# On Categorical Models of GoI
# Lecture 3

Esfandiar Haghverdi

School of Informatics and Computing
Indiana University Bloomington
USA

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# In this lecture

- We shall discuss generalizations of GoI interpretation to multi-object setting.
- I shall follow the papers: Haghverdi & Scott (CSL 2005), Haghverdi (ICALP 2006).

# Recall: GoI, main idea

Example:

$$\frac{A \vdash A \quad A \vdash A}{A \vdash A} \qquad \succ \qquad A \vdash A$$

$$id_A \circ id_A \qquad = \quad id_A \;\; \text{(Static!)}$$

More generally:

$\Pi \succ \Pi'$, then $[\![\Pi]\!] = [\![\Pi']\!]$ (Static!)

$\Pi \succ \Pi'$, then $\theta(\Pi) \neq \theta(\Pi')$, yet

$EX(\theta(\Pi), \sigma) = EX(\theta(\Pi'), \tau)$ (Dynamic!)

# GoI categorically

- $(\mathbb{C}, U, T)$ where
- $\mathbb{C}$ a traced UDC,
- $U$ a reflexive object ($U \otimes U \lhd U$, ...)
- $T$ a traced endofunctor ($T \otimes T \lhd T$, ...)
- GoI for MELL à la Girard is completely captured, including $C^*$-algebraic implementation.
- Dictionary:

| | |
|---|---|
| execution formula | trace |
| orthogonality | nilpotency |
| datum | special morphisms |
| algorithm | special morphisms |

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- First Problem: There are no non-trivial reflexive objects.
- Second Problem: It is not traced!
- Hmm ...
- Work in a typed setting, no need for reflexive objects.
- Allow for partial trace.

$$Tr\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = A + B(I - D)^{-1}C.$$

# What is new?

| *Old* | *New* |
| --- | --- |
| single reflexive object | multiple objects |
| UDC, "sum like" monoidal product | arbitrary monoidal product |
| traced category | *partially* traced category |
| nilpotency | abstract orthogonality |
| GoI for MLL | MGoI for MLL |

# Partial trace (Trace Class)

(Cf. Abramsky, Blute, Panangaden 1999, Jeffrey 1998, Blute, Cockett, Seely 1999, Plotkin MFPS 2003.)
A *Trace Class* in SMC $\mathbb{C}$:

$$\mathbb{T}^U_{X,Y} \subseteq \mathbb{C}(X \otimes U, Y \otimes U)$$

$$Tr^U_{X,Y} : \mathbb{T}^U_{X,Y} \longrightarrow \mathbb{C}(X, Y)$$

subject to

- **Naturality** in $X$ and $Y$: For any $f \in \mathbb{T}^U_{X,Y}$ and $g : X' \longrightarrow X$ and $h : Y \longrightarrow Y'$,

$$(h \otimes 1_U) f (g \otimes 1_U) \in \mathbb{T}^U_{X',Y'}, \text{ and}$$

$$Tr^U_{X',Y'}((h \otimes 1_U) f (g \otimes 1_U)) = h\, Tr^U_{X,Y}(f)\, g$$

- **Dinaturality** in $U$:
  For any $f : X \otimes U \longrightarrow Y \otimes U'$, $g : U' \longrightarrow U$,

$$(1_Y \otimes g)f \in \mathbb{T}^{U}_{X,Y} \;\text{ iff }\; f(1_X \otimes g) \in \mathbb{T}^{U'}_{X,Y}, \text{and}$$

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- **Vanishing I**: $\mathbb{T}^I_{X,Y} = \mathbb{C}(X \otimes I, Y \otimes I)$ and for $f \in \mathbb{T}^I_{X,Y}$

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- **Vanishing II**: For any $g : X \otimes U \otimes V \longrightarrow Y \otimes U \otimes V$, if $g \in \mathbb{T}^V_{X\otimes U, Y\otimes U}$, then

  $$g \in \mathbb{T}^{U\otimes V}_{X,Y} \text{ iff } Tr^V_{X\otimes U, Y\otimes U}(g) \in \mathbb{T}^U_{X,Y}, \text{and}$$

  $$Tr^{U\otimes V}_{X,Y}(g) = Tr^U_{X,Y}(Tr^V_{X\otimes U, Y\otimes U}(g)).$$

- **Superposing**: For any $f \in \mathbb{T}^U_{X,Y}$ and $g : W \longrightarrow Z$,

$$g \otimes f \in \mathbb{T}^U_{W \otimes X, Z \otimes Y}, \text{ and}$$

$$Tr^U_{W \otimes X, Z \otimes Y}(g \otimes f) = g \otimes Tr^U_{X,Y}(f).$$

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- **Yanking**: $\quad s_{UU} \in \mathbb{T}^U_{U,U}, \quad$ and

$$Tr^U_{U,U}(s_{U,U}) = 1_U.$$

# Examples of trace classes

- $(\mathbf{FDVec_k}, \oplus, \mathbf{0})$: symmetric monoidal, additive
  $f : \oplus_I X_i \longrightarrow \oplus_J Y_j$, $f = [f_{ij}]$, where $f_{ij} : X_j \longrightarrow Y_i$.
  $f : X \oplus U \longrightarrow Y \oplus U$ is trace class iff $(I - f_{22})$ is invertible

  $$Tr^U_{X,Y}(f) = f_{11} + f_{12}(I - f_{22})^{-1} f_{21}.$$

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  $*$ Uses linear algebra facts for block matrices (Schur's complement, etc.)
- $(\mathbf{FDHilb_k}, \oplus)$: finite dimensional Hilbert spaces and bounded linear maps.

