On Extension of Parameterized Coinduction

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Background

Formal Verification of (Co)Inductive Properties using Proof Assistants

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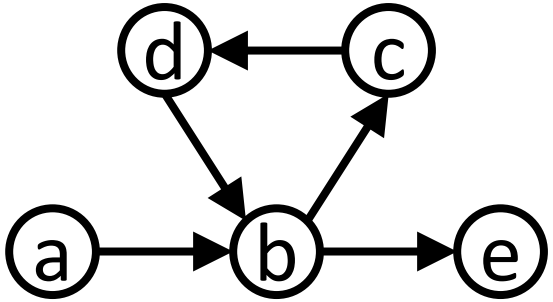
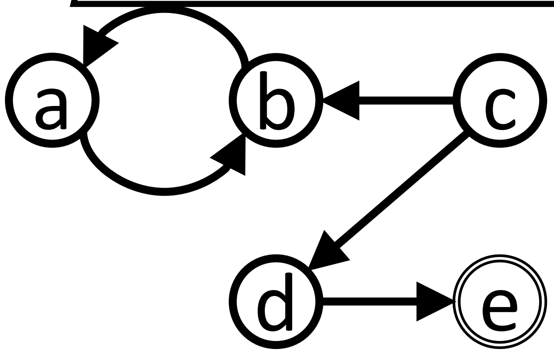


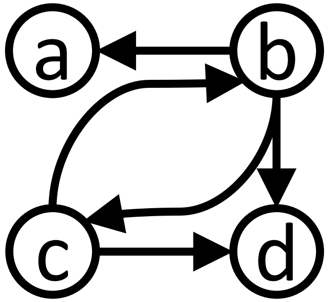
Infinite Path

$\{a\} \subseteq \nu Z. \diamond Z$



Reachability

$\{c\} \subseteq \mu Z. \{e\} \cup \diamond Z$



Bisimulation

$\{(b, c)\} \subseteq \nu R. \operatorname{sim}(R)$

[Hur *et al.*, POPL '13]

Parameterized coinduction
for coinductive properties

Parameterized induction
for inductive properties

No-discovery characterization

How convenient is it?
Mathematical characterization

Our
contribution

[Hur *et al.*, POPL '13]

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Complete lattice and monotone function

Def.

$(C, \sqsubseteq)$: complete lattice *i.e.*

- $\forall A \subseteq C.\ \forall x \in C.\ x \sqsubseteq \sqcap A \Leftrightarrow (\forall y \in A.\ x \sqsubseteq y)$
- $\forall A \subseteq C.\ \forall x \in C.\ \bigsqcup A \sqsubseteq x \Leftrightarrow (\forall y \in A.\ y \sqsubseteq x)$
- $x \sqcup y = \bigsqcup\{x, y\}, \bot = \bigsqcup \emptyset$
- $x \sqcap y = \sqcap \{x, y\}, \top = \sqcap \emptyset$

$f : C \to C$: monotone *i.e.*

- $\forall x, y \in C.\ x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$

Tarski coinduction

Def.

$\nu f = \max\{x \in C \mid x = f(x)\}$ greatest fixpoint

Lem.

$f(\nu f) = \nu f$

$\forall x.\ x \sqsubseteq f(x) \Rightarrow x \sqsubseteq \nu f$ (Tarski)

Coq's built-in coinduction

Parameterized coinduction [Hur *et al.*, POPL '13]

Def.

$$G_f(x) = \nu y.\ f(x \sqcup y)$$ parameterized greatest fixpoint

Lem.

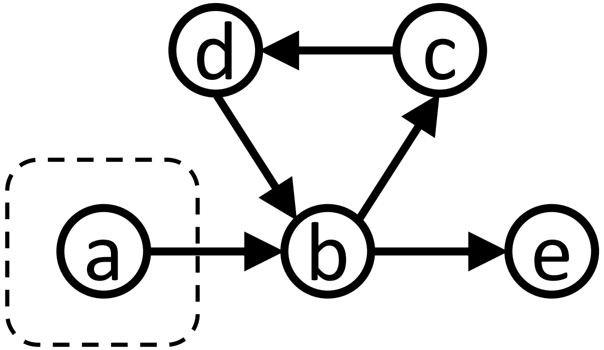
$\forall x, y \in C.\ y \sqsubseteq f\left(x \sqcup G_f(x)\right) \Rightarrow y \sqsubseteq G_f(x)$ (Unfolding)

$\forall x, y \in C.\ y \sqsubseteq G_f(x \sqcup y) \Rightarrow y \sqsubseteq G_f(x)$ (Accumulation)

Implemented on Coq in [Hur *et al.*, POPL '13]

