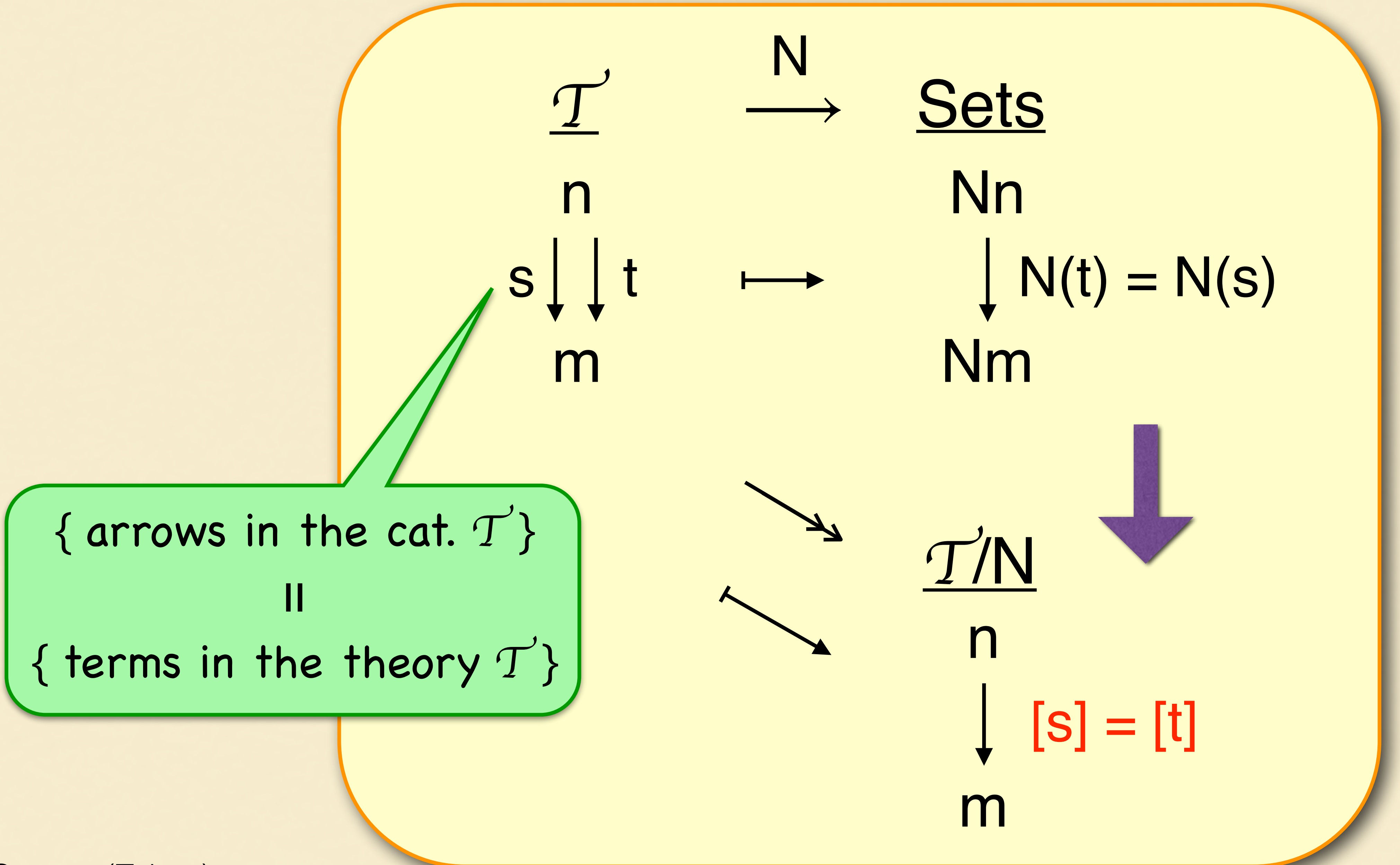

Coincidence of two quotients

<u>Thm.</u>

Given a monad T: Sets → Sets

a $\mathcal{EM}$-alg. $\alpha: TA \to A$ w/ cong.equiv.rel $\sim_A \subseteq A \times A$,

Then, we have $\mathcal{T}(T/_{(\alpha, \sim_A)}) \simeq (\mathcal{T}_T)/Q_{(\alpha, \sim_A)}$ as in:

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- Metric on prob. measure/distribution
  - Quantitative behavioural equiv. betw. prob. sys. [Breugel, Worrel]