$M = \left[\begin{array}{cc} A & B \\ C & D \end{array}\right]$
$A\ (m \times m), B\ (m \times n), C\ (n \times m),\ D\ (n \times n)$.
If $D$ is invertible,
then $M$ is invertible iff $A - BD^{-1}C$ (the *Schur Complement of $D$*) is invertible.

$A\ (m \times n)$ and $B\ (n \times m)$
$(I_m - AB)$ is invertible iff $(I_n - BA)$ is invertible.
Moreover $(I_m - AB)^{-1}A = A(I_n - BA)^{-1}$.

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- product: $(M \times N, d_{M\times N})$, *max* metric:
  $d_{M\times N}((m, n), (m', n')) = max\{d_M(m, m'), d_N(n, n')\}$.

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  $d_{M \times N}((m, n), (m', n')) = max\{d_M(m, m'), d_N(n, n')\}$.
- $f : X \times U \longrightarrow Y \times U$ is trace class iff

  $$\forall x \in X. \exists! u. \exists y. f(x, u) = (y, u).$$

  That is, the induced map $: U \longrightarrow U$ has a unique fixed point.
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  and
- $Tr^U_{X,Y}(f)(x) = y$.

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- $a \in A$ be the unique fixed point of $gf : A \longrightarrow A$ and $b \in B$ be the unique fixed point of $fg : B \longrightarrow B$.
- Then $f(a) = b$ and $g(b) = a$.

$(\mathbf{Sets}, \times)$: sets and mappings.

Nonexample: $(\mathbf{Rel}, \times)$, sets and relations.

(Cf. Hyland and Schalk 2003)

### Definition

Let $\mathbb{C}$ be a traced symmetric monoidal category. An *orthogonality relation* on $\mathbb{C}$ is a family of relations $\perp_{UV}$ between morphisms $u : V \longrightarrow U$ and $x : U \longrightarrow V$

$$V \xrightarrow{u} U \perp_{UV} U \xrightarrow{x} V$$

subject to the following axioms:

- *Isomorphism* : Let $f : U \otimes V' \longrightarrow V \otimes U'$ and $\hat{f} : U' \otimes V \longrightarrow V' \otimes U$ be such that $Tr^{V'}(Tr^{U'}((1 \otimes 1 \otimes s_{U',V'})\alpha^{-1}(f \otimes \hat{f})\alpha)) = s_{U,V}$ and $Tr^{V}(Tr^{U}((1 \otimes 1 \otimes s_{U,V})\alpha^{-1}(\hat{f} \otimes f)\alpha)) = s_{U',V'}$. Here $\alpha = (1 \otimes 1 \otimes s)(1 \otimes s \otimes 1)$ with $s$ at appropriate types.

Then for all $u : V \longrightarrow U$ and $x : U \longrightarrow V$,

$$u \perp_{UV} x$$

iff

$$Tr^{U}_{V',U'}(s_{U,U'}(u \otimes 1_{U'})fs_{V',U}) \perp_{U'V'} Tr^{V}_{U',V'}((1_{V'} \otimes x)\hat{f});$$

that is, orthogonality is invariant under isomorphism.

- *Precise Tensor:*
  For all $u : V \longrightarrow U$, $v : V' \longrightarrow U'$ and $h : U \otimes U' \longrightarrow V \otimes V'$,

$$(u \otimes v) \perp_{U \otimes U', V \otimes V'} h.$$

$$\text{iff}$$

$$u \perp_{UV} Tr^{U'}_{U,V}((1_V \otimes v)h) \text{ and } v \perp_{U'V'} Tr^{U}_{U',V'}(s_{U,V'}(u \otimes 1_{V'})hs_{U',U})$$

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- *Identity* : For all $u : V \longrightarrow U$ and $x : U \longrightarrow V$,

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- *Symmetry* : For all $u : V \longrightarrow U$ and $x : U \longrightarrow V$,

$$u \perp_{UV} x \text{ iff } x \perp_{VU} u.$$

# Examples:

- Orthogonality defined by trace class:
  $(\mathbb{C}, \otimes, I, Tr)$ partially traced category,
  $f : A \longrightarrow B$ and $g : B \longrightarrow A$

$$f \perp_{BA} g \text{ iff } gf \in \mathbb{T}^{A}_{I,I}$$

  (cf. *Focussed orthogonality* of Hyland and Schalk)

  (cf. Polarity definition of Girard: Pole = trace class)

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- $\mathbf{FDVec_k}$ . For $A \in \mathbf{FDVec_k}$ , $f, g \in End(A)$, define $f \perp g$ iff $I - gf$ is invertible.

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- $\mathbf{FDVec_k}$ . For $A \in \mathbf{FDVec_k}$ , $f, g \in End(A)$, define $f \perp g$ iff $I - gf$ is invertible.
- **CMet**. Let $M \in$ **CMet**. For $f, g \in End(M)$, define $f \perp g$ iff $gf$ has a unique fixed point.

# MGoI for MLL: formulas

$A$, object of $\mathbb{C}$, $X \subseteq End(A)$

$$X^{\perp} = \{f \in End(A) \mid \forall g \in X, f \perp g\}.$$

$$\mathcal{T}(A) = \{X \subseteq End(A) \mid X^{\perp\perp} = X\}.$$

- $[\![ \mathbf{1} ]\!] = [\![ \perp ]\!] = I$ where $I$ is the unit of $\mathbb{C}$.
- $[\![ \alpha^{\perp} ]\!] = [\![ \alpha ]\!]$, $\alpha$ atomic.
- $[\![ A \parr B ]\!] = [\![ A \otimes B ]\!] = [\![ A ]\!] \otimes [\![ B ]\!]$.