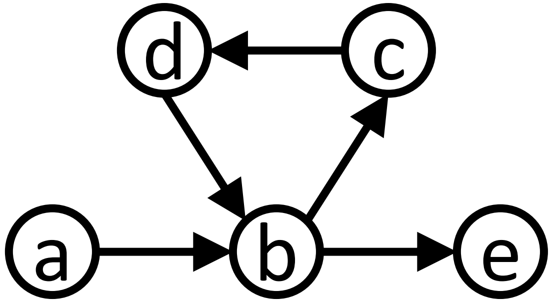
Example

Tarski



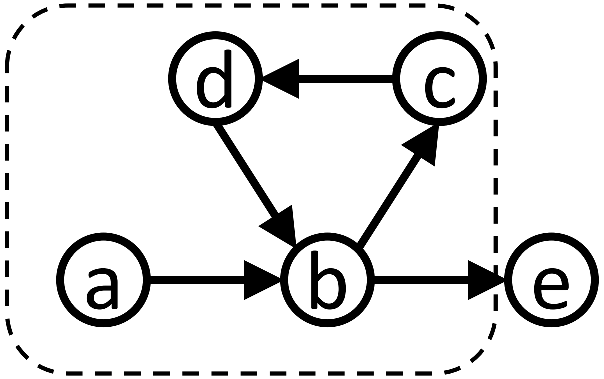
$\{a\} \subseteq \nu Z. \diamond Z$

PaCo



Example

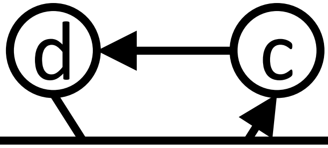
Tarski



$\{a\} \subseteq \nu Z. \diamond Z$

$\Leftarrow \{a, b, c, d\} \subseteq \nu Z. \diamond Z$

PaCo



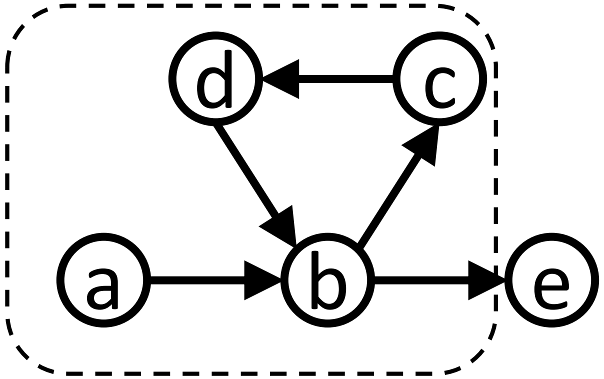
Transitivity

$x \sqsubseteq \nu f$

$\Leftarrow x \sqsubseteq x', x' \sqsubseteq \nu f$

Example

Tarski



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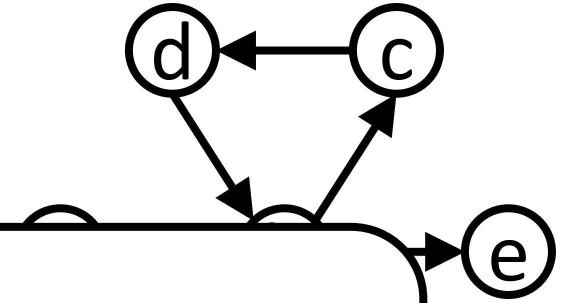


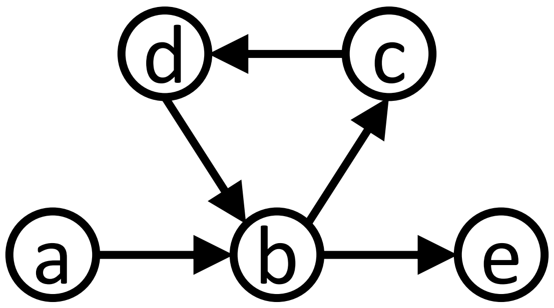
Tarski

$x \sqsubseteq \nu f$

$\Leftarrow x \sqsubseteq f(x)$

Example

Tarski



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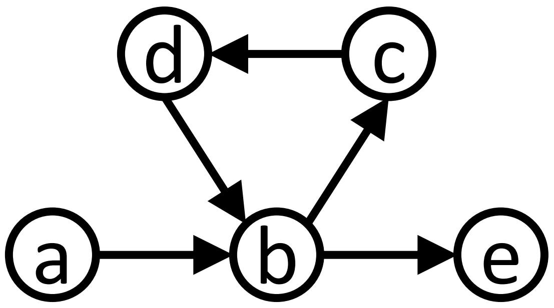
$\Leftarrow \{a, b, c, d\} \subseteq \{a, b, c, d\}$

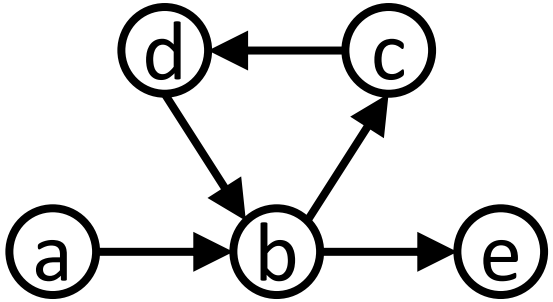
$\square$

PaCo



Example

Tarski



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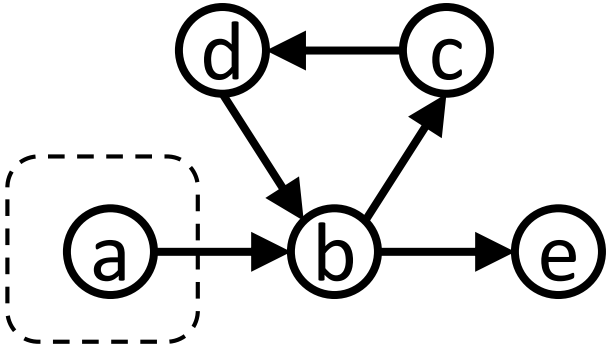
$\Leftarrow \{a, b, c, d\} \subseteq \{a, b, c, d\}$ □