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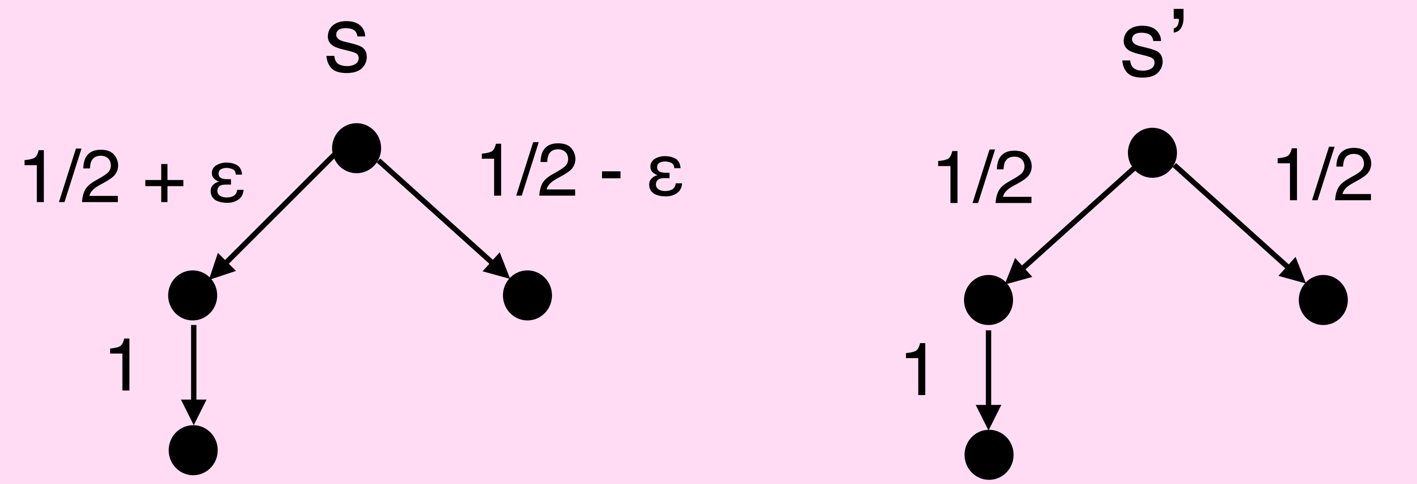

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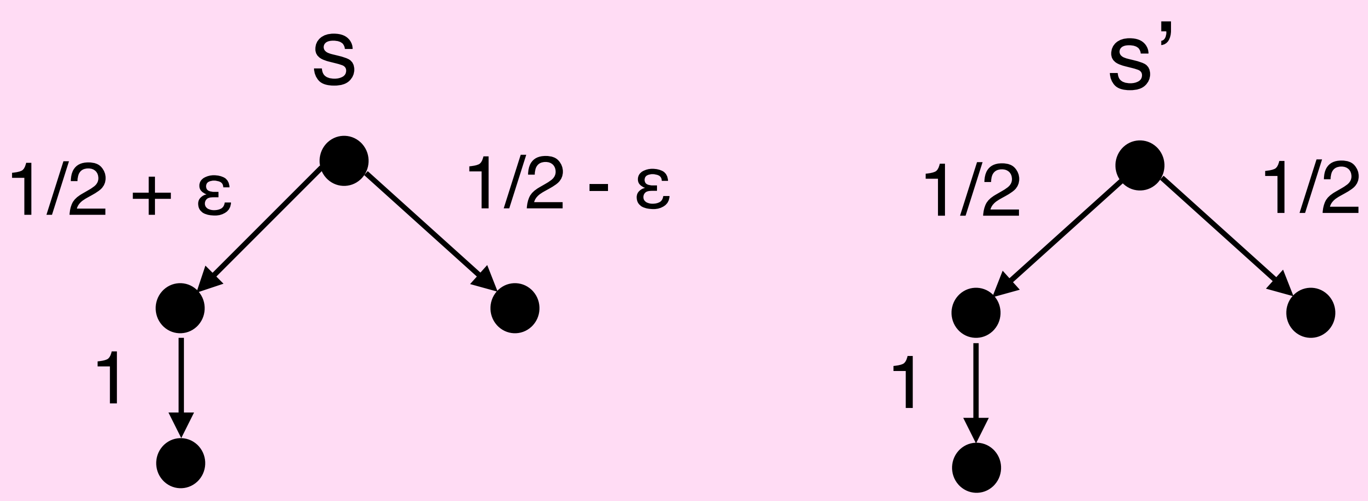