MGoI-interpretation for formulas, $\theta(A)$:

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- For any formula $A$, $\theta A^{\perp} = (\theta A)^{\perp}$,
- $\theta(A) \subseteq End(\llbracket A \rrbracket)$,
- $\theta(A)^{\perp\perp} = \theta(A)$.

# MGoI for MLL: proofs

$\Pi$ a proof of $\vdash [\Delta], \Gamma$. $\theta(\Pi) \in End(\otimes[\![\Gamma]\!] \otimes [\![\Delta]\!])$ :

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- *axiom*: $\vdash A, A^{\perp}$, suppose $[\![A]\!] = V$.
  Then, $\theta(\Pi) : V \otimes V \longrightarrow V \otimes V$, which is defined to be $s_{V,V}$

- *cut* rule on $\Pi'$ and $\Pi''$:

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta'], \Gamma', A\end{array} \quad \begin{array}{c}\Pi'' \\ \vdots \\ \vdash [\Delta''], A^{\perp}, \Gamma''\end{array}}{\vdash [\Delta', \Delta'', A, A^{\perp}], \Gamma', \Gamma''} \; cut$$

$\theta(\Pi) = \tau^{-1}(\theta(\Pi') \otimes \theta(\Pi''))\tau$,
$\Gamma' \otimes \Gamma'' \otimes \Delta' \otimes \Delta'' \otimes A \otimes A^{\perp} \xrightarrow{\tau} \Gamma' \otimes A \otimes \Delta' \otimes A^{\perp} \otimes \Gamma'' \otimes \Delta''$

- *exchange* rule:

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma'\end{array}}{\vdash [\Delta], \Gamma} \; \textit{exchange}$$

$\Gamma' = \Gamma'_1, A_i, A_{i+1}, \Gamma'_2$, $\Gamma = \Gamma'_1, A_{i+1}, A_i, \Gamma'_2$. Then,

$$\theta(\Pi) = \tau^{-1}\theta(\Pi')\tau$$

$\tau = 1_{\Gamma'_1} \otimes s \otimes 1_{\Gamma'_2 \otimes \Delta}$.

- *par* rule:

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma', A, B\end{array}}{\vdash [\Delta], \Gamma', A \parr B} \parr \quad . \quad \text{Then } \theta(\Pi) = \theta(\Pi')$$

- *times* rule:

$$\dfrac{\dfrac{\Pi'}{\vdots}\quad \dfrac{\Pi''}{\vdots}}{}$$

$$\dfrac{\vdash [\Delta'], \Gamma', A \quad \vdash [\Delta''], \Gamma'', B}{\vdash [\Delta', \Delta''], \Gamma', \Gamma'', A \otimes B} \otimes$$

Then $\theta(\Pi) = \tau^{-1}(\theta(\Pi') \otimes \theta(\Pi''))\tau$,
$\Gamma' \otimes \Gamma'' \otimes A \otimes B \otimes \Delta' \otimes \Delta'' \xrightarrow{\tau} \Gamma' \otimes A \otimes \Delta' \otimes \Gamma'' \otimes B \otimes \Delta''$.

# Examples

$\Pi$:

$$\frac{\vdash A, A^{\perp} \quad \vdash A, A^{\perp}}{\vdash [A^{\perp}, A], A, A^{\perp}}\ cut$$

Then,

$$\theta(\Pi) = \tau^{-1}(s \otimes s)\tau = s_{V \otimes V, V \otimes V}$$

where $\tau = (1 \otimes 1 \otimes s)(1 \otimes s \otimes 1)$ and
$[\![ A ]\!] = [\![ A^{\perp} ]\!] = V$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash B, B^\perp \quad \vdash C, C^\perp}{\vdash B, C, B^\perp \otimes C^\perp}}{\vdash B, B^\perp \otimes C^\perp, C}}{\vdash B^\perp \otimes C^\perp, B, C}}{\vdash B^\perp \otimes C^\perp, B \parr C}\ .$$

with $[\![ B ]\!] = [\![ B^\perp ]\!] = V$ and $[\![ C ]\!] = [\![ C^\perp ]\!] = W$.
$\theta(\Pi) = s_{V\otimes W, V\otimes W}$

# Proofs are permutations

## Proposition

*Let $\Pi$ be an MLL proof of $\vdash [\Delta], \Gamma$ where $|\Delta| = 2m$ and $|\Gamma| = n$ (counting occurrences of propositional variables). Then $\theta(\Pi)$ is a fixed-point free involutive permutation on $n + 2m$ objects of $\mathbb{C}$. That is $\theta(\Pi) : V_1 \otimes \cdots \otimes V_{n+2m} \longrightarrow V_1 \otimes \cdots \otimes V_{n+2m}$ induces a permutation $\pi$ on $\{1, 2 \cdots, n + 2m\}$ and*

- $\pi^2 = 1$
- *For all $i \in \{1, 2, \cdots, n + 2m\}$, $\pi(i) \neq i$.*
- *For all $i \in \{1, 2, \cdots, n + 2m\}$, $V_i = V_{\pi(i)}$.*