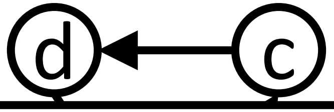
PaCo



$\{a\} \subseteq G_{\diamond}(\emptyset)$

Example

Tarski



Unfolding

$y \sqsubseteq G_f(x)$

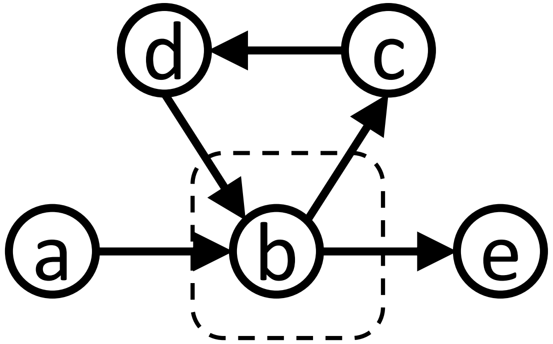
$\Leftarrow y \sqsubseteq f\left(x \sqcup G_f(x)\right)$

$\Leftarrow \{a,b,c,d\} \subseteq \Diamond\{a,b,c,d\}$

$\Leftarrow \{a,b,c,d\} \subseteq \{a,b,c,d\}$

□

PaCo

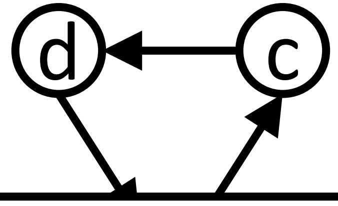


$\{a\} \subseteq G_{\Diamond}(\emptyset)$

$\Leftarrow \{b\} \subseteq G_{\Diamond}(\emptyset)$

Example

Tarski



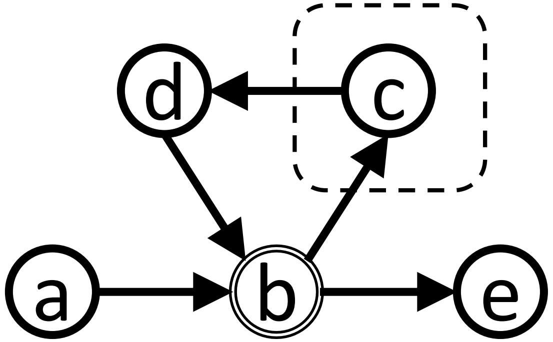
Accumulation

$y \sqsubseteq G_f(x)$

$\Leftarrow y \sqsubseteq G_f(x \sqcup y)$

$\Leftarrow \{a, b, c, d\} \subseteq \{a, b, c, d\}$ □

PaCo



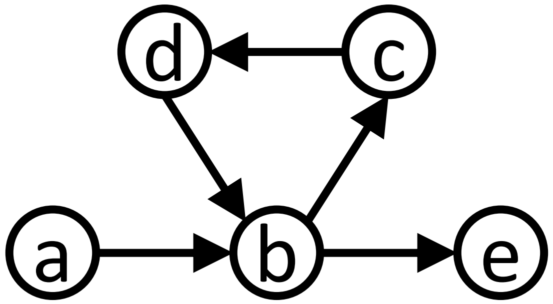
$\{a\} \subseteq G_\diamond(\emptyset)$

$\Leftarrow \{b\} \subseteq G_\diamond(\emptyset)$

$\Leftarrow \{c\} \subseteq G_\diamond(\{b\})$

Example

Tarski



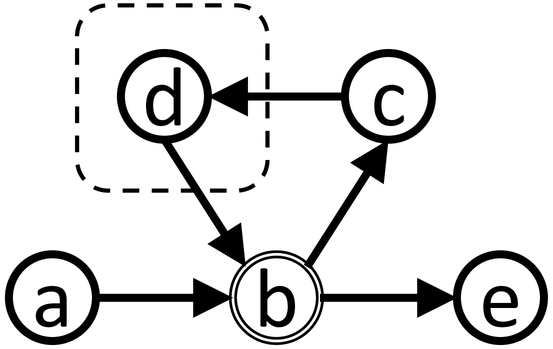
$\{a\} \subseteq \nu Z. \diamond Z$

$\Leftarrow \{a, b, c, d\} \subseteq \nu Z. \diamond Z$

$\Leftarrow \{a, b, c, d\} \subseteq \diamond \{a, b, c, d\}$

$\Leftarrow \{a, b, c, d\} \subseteq \{a, b, c, d\}$ $\square$

PaCo



$\{a\} \subseteq G_\diamond(\emptyset)$

$\Leftarrow \{b\} \subseteq G_\diamond(\emptyset)$

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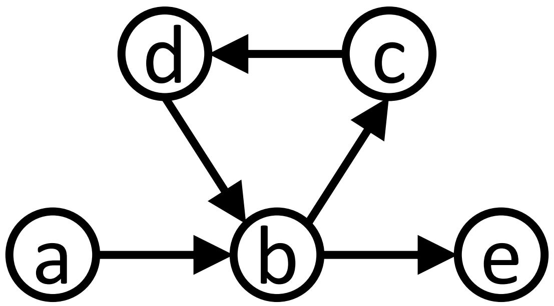
Example