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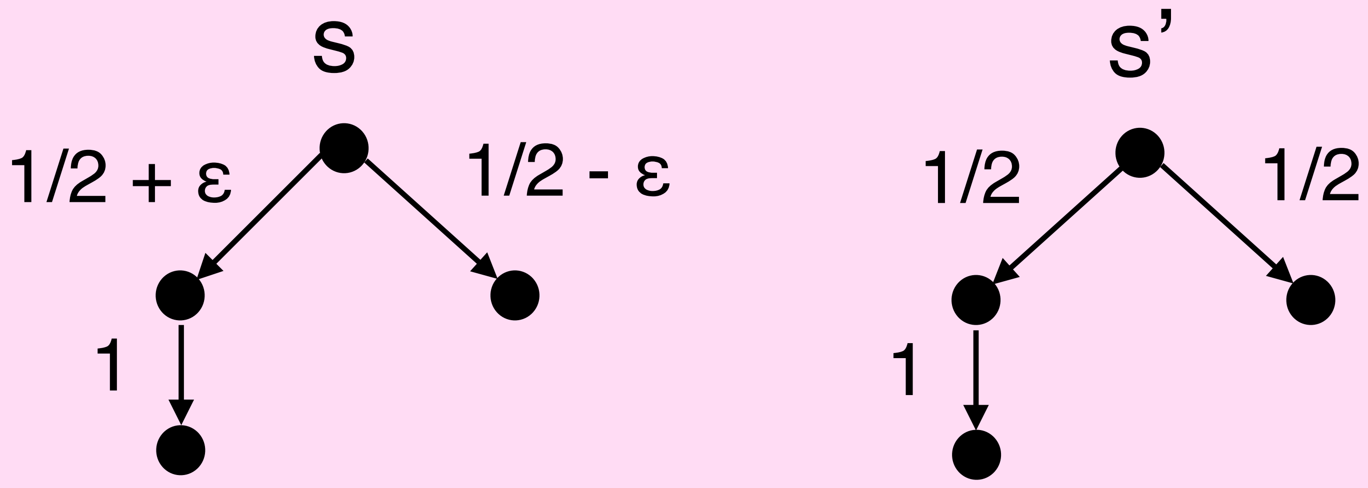

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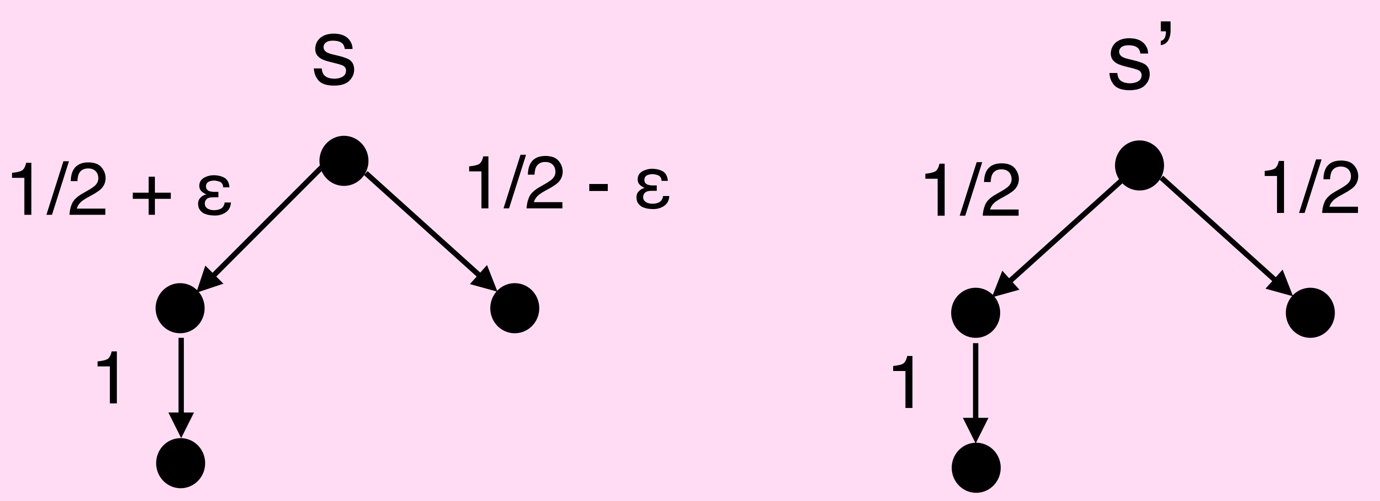

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Ans2. [Breugel, Worrel] : No... but the distance is $c \cdot \varepsilon$.