**Theorem (Completeness)**

*Let $M$ be a fixed-point free involutive permutation from $V_1 \otimes \cdots \otimes V_n \longrightarrow V_1 \otimes \cdots \otimes V_n$ (induced by a permutation $\mu$ on $\{1, 2, \cdots, n\}$) where $n > 0$ is an even integer, $V_i = [\![ A_i ]\!]$, and $V_i = V_{\mu(i)}$ for all $i = 1, \cdots, n$. Then there is a provable MLL formula $\varphi$ built from the $A_i$, with a proof $\Pi$ such that $\theta(\Pi) = M$.*

# An example

$\mu = (1,4)(2,3)$ on $\{1,2,3,4\}$.
$\varphi(A_1, A_2, A_3, A_4) = \varphi(A_1, A_2, A_2^\perp, A_1^\perp) = ((A_1 \otimes A_2) \parr A_2^\perp) \parr A_1^\perp$
One possible $\Pi$ is (ignoring exchange):

$$\dfrac{\dfrac{\dfrac{\vdash A_1, A_1^\perp \qquad \vdash A_2, A_2^\perp}{\vdash A_1 \otimes A_2, A_1^\perp, A_2^\perp}}{\vdash (A_1 \otimes A_2) \parr A_2^\perp, A_1^\perp}}{\vdash ((A_1 \otimes A_2) \parr A_2^\perp) \parr A_1^\perp}$$

# MGoI for MLL: cut-elimination

$\Pi$ a proof of $\vdash [\Delta], \Gamma$, and $\sigma = s \otimes \cdots \otimes s$ ($m$ times) models $\Delta$, with $|\Delta| = 2m$.

$$EX(\theta(\Pi), \sigma) = Tr^{\otimes\Delta}_{\otimes\Gamma, \otimes\Gamma}((1 \otimes \sigma)\theta(\Pi)).$$

$EX(\theta(\Pi), \sigma) : \otimes\Gamma \longrightarrow \otimes\Gamma$, when it exists.
We prove the execution formula always exists for any MLL proof $\Pi$.

# Example

$\Pi$:

$$\frac{\vdash A, A^{\perp} \quad \vdash A, A^{\perp}}{\vdash [A^{\perp}, A], A, A^{\perp}} \; cut$$

Then,

$$\theta(\Pi) = \tau^{-1}(s \otimes s)\tau = s_{V \otimes V, V \otimes V}$$

where $\tau = (1 \otimes 1 \otimes s)(1 \otimes s \otimes 1)$ and
$[\![A]\!] = [\![A^{\perp}]\!] = V$.
$\sigma = s$, $(m = 1)$.
$EX(\theta(\Pi), \sigma) = Tr((1 \otimes s_{V,V})s_{V \otimes V, V \otimes V}) = s_{V,V}$.
MGoI int. of the cut-free proof of $\vdash A, A^{\perp}$.

# Associativity of cut

**Lemma**

*Let $\Pi$ be a proof of $\vdash [\Gamma, \Delta], \Lambda$ and $\sigma$ and $\tau$ be the morphisms representing the cut-formulas in $\Gamma$ and $\Delta$ respectively. Then*

$$EX(\theta(\Pi), \sigma \otimes \tau) = EX(EX(\theta(\Pi), \tau), \sigma)$$

$$= EX(EX((1 \otimes s)\theta(\Pi)(1 \otimes s), \sigma), \tau)$$

*whenever all traces exist.*

# The big picture

$$
\begin{array}{ccc}
\text{proof} & \rightsquigarrow & \text{algorithm} \\
\text{cut-elim.} \downarrow & & \downarrow \text{computation} \\
\text{cut-free proof} & \rightsquigarrow & \text{datum}
\end{array}
$$

$$
\begin{array}{ccc}
\Pi & \rightsquigarrow & \theta(\Pi) \\
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\Pi' & \rightsquigarrow & \theta(\Pi') = EX(\theta(\Pi), \sigma)
\end{array}
$$

# Datum & Algorithm, *simplified*

Let $\Gamma = A_1, A_2$ and $V_i = [\![ A_i ]\!]$.
A *datum of type* $\theta\Gamma$:
$M : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2$ s.t. for any $\alpha_i \in \theta(A_i^\perp)$,

$$\alpha_1 \otimes \alpha_2 \perp M$$

and

$$M \centerdot \alpha_1 := Tr^{V_1}(s^{-1}_{V_2,V_1}(\alpha_1 \otimes 1_{V_2}) M s_{V_2,V_1})$$

and

$$M \hat{\centerdot} \alpha_2 := Tr^{V_2}((1 \otimes \alpha_2) M)$$

both exist.

## Lemma

*$M$ is a datum of type $\theta(A_1, A_2)$ iff for all $\alpha_i \in \theta(A_i^\perp)$, $M \cdot \alpha_1$ and $M \hat{\cdot} \alpha_2$ both exist and are in $\theta(A_2)$ and $\theta(A_1)$ respectively.*

An *algorithm of type* $\theta\Gamma$:
$M : V_1 \otimes V_2 \otimes [\![ \Delta ]\!] \longrightarrow V_1 \otimes V_2 \otimes [\![ \Delta ]\!]$
$\Delta = B_1, B_2, \cdots, B_{2m}$ , $B_{i+1} = B_i^{\perp}$
$i = 1, 3, \cdots, 2m-1$
if $\sigma := \otimes_{i=1,odd}^{2m-1} \; s_{[\![ B_i ]\!],[\![ B_{i+1} ]\!]}$,
$EX(M, \sigma)$ exists and is a datum of type $\theta\Gamma$.

# Main Theorems

Theorem (Convergence)

*Let* $\Pi$ *be an MLL proof of a sequent* $\vdash [\Delta], \Gamma$. *Then* $\theta(\Pi)$ *is an algorithm of type* $\theta\Gamma$.