Tarski



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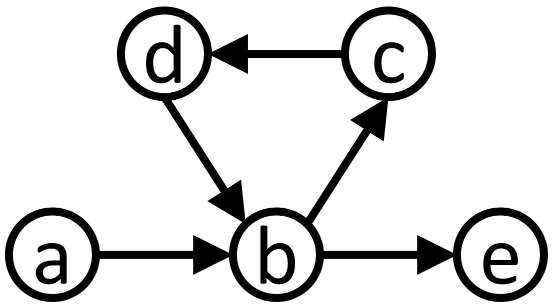
□

PaCo



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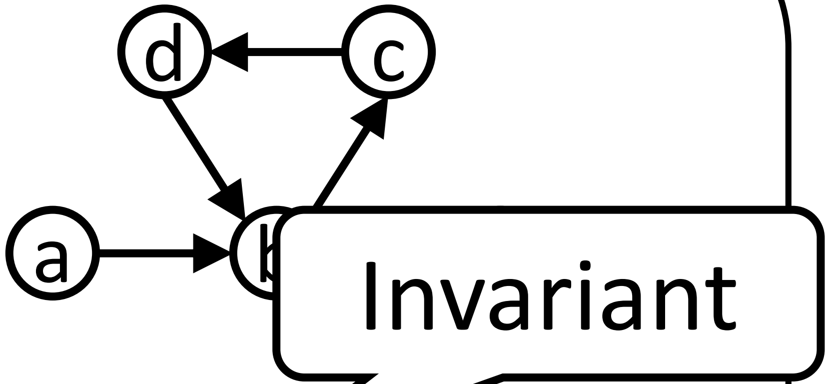
□

Example

Tarski



$\{a\} \subseteq \nu Z.\diamond Z$

$\Leftarrow \{a, b, c, d\} \subseteq \nu Z.\diamond Z$

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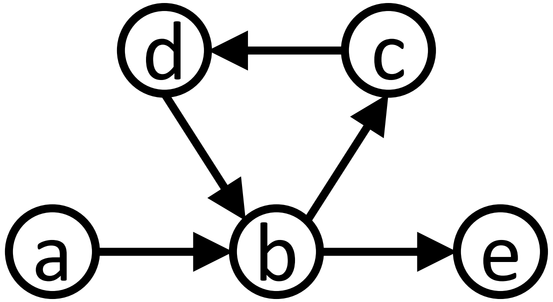
PaCo



No Invariant

$\{a\} \subseteq G_\diamond(\emptyset)$

$\Leftarrow \{b\} \subseteq G_\diamond(\emptyset)$

$\Leftarrow \{c\} \subseteq G_\diamond(\{b\})$

$\Leftarrow \{d\} \subseteq G_\diamond(\{b\})$

$\Leftarrow \{b\} \subseteq \{b\}$ $\square$

Functional characterization

Def.