Kantorovich metric def.

usual categorical setting ($d_X : X \times X \to [0,1]$)

Def. [Kantorovich]

For (X, d_X) : (1-bounded) metric sp.

$\mathcal{B}(X)$: the set of Borel prob. meas. on X,

the Kantorovich metric d_K on $\mathcal{B}(X)$ is defined by:

$$d_K(\mu, \mu') = \sup_{\substack{f : X \to [0,1] \\ \text{non. exp.}}} \left| \int f \, d\mu - \int f \, d\mu' \right|$$

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- (intuition) d_K is the dual of transportation problem.

Thm. [Kantorovich]

We also define a metric d_L on $\mathcal{B}(X)$ by:

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Then, $d_K = d_L$.

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minimize the cost $c = \sum \mathrm{dist}_i \times \mathrm{mass}_i$

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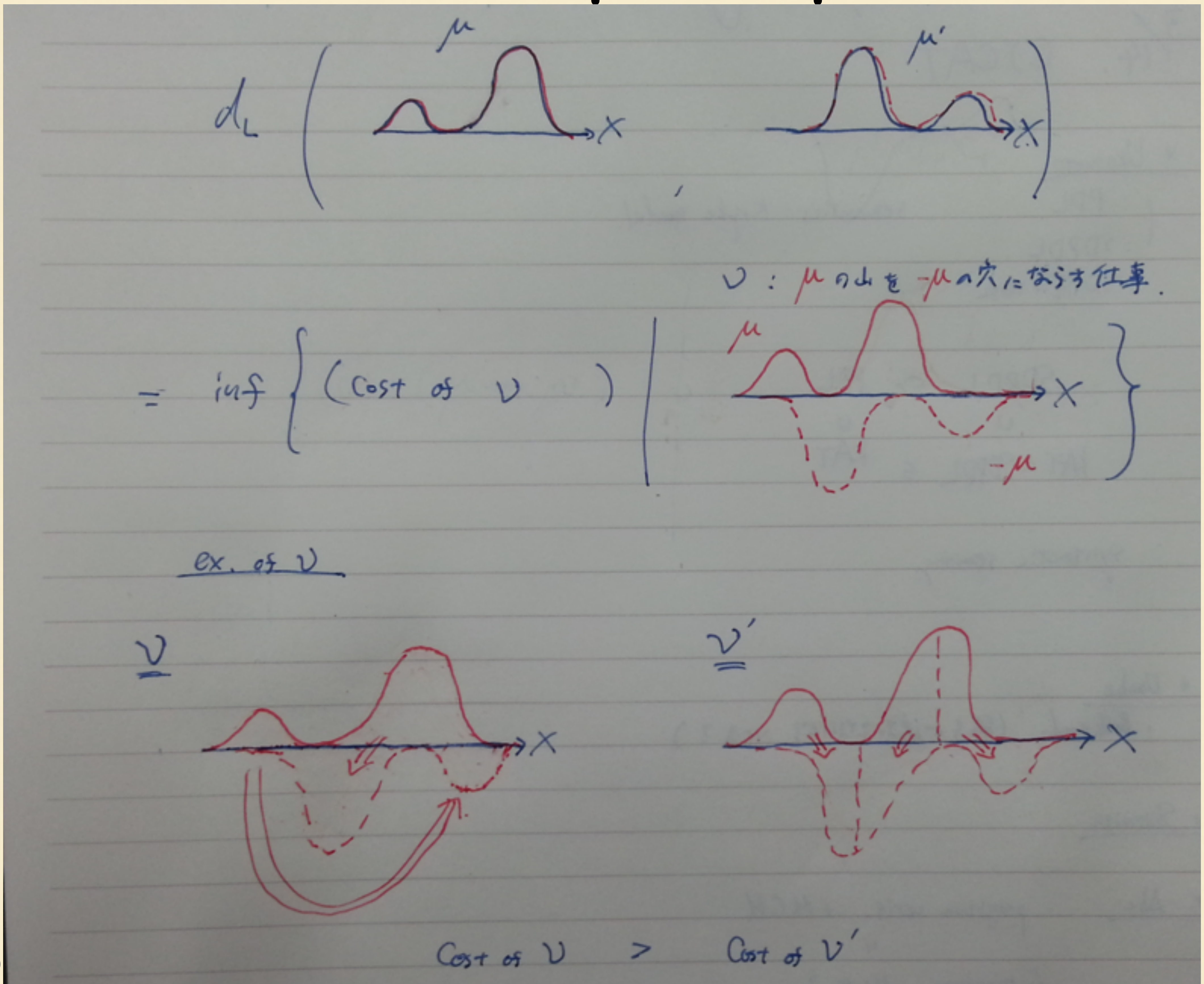