Corollary (Existence of Dynamics)

*Let* $\Pi$ *be an MLL proof of a sequent* $\vdash [\Delta], \Gamma$. *Then* $EX(\theta(\Pi), \sigma)$ *exists.*

## Theorem (Invariance)

*Let $\Pi$ be an MLL proof of a sequent $\vdash [\Delta], \Gamma$. Then,*

- *If $\Pi$ reduces to $\Pi'$ by any sequence of cut-eliminations, then $EX(\theta(\Pi), \sigma) = EX(\theta(\Pi'), \tau)$. So $EX(\theta(\Pi), \sigma)$ is an invariant of reduction.*
- *In particular, if $\Pi'$ is any cut-free proof obtained from $\Pi$ by cut-elimination, then $EX(\theta(\Pi), \sigma) = \theta(\Pi')$.*

# Future Work

- System theoretic insights.
- Algorithmic and convergence properties of various trace formulas. (traced UDC based models and complexity analysis.)

# Beyond multiplicatives

| *Old (GoI)* | *New (MGoI)* |
|---|---|
| single reflexive object | multiple objects |
| needs monoidal retractions | needs monoidal retractions |
| UDC, "sum like" monoidal product | arbitrary monoidal product |
| traced category | *partially* traced category |
| nilpotency | *compatible* abstract orthogonality |
| GoI for MELL | MGoI for MELL |

# Monoidal $*$-Categories

(Cf. Abramsky, Blute, Panangaden 99, Longo, Roberts, Doplicher, mid 80's)

- $\mathbb{C}$ monoidal category, $(\ )^* : \mathbb{C}^{op} \longrightarrow \mathbb{C}$ strict symmetric monoidal functor, strictly involutive, the identity on objects.

Note that the conditions above imply $(f \otimes g)^* = f^* \otimes g^*$, and $s^*_{A,B} = s_{B,A}$.

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- $f : A \longrightarrow B$ is *partial isometry* if $f^* f f^* = f^*$ or equivalently if $f f^* f = f$.
- $f : A \longrightarrow A$ is *partial symmetry* if it is a Hermitian partial isometry. That is, if $f^* = f$ and $f^3 = f$.

Note that the conditions above imply $(f \otimes g)^* = f^* \otimes g^*$, and $s^*_{A,B} = s_{B,A}$.

# Examples

- $(\mathbf{Hilb}, \otimes)$: $f : H \longrightarrow K$, $f^* : K \longrightarrow H$ is given by the adjoint of $f$, defined uniquely by $\langle f(x), y\rangle = \langle x, f^*(y)\rangle$.
- $(\mathbf{Hilb}, \oplus)$: with the same definition for the $(\ )^*$ functor.
- $(\mathbf{Rel}, \times)$: $f : X \longrightarrow Y$, $f^* = \overline{f}$ where $\overline{f}$ is the converse relation.
- $(\mathbf{Rel}, \oplus)$: with the same definition for the $(\ )^*$ functor.
- $(\mathbf{PInj}, \uplus)$: $f : X \longrightarrow Y$, $f^* = f^{-1}$.

# GoI category

- $(\mathbb{C}, T, \perp)$

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- $(\mathbb{C}, T, \perp)$
- $\mathbb{C}$ partially traced *-category
- $T = (T, \psi, \psi_I) : \mathbb{C} \longrightarrow \mathbb{C}$ traced symmetric monoidal functor: if $f \in \mathbb{T}^U_{X,Y}$, then $\psi^{-1}_{Y,U} T(f) \psi_{X,U} \in \mathbb{T}^{TU}_{TX,TY}$, and

$$Tr^{TU}_{TX,TY}(\psi^{-1}_{Y,U} T(f) \psi_{X,U}) = T(Tr^U_{X,Y}(f))$$

# GoI category

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- $\perp$ is an orthogonality relation on $\mathbb{C}$.

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$$Tr^{TU}_{TX,TY}(\psi^{-1}_{Y,U} T(f) \psi_{X,U}) = T(Tr^U_{X,Y}(f))$$

- $\perp$ is an orthogonality relation on $\mathbb{C}$.
- The following natural retractions exist:
  - $\mathcal{K}_I \lhd T \ (w, w^*)$
  - $Id \lhd T \ (d, d^*)$
  - $T^2 \lhd T \ (e, e^*)$
  - $T \otimes T \lhd T \ (c, c^*)$

The orthogonality relation is *GoI compatible*:

(c1) For all $f : V \longrightarrow U$, $g : U \longrightarrow V$,

$$f \perp_{U,V} g \text{ implies } d_U f d_V^* \perp_{TU,TV} Tg.$$

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(c1) For all $f : V \longrightarrow U$, $g : U \longrightarrow V$,

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(c2) For all $f : U \longrightarrow U$ and $g : I \longrightarrow I$,

$$w_U g w_U^* \perp_{TU,TU} Tf.$$

The orthogonality relation is *GoI compatible*:

(c1) For all $f : V \longrightarrow U$, $g : U \longrightarrow V$,

$$f \perp_{U,V} g \text{ implies } d_U f d_V^* \perp_{TU,TV} Tg.$$

(c2) For all $f : U \longrightarrow U$ and $g : I \longrightarrow I$,

$$w_U g w_U^* \perp_{TU,TU} Tf.$$

(c3) For all $f : TV \otimes TV \longrightarrow TU \otimes TU$ and $g : U \longrightarrow V$,

$$f \perp_{TU \otimes TU, TV \otimes TV} Tg \otimes Tg \text{ implies } c_U f c_V^* \perp_{TU,TV} Tg.$$

- The functor $T$ commute with $(\,)^*$, that is $(T(f))^* = T(f^*)$.
- $\psi^* = \psi^{-1}$ and $\psi_I^* = \psi_I^{-1}$.
- For example, let $(\mathbb{C}, \otimes, I, Tr)$, $A$ and $B$ be objects of $\mathbb{C}$. For $f : A \longrightarrow B$ and $g : B \longrightarrow A$, $f \perp_{BA} g$ iff $gf \in \mathbb{T}^A_{I,I}$. Then, $\perp$ is GoI compatible.