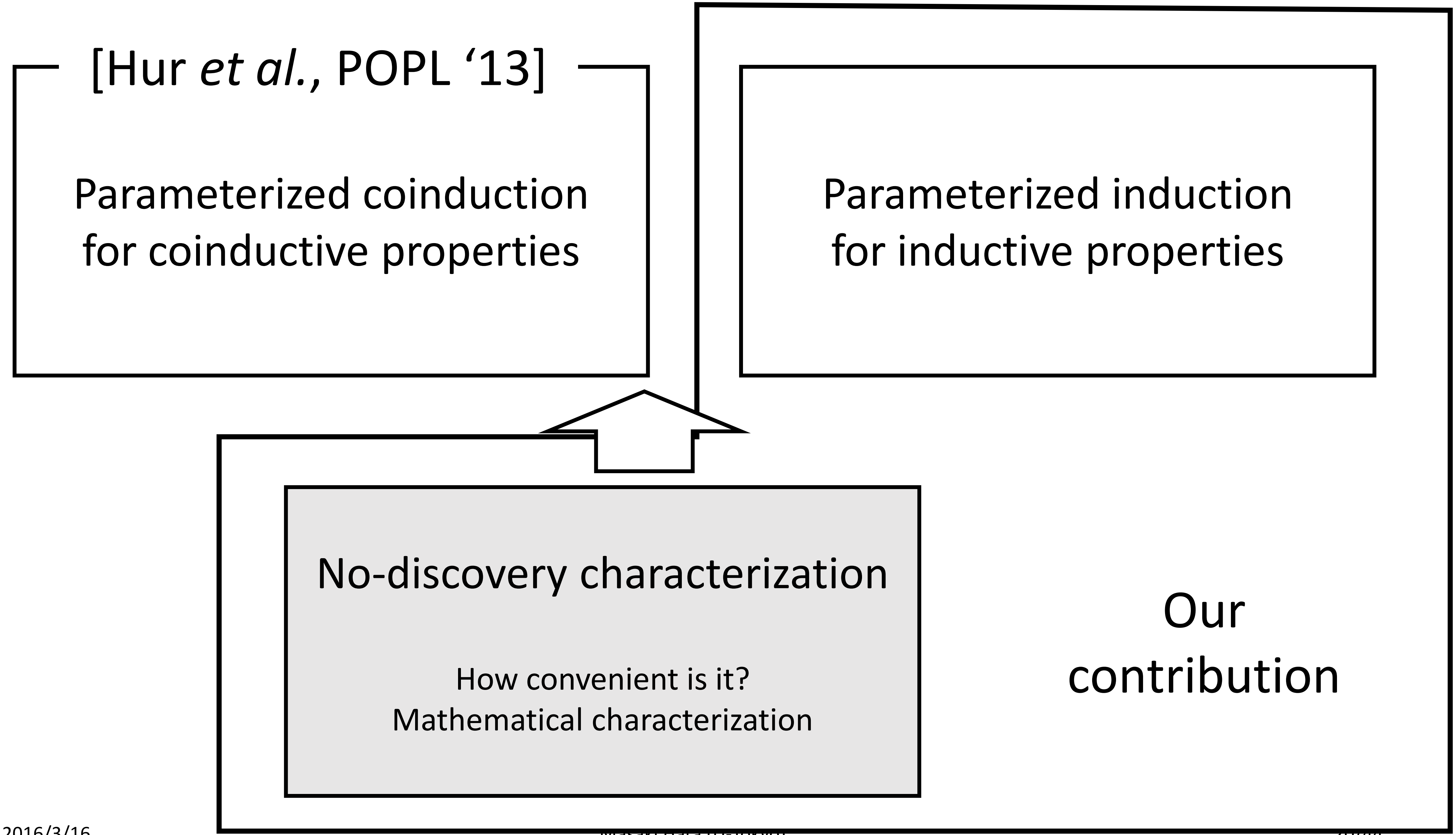
For $F : C \to C$ a function,

- F satisfies *Unfolding*, if $\forall x, y \in C.\ y \sqsubseteq f\big(x \sqcup F(x)\big) \Rightarrow y \sqsubseteq F(x)$
- F satisfies *Accumulation*, if $\forall x, y \in C.\ y \sqsubseteq F(x \sqcup y) \Rightarrow y \sqsubseteq F(x)$

Thm.

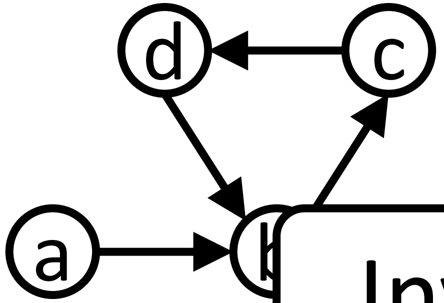
For F satisfying Unfolding and Accumulation,

$G_f(x) \sqsubseteq F(x)$.


[Hur *et al.*, POPL ‘13]
Parameterized coinduction for coinductive properties
Parameterized induction for inductive properties
No-discovery characterization
How convenient is it?
Mathematical characterization
Our contribution

Review : comparison of two proofs

Tarski



Invariant

$\{a\} \subseteq \nu Z. \diamond Z$

$\Leftarrow \{a, b, c, d\} \subseteq \nu Z. \diamond Z$

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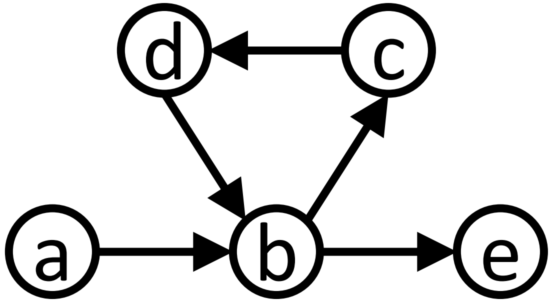
$\Leftarrow \{a, b, c, d\} \subseteq \{a, b, c, d\}$ □

PaCo



No Invariant

$\{a\} \subseteq G_\diamond(\emptyset)$

$\Leftarrow \{b\} \subseteq G_\diamond(\emptyset)$

$\Leftarrow \{c\} \subseteq G_\diamond(\{b\})$

$\Leftarrow \{d\} \subseteq G_\diamond(\{b\})$

$\Leftarrow \{b\} \subseteq \{b\}$ □

Review : comparison of two proofs

Tarski

PaCo

Question:

1. What is invariant discovery?
2. Doesn't PaCo need invariant discovery?

$\{a\} \subseteq$

$\Leftarrow \{a, b$

$\Leftarrow \{a, b,$

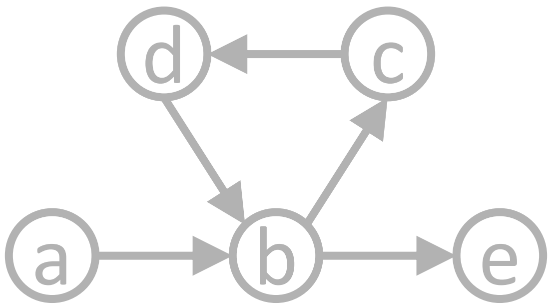
$\Leftarrow \{a, b, c, d\} \subseteq \{a, b, c, d\}$ □

ant

$\Leftarrow \{b\} \subseteq \{b\}$ □

Invariant discovery : informal definition

Tarski



$\{a\} \subseteq \nu Z. \diamond Z$

$\Leftarrow \{a,b,c,d\} \subseteq \nu Z. \diamond Z$

$\Leftarrow \{a,b,c,d\} \subseteq \diamond \{a,b,c,d\}$

$\Leftarrow \{a,b,c,d\} \subseteq \{a,b,c,d\}$ □

Def. (Informal)