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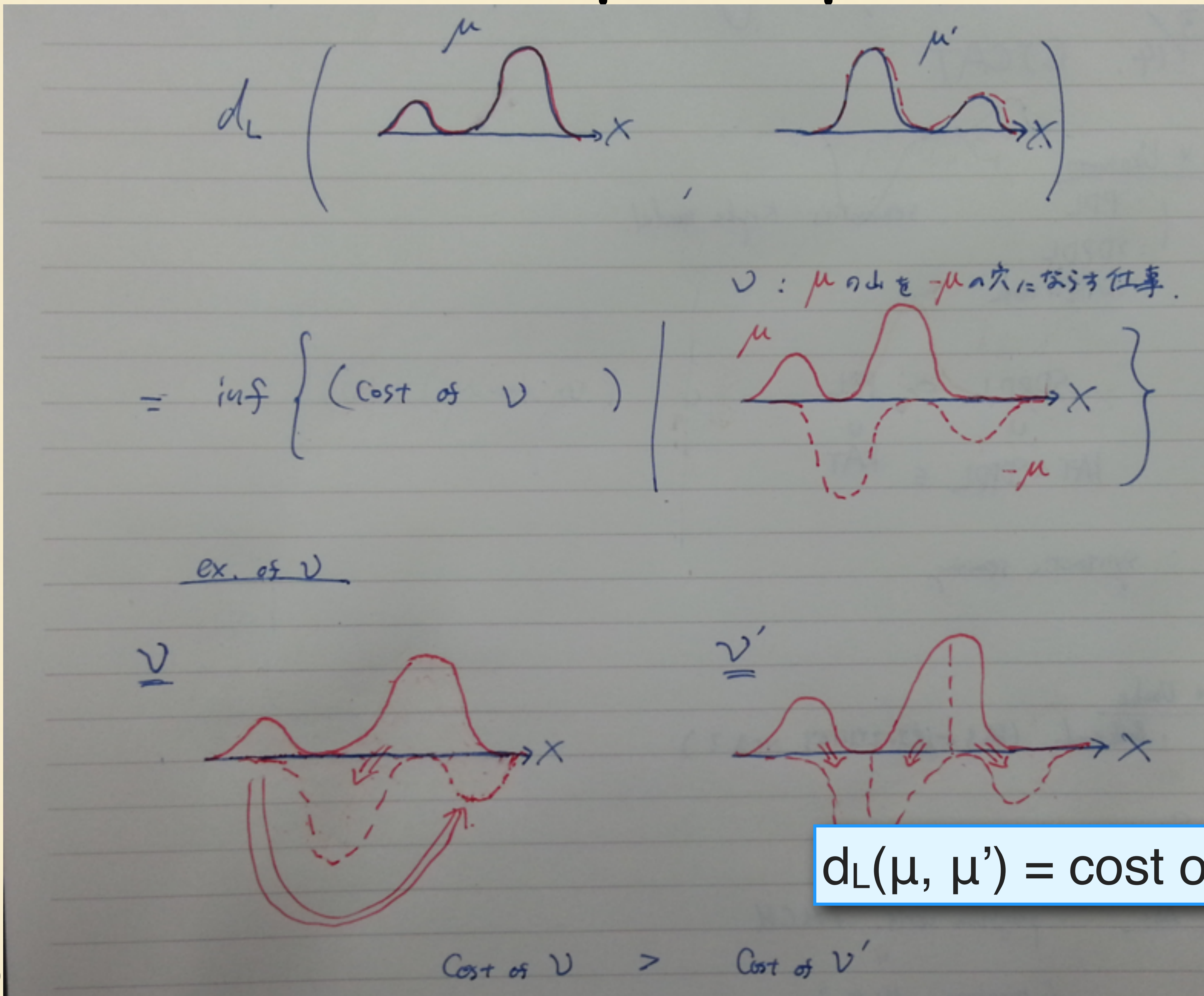