# Examples:

- $(PInj, \uplus, \mathbb{N} \times -, \perp)$
  $f \perp g$ iff $gf$ is nilpotent.
  Retractions as before.

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# Examples:

- $(PInj, \uplus, \mathbb{N} \times -, \perp)$
  $f \perp g$ iff $gf$ is nilpotent.
  Retractions as before.
- $(Rel, \oplus, \mathbb{N} \times -, \perp)$.
- $(Hilb, \oplus, \ell^2 \otimes -, \perp)$, where Hilb is the category of Hilbert spaces and bounded linear maps.
  Where $\ell^2$ is the space of square summable sequences.
  $f \perp g$ iff $(1 - gf)$ is invertible.

# MGoI for MELL: formulas

$A$, object of $\mathbb{C}$, $X \subseteq End(A)$

$$X^{\perp} = \{f \in End(A) \,|\, \forall g \in X, f \perp g\}.$$

$$\mathcal{T}(A) = \{X \subseteq End(A) \,|\, X^{\perp\perp} = X\}.$$

- $[\![ !A ]\!] = [\![ ?A ]\!] = T[\![ A ]\!]$.

$\theta(A)$:

- $\theta(!A) = \{ Ta \mid a \in \theta(A)\}^{\perp\perp}$
- $\theta(?A) = \{ Ta \mid a \in \theta(A^{\perp})\}^{\perp}$

*FACTS*

(i) for any formula $A$,
$\theta A^{\perp} = (\theta A)^{\perp}$,
(ii) $\theta(A) \subseteq End(\llbracket A \rrbracket)$,
(iii) $\theta(A)^{\perp\perp} = \theta(A)$.

# MGoI for MELL: proofs

- $\Pi$, a proof of $\vdash [\Delta], \Gamma$
- $\theta(\Pi) \in End(\otimes[\![\Gamma]\!] \otimes [\![\overline{\Delta}]\!])$,
- with $\Delta = B_1, B_1^{\perp}, \cdots B_m, B_m^{\perp}$, $[\![\overline{\Delta}]\!] = T^k([\![B_1]\!] \otimes \cdots \otimes [\![B_m^{\perp}]\!])$, for some non-negative integer $k$.
- $T^0$ is the identity functor.
- MLL case can be recovered easily by letting $k = 0$.

- $\Pi$ is the axiom $\vdash \mathbf{1}$, then $\theta(\Pi) = 1_I$.
- $\Pi$ is obtained using the $\perp$ *rule* applied to the proof $\Pi'$ of $\vdash [\Delta], \Gamma'$. Then $\theta(\Pi) = \theta(\Pi') \otimes 1_I = \theta(\Pi')$.

- $\Pi$ is an *axiom* $\vdash A, A^\perp$, $\theta(\Pi) := s_{V,V}$ where $[\![ A ]\!] = [\![ A^\perp ]\!] = V$.

- $\Pi$ is an *axiom* $\vdash A, A^\perp$, $\theta(\Pi) := s_{V,V}$ where $[\![ A ]\!] = [\![ A^\perp ]\!] = V$.
- $\Pi$ is obtained using the *cut* rule on $\Pi'$ and $\Pi''$ that is,

$$\dfrac{\begin{array}{c}\Pi'\\ \vdots\\ \vdash [\Delta'], \Gamma', A\end{array} \quad \begin{array}{c}\Pi''\\ \vdots\\ \vdash [\Delta''], A^\perp, \Gamma''\end{array}}{\vdash [\Delta', \Delta'', A, A^\perp], \Gamma', \Gamma''}\ cut$$

Define $\theta(\Pi) = \tau^{-1}(\theta(\Pi') \otimes \theta(\Pi''))\tau$, where $\tau$ is the permutation
$\Gamma' \otimes \Gamma'' \otimes \overline{\Delta'} \otimes \overline{\Delta''} \otimes A \otimes A^\perp \xrightarrow{\tau} \Gamma' \otimes A \otimes \overline{\Delta'} \otimes A^\perp \otimes \Gamma'' \otimes \overline{\Delta''}$.

$\Pi$ is obtained using the *exchange* rule on the formulas $A_i$ and $A_{i+1}$ in $\Gamma'$. That is $\Pi$ is of the form

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma'\end{array}}{\vdash [\Delta], \Gamma} \; exchange$$

where $\Gamma' = \Gamma_1', A_i, A_{i+1}, \Gamma_2'$ and $\Gamma = \Gamma_1', A_{i+1}, A_i, \Gamma_2'$. Then, $\theta(\Pi)$ is obtained from $\theta(\Pi')$ by interchanging the rows $i$ and $i+1$. So, $\theta(\Pi) = \tau^{-1}\theta(\Pi')\tau$, where $\tau = 1_{\Gamma_1'} \otimes s \otimes 1_{\Gamma_2' \otimes \overline{\Delta}}$.