Invariant Discovery is a proof like this:

$x \sqsubseteq \nu f$

$\Leftarrow x \sqsubseteq x', \; x' \sqsubseteq \nu f$

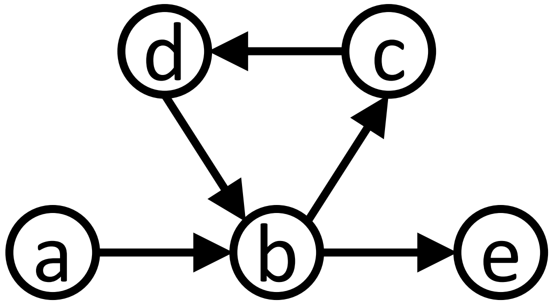
$\Leftarrow x \sqsubseteq x', \; x' \sqsubseteq f(x')$

Invariant discovery : example

	With invariant	**Without invariant**	
		Finite	Infinite
Tarski	✓	✘	✘
PaCo	✓	✓	✘

Invariant discovery : example

- Proof with invariant discovery
- $\{a\} \subseteq \nu Z. \diamond Z$
  $\Leftarrow \{a, b, c, d\} \subseteq \nu Z. \diamond Z$
  $\Leftarrow \{a, b, c, d\} \subseteq \diamond \{a, b, c, d\}$
  $\Leftarrow \{a, b, c, d\} \subseteq \{a, b, c, d\}$ □

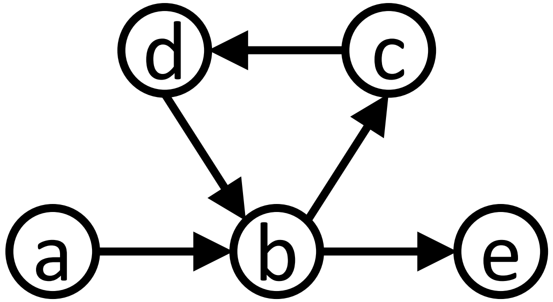


	With invariant	Without invariant	
		Finite	Infinite
Tarski	✓	×	×
PaCo	✓	✓	×

Invariant discovery : example

- Proof without invariant discovery : Tarski case
- $\{a\} \subseteq \nu Z. \diamond Z$
  $\Leftarrow \{b\} \subseteq \nu Z. \diamond Z$
  $\Leftarrow \{b\} \subseteq \diamond \{b\}$
  $\Leftrightarrow \{b\} \subseteq \{a, d\}$

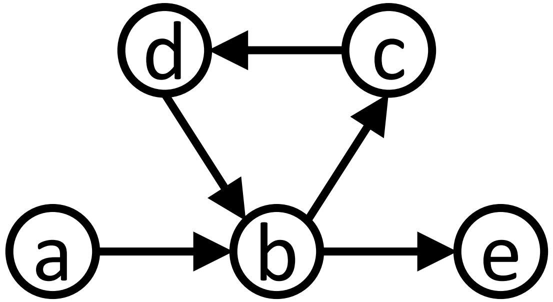
Unsuccessful proof



	With invariant	Without invarian	
		Finite	Infinite
Tarski	✓	×	×
PaCo	✓	✓	×

Invariant discovery : example

- Proof without invariant discovery : Tarski case
- $\{a\} \subseteq \nu Z. \diamond Z$
  $\Leftarrow \{b\} \subseteq \nu Z. \diamond Z$
  $\Leftarrow \{c\} \subseteq \nu Z. \diamond Z$
  $\Leftarrow \{d\} \subseteq \nu Z. \diamond Z$
  $\Leftarrow \quad \vdots$