$d_L(\mu, \mu') = \text{cost of } v'$

Observation metric in $\mathcal{T}_{\mathcal{D}}$

- fix the Law. th. as $\mathcal{T}_{\mathcal{D}}$ (the th. of convex sp.)
- replace Sets w/ Met (the cod. of model)
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measure the dist. in Met

- not only adding equations, we can also induce a distance betw. two terms μ, μ' by:

$$d_O(\mu, \mu') := d_{(Nn \Rightarrow N1)}(N(\mu), N(\mu'))$$

Coincidence of distances

- For now, we deal with only "finite, discr." case…

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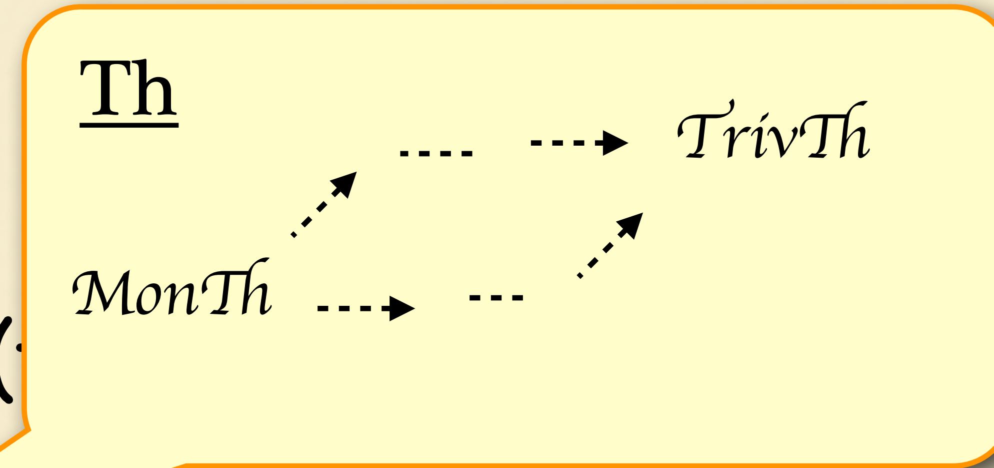

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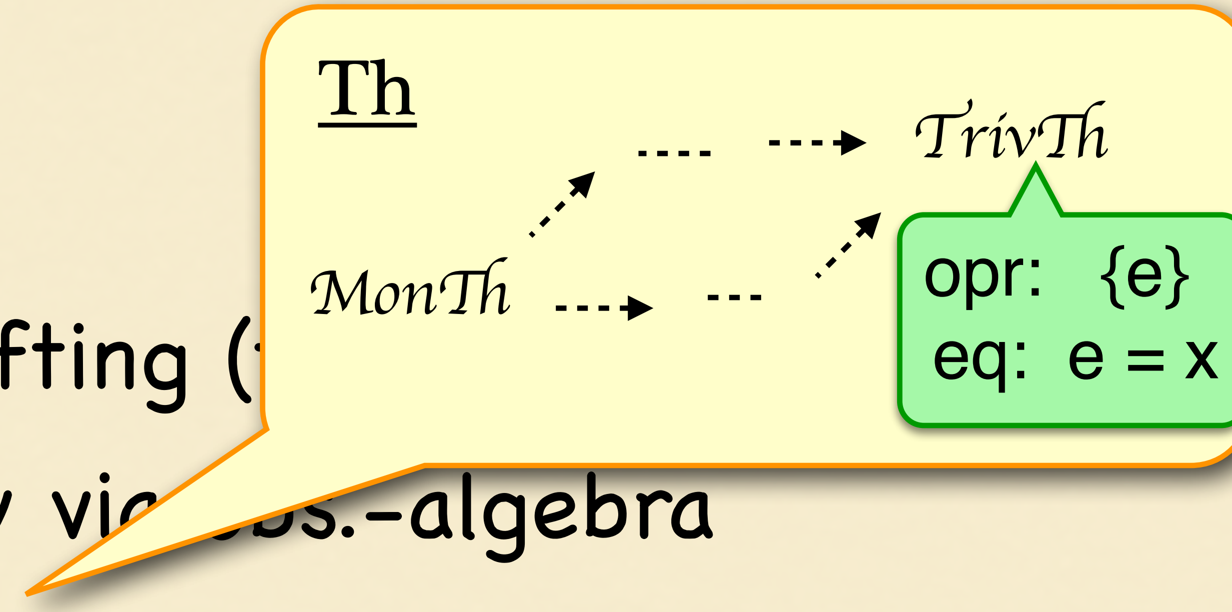

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