$\Pi$ is obtained using an application of the *par* rule, that is $\Pi$ is of the form:

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma', A, B\end{array}}{\vdash [\Delta], \Gamma', A \parr B}\,\parr \quad . \quad \text{Then } \theta(\Pi) = \theta(\Pi').$$

$\Pi$ is obtained using an application of the *times* rule, that is $\Pi$ is of the form:

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta'], \Gamma', A\end{array} \quad \begin{array}{c}\Pi'' \\ \vdots \\ \vdash [\Delta''], \Gamma'', B\end{array}}{\vdash [\Delta', \Delta''], \Gamma', \Gamma'', A \otimes B} \otimes$$

Then $\theta(\Pi) = \tau^{-1}(\theta(\Pi') \otimes \theta(\Pi''))\tau$, where $\tau$ is the permutation $\Gamma' \otimes \Gamma'' \otimes A \otimes B \otimes \overline{\Delta'} \otimes \overline{\Delta''} \xrightarrow{\tau} \Gamma' \otimes A \otimes \overline{\Delta'} \otimes \Gamma'' \otimes B \otimes \overline{\Delta''}$.

$\Pi$ is obtained from $\Pi'$ by an *of course* rule, that is $\Pi$ has the form :

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], ?\Gamma', A\end{array}}{\vdash [\Delta], ?\Gamma', !A} \text{ of course}$$

- $\theta(\Pi) = (e_{\Gamma'} \otimes 1_{TA} \otimes 1_{\overline{\Delta}})\varphi^{-1} T([\![ \Pi' ]\!])\varphi(e^*_{\Gamma'} \otimes 1_{TA} \otimes 1_{\overline{\Delta}})$,
- where $TT \lhd T\,(e, e^*)$,
- with $\Gamma' = A_1, \cdots, A_n$, $e_{\Gamma'} = e_{A_1} \otimes \cdots \otimes e_{A_n}$, similarly for $e^*$, and
- $\varphi$ is the canonical isomorphism $\varphi : T^2(\Gamma') \otimes TA \otimes T(\overline{\Delta}) \longrightarrow T(T(\Gamma') \otimes A \otimes \overline{\Delta})$ is defined using
- the isomorphism $\psi_{X,Y} : TX \times TY \longrightarrow T(X \otimes Y)$. With $\Gamma' = A_1, \cdots, A_n$, $T(\Gamma')$ is a shorthand for $TA_1 \otimes \cdots \otimes TA_n$, and $\overline{\Delta}$ is as before.

$\Pi$ is obtained from $\Pi'$ by the *dereliction* rule, that is, $\Pi$ is of the form:

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma', A\end{array}}{\vdash [\Delta], \Gamma', ?A}\ \text{dereliction}$$

Then $\theta(\Pi) = (1_{\Gamma'} \otimes d_A \otimes 1_{\overline{\Delta}})\theta(\Pi')(1_{\Gamma'} \otimes d_A^* \otimes 1_{\overline{\Delta}})$ where $Id \lhd T\,(d, d^*)$.

$\Pi$ is obtained from $\Pi'$ by the *weakening* rule, that is, $\Pi$ is of the form:

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma'\end{array}}{\vdash [\Delta], \Gamma', ?A} \text{ weakening}$$

Then $\theta(\Pi) = (1_{\Gamma'} \otimes w_A \otimes 1_{\overline{\Delta}})\theta(\Pi')(1_{\Gamma'} \otimes w_A^* \otimes 1_{\overline{\Delta}})$, where $\mathcal{K}_I \lhd T\,(w, w^*)$.

$\Pi$ is obtained from $\Pi'$ by the *contraction* rule, that is, $\Pi$ is of the form :

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma', ?A, ?A\end{array}}{\vdash [\Delta], \Gamma', ?A}\ \text{contraction}$$

Then $\theta(\Pi) = (1_{\Gamma'} \otimes c_A \otimes 1_{\overline{\Delta}})\theta(\Pi')(1_{\Gamma'} \otimes c_A^* \otimes 1_{\overline{\Delta}})$, where $T \otimes T \lhd T\,(c, c^*)$.

# Example

Consider the following proof

$$\dfrac{\dfrac{\dfrac{\vdash A, A^{\perp}}{\vdash A, ?A^{\perp}}}{\vdash !A, ?A^{\perp}} \quad \vdash B, B^{\perp}}{\vdash !A \otimes B, ?A^{\perp} \parr B^{\perp}}$$

Given $[\![ A ]\!] = V$ and $[\![ B ]\!] = W$, we have $\theta(\Pi) =$
$(1 \otimes s \otimes 1)(1 \otimes e \otimes 1 \otimes 1)(\psi^{-1} T(h)\psi \otimes s)(1 \otimes e^* \otimes 1 \otimes 1)(1 \otimes s \otimes 1)$
where $h = (1 \otimes d_V)s(1 \otimes d_V^*)$.

# Proofs as partial symmetries

### Proposition

*Let* $\Pi$ *be an* **MELL** *proof of* $\vdash [\Delta], \Gamma$. *Then* $\theta(\Pi)$ *is a partial symmetry.*

### Proof.

By induction on the length of the proofs, noting that the functor $(\ )^*$ is a strict symmetric monoidal functor, $T(f)^* = T(f^*)$, $\psi^* = \psi^{-1}$, and $\psi_I^* = \psi_I^{-1}$. □

# A calculation

For example:
$\theta(\Pi) =$
$(1 \otimes s \otimes 1)(1 \otimes e_V \otimes 1 \otimes 1)(\psi^{-1} T(h)\psi \otimes s)(1 \otimes e_V^* \otimes 1 \otimes 1)(1 \otimes s \otimes 1)$
where $h = (1 \otimes d_V)s(1 \otimes d_V^*)$.