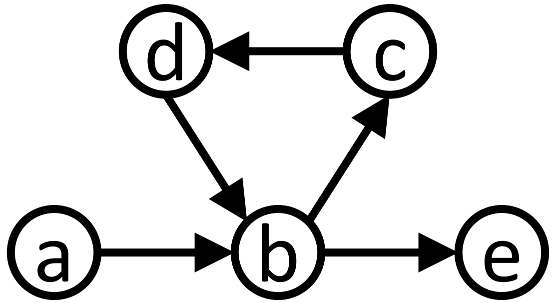


Unsuccessful proof

	With invariant	**Without invariant**	
		Finite	Infinite
Tarski	✓	✗	✗
PaCo	✓	✓	✗

Invariant discovery : example

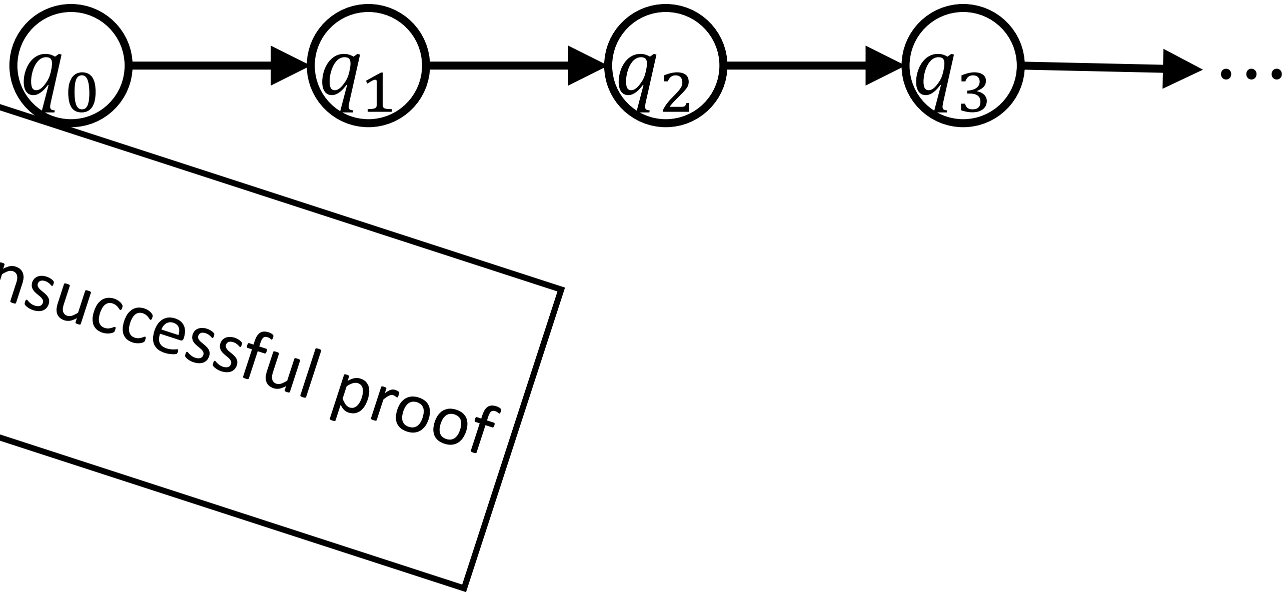
- Proof without invariant discovery : PaCo case
- $\{a\} \subseteq G_\diamond(\emptyset)$
  $\Leftarrow \{b\} \subseteq G_\diamond(\emptyset)$
  $\Leftarrow \{c\} \subseteq G_\diamond(\{b\})$
  $\Leftarrow \{d\} \subseteq G_\diamond(\{b\})$
  $\Leftarrow \{b\} \subseteq \{b\}$ $\square$



	With invariant	Without invariant	
		Finite	Infinite
Tarski	✓	✗	✗
PaCo	✓	✓	✗

Invariant discovery : example

- Proof without invariant discovery : PaCo case
- $\{q_0\} \subseteq G_\diamond(\emptyset)$
  $\Leftarrow \{q_1\} \subseteq G_\diamond(\{q_0\})$
  $\Leftarrow \{q_2\} \subseteq G_\diamond(\{q_0, q_1\})$
  $\Leftarrow \{q_3\} \subseteq G_\diamond(\{q_0, q_1, q_2\})$
  $\Leftarrow \vdots$



	With invariant	Without invariant	
		Finite	Infinite
Tarski	✓	×	×
PaCo	✓	✓	×

Invariant discovery : example

	With invariant	Without invariant	
		Finite	Infinite
Tarski	✓	✗	✗
PaCo	✓	✓	✗

Aim: to make these rigorous

Invariant discovery : analysis

$\{a\} \subseteq \nu Z. \diamond Z$
$\Leftarrow \{a, b, c, d\} \subseteq \nu Z. \diamond Z$

Invariant discovery uses <u>transitivity</u>

<u>Forget transitivity</u>

$y \sqsubseteq G_f(x) \longrightarrow P(x, y)$

Invariant discovery : assumption

Asm.

From now on, we assume that for all x, $\lambda y.\ f(x \sqcup y)$ preserves directed infima.

Invariant discovery : definition

Def.

For $P(x, y)$: predicate,

- Downward closure :

$$\forall y, y' \in C.\ y \sqsubseteq y', P(x, y') \quad \Rightarrow \quad P(x, y)$$

cf. transitivity $y \sqsubseteq y', y' \sqsubseteq G_f(x) \ \Rightarrow\ y \sqsubseteq G_f(x)$

- Join closure :

$$\forall x \in C.\ \forall Y \subseteq C. \big(\forall y \in Y.\ P(x, y)\big) \ \Rightarrow\ P(x, \bigsqcup Y)$$

Invariant discovery : definition

Def.

For $P(x, y)$: predicate,

- Unfolding : for all x, y, z, $P(x, y)$ holds if
  - z is minimal among $\{ z' \mid y \sqsubseteq f(z') \}$
  - $P(x, z)$.
- Accumulation : $\forall x, y.\ P(x \sqcup y, y) \Rightarrow P(x, y)$
- Tarski : $\forall x, y.\ y \sqsubseteq f(x \sqcup y) \Rightarrow P(x, y)$

Invariant discovery : definition

Thm.

- $y \sqsubseteq \nu f$

  $\Leftrightarrow \quad \forall P.$ Downward closure, Join closure, Unfolding, Tarski

  $\Rightarrow P(\bot, y)$

- $y \sqsubseteq G_f(x)$

  $\Leftrightarrow \quad \forall P.$ Downward closure, Join closure, Unfolding, Accumulation

  $\Rightarrow P(x, y)$

Invariant discovery : definition

Def.

- $y \sqsubseteq \nu f$ is *no-discovery Tarski provable*
  $\Leftrightarrow \quad \forall P.$ Join closure, Unfolding, Tarski
  $\Rightarrow P(\bot, y)$
- $y \sqsubseteq G_f(x)$ is *no-discovery PaCo provable*
  $\Leftrightarrow \quad \forall P.$ Join closure, Unfolding, Accumulation
  $\Rightarrow P(x, y)$

Invariant discovery : main theorem

Thm.

- For finite lattices, $y \sqsubseteq G_f(x)$ is no-discovery PaCo provable, if it holds.
- For infinite lattices, $y \sqsubseteq G_f(x)$ is not necessarily no-discovery PaCo provable.
- $y \sqsubseteq \nu f$ is not necessarily no-discovery Tarski provable.

Invariant discovery : main theorem

Thm.

- For finite lattices, $y \sqsubseteq G_f(x)$ is no-discovery PaCo provable, if it holds.
- For infinite lattices, $y \sqsubseteq G_f(x)$ is not necessarily no-discovery PaCo provable.
- $y \sqsubseteq \nu f$ is not necessarily no-discovery Tarski provable.

	With invariant	**Without invariant**	
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Tarski	✓	✗	✗
PaCo	✓	✓	✗

[Hur *et al.*, POPL '13]

Parameterized coinduction
for coinductive properties

Parameterized induction
for inductive properties

No-discovery characterization

How convenient is it?
Mathematical characterization

Our
contribution

Parameterized induction

Def.

$L_f(x) = \mu y.\ f(x \sqcup y)$ parameterized least fixpoint

Lem.

$\forall x, y.\ y \sqsubseteq f\left(x \sqcup L_f(x)\right) \Rightarrow\ y \sqsubseteq L_f(x)$ (Unfolding)

$\forall x, y : \alpha' \to C$ (x increasing).

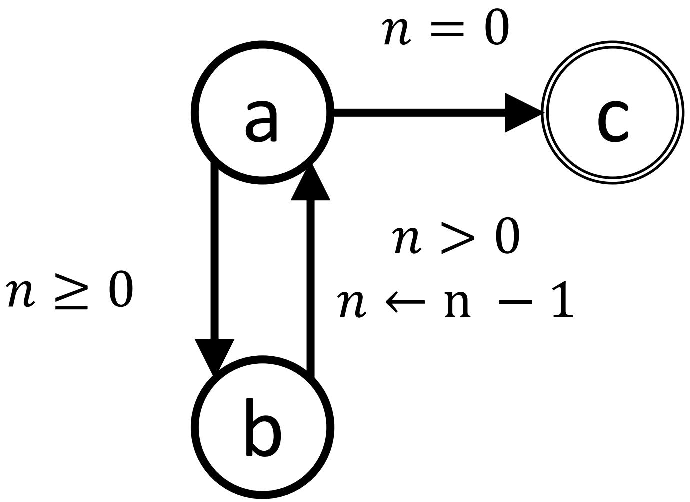
$$\left(\forall \alpha. y(\alpha) \sqsubseteq L_f\left(x(\alpha) \sqcup \bigsqcup_{\beta<\alpha} y(\beta)\right)\right)$$

$$\Rightarrow\ \left(\forall \alpha.\ y(\alpha) \sqsubseteq L_f\big(x(\alpha)\big)\right)$$

(Indexed Accumulation)

Parameterized induction : pushdown example

- $\forall n. \{a(n)\} \subseteq L_f(\emptyset)$
  $\Leftarrow \forall n. \{a(n)\} \subseteq L_f(\{a(m) | m < n\})$
  $\Leftarrow \forall n. \{b(n)\} \subseteq L_f(\{a(m) | m < n + 1\})$
  $\Leftarrow \forall n. \{a(n)\} \subseteq \{a(m) | m < n + 1\}$ □



$f(Z) = \{c(n)\} \cup \diamond Z$

Conclusion & Future Work

- No-discovery characterization
  - Gave mathematical framework to characterize power of parameterized coinduction
  - No-go theorem : perhaps useful for future proof-technique development.
- Future work
  - Syntactic Approach
  - Trying non-Boolean examples

Conclusion & Future Work

- **Parameterized Induction**
  - Applied PaCo-like strategy to inductive properties.
  - Perhaps useful for pushdown systems
- Future Work
  - Implementation to a proof assistant
  - Generalization to mixed μ/ν formulas

[Hur *et al.*, POPL '13]

Parameterized coinduction
for coinductive properties

Parameterized induction
for inductive properties

No-discovery characterization

How convenient is it?
Mathematical characterization

Our
contribution