Then $\theta(\Pi)^* = \theta(\Pi)$ as $h^* = h$
$s_{V,W}^* = s_{V,W}^{-1} = s_{W,V}$
$T(h)^* = T(h^*)$ and
$\psi^* = \psi^{-1}$.

# MGoI for MELL: cut-elimination

$\Pi$ a proof of $\vdash [\Delta], \Gamma$, and $\sigma = T^k(s \otimes \cdots \otimes s)$ ($m$ times) models $\Delta$, with $|\Delta| = 2m$, and $k$ a non-negative integer.

$$EX(\theta(\Pi), \sigma) = Tr^{\otimes\overline{\Delta}}_{\otimes\Gamma,\otimes\Gamma}((1 \otimes \sigma)\theta(\Pi))$$

$EX(\theta(\Pi), \sigma) : \otimes\Gamma \longrightarrow \otimes\Gamma$, when it exists.
We prove the execution formula always exists for any MELL proof $\Pi$.

# The big picture

$$\begin{array}{ccc} & \text{proof} & \rightsquigarrow & \text{algorithm} \\ \text{cut-elim.} \downarrow & & & \downarrow \text{computation} \\ & \text{cut-free proof} & \rightsquigarrow & \text{datum} \end{array}$$

$$\begin{array}{cccc} & \Pi & \rightsquigarrow & \theta(\Pi) \\ \text{cut-elim.} \downarrow & & & \downarrow \text{computation} \\ & \Pi' & \rightsquigarrow & \theta(\Pi') = EX(\theta(\Pi), \sigma) \end{array}$$

# Datum & Algorithm, *simplified*

Let $\Gamma = A_1, A_2$ and $V_i = [\![ A_i ]\!]$.
A *datum of type* $\theta\Gamma$:
$M : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2$ s.t. for any $\alpha_i \in \theta(A_i^{\perp})$,

$$\alpha_1 \otimes \alpha_2 \perp M$$

and

$$M \cdot \alpha_1 := Tr^{V_1}(s_{V_2,V_1}^{-1}(\alpha_1 \otimes 1_{V_2}) M s_{V_2,V_1})$$

and

$$M \hat{\cdot} \alpha_2 := Tr^{V_2}((1 \otimes \alpha_2) M)$$

both exist.

### Lemma

*$M$ is a datum of type $\theta(A_1, A_2)$ iff for all $\alpha_i \in \theta(A_i^{\perp})$, $M \cdot \alpha_1$ and $M \hat{\cdot} \alpha_2$ both exist and are in $\theta(A_2)$ and $\theta(A_1)$ respectively.*

An *algorithm of type* $\theta\Gamma$:
$M : V_1 \otimes V_2 \otimes \llbracket \overline{\Delta} \rrbracket \longrightarrow V_1 \otimes V_2 \otimes \llbracket \overline{\Delta} \rrbracket$

$\Delta = B_1, B_1^{\perp}, \cdots, B_m, B_m^{\perp}$,
if $\sigma : T^k(\otimes_{i=1}^{2m} \llbracket B_i \rrbracket) \longrightarrow T^k(\otimes_{i=1}^{2m} \llbracket B_i \rrbracket)$ defined as $T^k(\otimes_{i=1,odd}^{2m-1} s_{\llbracket B_i \rrbracket, \llbracket B_{i+1} \rrbracket})$, for some non-negative integer $k$, $EX(M, \sigma)$ exists and is a datum of type $\theta\Gamma$.

# Main Theorems

**Theorem (Convergence)**

*Let* $\Pi$ *be an MELL proof of a sequent* $\vdash [\Delta], \Gamma$*. Then* $\theta(\Pi)$ *is an algorithm of type ass* $\theta\Gamma$*.*

**Corollary (Existence of Dynamics)**

*Let* $\Pi$ *be an MELL proof of a sequent* $\vdash [\Delta], \Gamma$*. Then* $EX(\theta(\Pi), \sigma)$ *exists.*

## Theorem (Invariance)

*Let $\Pi$ be an MELL proof of a sequent $\vdash [\Delta], \Gamma$ such that $?A$ does not occur in $\Gamma$ for any formula $A$. Then,*

- *If $\Pi$ reduces to $\Pi'$ by any sequence of cut-elimination steps, then $EX(\theta(\Pi), \sigma) = EX(\theta(\Pi'), \tau)$. So $EX(\theta(\Pi), \sigma)$ is an invariant of reduction.*
- *In particular, if $\Pi'$ is any cut-free proof obtained from $\Pi$ by cut-elimination, then $EX(\theta(\Pi), \sigma) = EX(\theta(\Pi'), 1_I) = \theta(\Pi')$.*

# In conclusion

- $(\mathbf{Hilb}, \oplus)$ is partially traced.
- GoI Categories $(\mathbb{C}, \otimes, (\,)^*, Tr, T, \perp)$.
- Compatibility conditions for $\perp$.
- Proofs are partial symmetries.
- No completeness or charaterization theorem yet :-(((
- Infinity sneaks in!
- Soundness theorem.
- Relating to Doplicher, Longo, Roberts work??

# Things we did not talk about

- Full completeness theorem for MLL (Thesis, TLCA 01)
- Proofs as Polynomials (ENTCS 2008)
- From GoI semantics to denotational semantics (CTCS 04, work in progress)
- Relation to path-based semantics, $\Lambda^*$-algebra (Thesis, MSCS 2000)