# GAMES FOR DISCRETE-TIME MARKOV CHAIN AND THEIR APPLICATION TO VERIFICATION
# 離散時間マルコフ連鎖に対するゲームと、その形式検証への応用

by

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## ABSTRACT

This paper formulates some notions and results about discrete-time Markov chains in terms of game-theoretic probability. Discrete-time Markov chain is a one of the models which are used to model systems' probabilistic behavior in formal verification. The theory of probability is needed for modeling of systems which have probabilistic behavior. While, after Kolmogorov, probabilities are most of the time formulated in the language of measure theory, a new formulation which uses the language of game theory has arisen; namely game-theoretic probability was proposed by Shafer and Vovk (2001). We demonstrate that some notions and results like fairness theorem are natural and simple in terms of game-theoretic probability.



## 論文要旨

本論文では、ゲーム論的確率によって離散時間マルコフ連鎖についてのいくつかの概念や証拠を定式化する。離散時間マルコフ連鎖は、検証のためにシステムの確率的挙動をモデル化するために使用されるモデルの一つである。確率論は確率的挙動を持つシステムのモデル化のために必要とされる。Kolmogorov 以後、ほとんどの場合において確率が測度論の言葉で定式化される中、ゲーム論の言葉を用いた新しい定式化が起こった。ゲーム論的確率は Shafer と Vovk によって提案された。Fairness theorem などのいくつかの証拠や概念は、ゲーム論的確率論を用いた定式化において、自然であり、簡潔であると考えられる。

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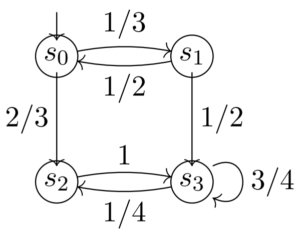
# Contents

# Chapter 1

# Introduction

## 1.1 Discrete-Time Markov Chain

Many systems nowadays utilize probabilistic mechanisms for resolving nondeterminism. Examples include the leader election protocol [10] and the crowds protocol [15]. The notion of *discrete-time Markov chain* (DTMC) is a mathematical model of such probabilistic systems. DTMCs are state transition systems—much like automata or labeled transition systems—whose transitions are chosen in a probabilistic manner.




Because of the importance of probabilistic systems, DTMCs as their models have been heavily studied. Specific research topics include: temporal logics for specification [8]; model checking [4]; and notions of (bi)simulation [4, 16].

*Temporal logics* are a family of modal logics that are used to formally express properties that a system is desired to satisfy (such properties are called *specifications*). The logic PCTL [8] for specification of DTMCs is obtained as a variation of CTL [5], a logic for labeled transition systems (in which choices are made nondeterministically as opposed to probabilistically). Specifically, PCTL is a branching-time temporal logic that replaces the path quantifiers $\exists$ and $\forall$ in CTL with the probabilistic operators $\mathbb{P}_{\geq p}$ and $\mathbb{P}_{>p}$ where $p \in [0, 1]$. Path formulas in PCTL are LTL formulas, much like in CTL.

*Model checking* is the problem whose input are a system (commonly a state transition system) and a specification (a temporal logic formula), and whose output is whether the system satisfies the specification or not. For labeled transition systems and CTL formulas there are many algorithms for model checking; see e.g. [4, 6]. There also exist some algorithms for model checking DTMCs [1, 4, 8, 14] and PCTL formulas. For example, the probabilities of reachability from a certain state to another can be a solution of a linear equation system [4]. Such algorithms have been implemented in tools like PRISM [9] or MRMC [11].

A *bisimulation* is a relation between states of two transition systems that expresses certain "equivalence": if two states are related by a bisimulation they behave in the same way. A *simulation* is a "one-sided" variation of a bimulation: if two states are related, one can mimic the behavior of the other (but not necessarily the other way round). These notions are useful in model checking because they witness logical equivalence or implication. Such notions have been formulated for probabilistic systems (including DTMCs), e.g. in [4, 16]. There are some algorithms for discovering/testing bisimulations or simulations in probabilistic systems [3, 21].

### 1.2 Game-Theoretic Probability

Measure-theoretic probability is a widely-used formulation of probabilities. The *game-theoretic probability proposed* Shafer and Vovk [18] is an alternative formulation of probabilities.

In the game-theoretic probability [18], probabilistic phenomena are described in terms of games between two players, called Skeptic and Reality. While it is necessary in the measure-theoretic probability that probability distributions are fixed explicitly, it is not necessary in the game-theoretic probability. Instead of fixing probability distributions beforehand, the notion of probability is expressed as Reality's behavior that tries to prevent Skeptic from winning. Informally, the more money Skeptic can make in an event, the less often this event happens.

In [18], two advantages of the game-theoretic probability are presented. First, the game-theoretic probability sometimes describes the results of probabilities more simply than the measure-theoretic probability, for example, the zero-one law [17], the strong law of large numbers [13] and the law of the iterated logarithm [12]. The main application area is finance in [18]. Second, it has potential ability to model some open systems influenced from outside of the systems, for example, quantum mechanics and Cox's regression model [7].

### 1.3 Contributions

Our aim is to apply the game-theoretic probability to model checking for probabilistic systems. In this thesis, we translate DTMCs to games. This translation is natural because one transition of states in DTMCs can be regarded as one round of games. We show two applications of those games.

Firstly, we prove the fairness theorem in terms of the game-theoretic probability. The fairness theorem is known as one of the critical theorems to model checking. This theorem states that if a certain state in a DTMC is visited infinitely often, the states that can be reached from this state after some steps are almost surely visited infinitely often. By the fairness theorem, we can prove that *qualitative* properties — properties that require that the probabilities of certain events are one — for events such as reachability and repeated reachability for finite DTMCs can be verified by using graph analysis [4]. We prove this theorem by following a simple strategy in the game.

Secondly, we describe a probabilistic simulation between two DTMCs based on Segala's [16]. It is the known theorem that this probabilistic simulation preserve where restricted PCTL formulas hold. We prove this in terms of the game-theoretic probability. In this game-theoretic proof, we construct a strategy in the game for one DTMC from a strategy in the game for the other DTMC by using weight functions. This expresses intuitive meaning of weight functions in those games.

# Chapter 2

# Discrete-Time Markov Chain

## 2.1 Discrete-Time Markov Chain

This section describes *discrete-time Markov chains (DTMCs)*, a model of probabilistic systems. We also define probabilities of events in DTMCs.

**Definition 2.1.1** (DTMC)**.** A *discrete-time Markov chain (DTMC)* is a tuple $(S, P, \iota_{\text{init}}, AP, L)$ where

- $S$ is a countable, nonempty set of states,
- $P\colon S \times S \to [0,1]$ is the *transition probability* such that for each state $s$: $\sum_{s' \in S} P(s, s') = 1$,
- $\iota_{\text{init}}\colon S \to [0,1]$ is the *initial distribution*, such that $\sum_{s \in S} \iota_{\text{init}}(s) = 1$,
- $AP$ is a set of *atomic propositions*, and
- $L\colon S \to 2^{AP}$ is a *labeling function*.

DTMCs are transition systems such that their choices of next states are decided at random following the transition probability $P$. A labeling function $L$ associates a state with the set of atomic propositions that hold at the state. We sometimes use states as atomic propositions. That is, we assume that $AP = S$ and $L(s) = \{s\}$ for each $s \in S$. In a DTMC, a state $s'$ can be a successor of $s$ when $P(s, s') > 0$. For each state $s$ in a DTMC, there exists at least on successor of this state $s$ because if the state $s$ have no successors, $\sum_{s' \in S} P(s, s') = 0 \neq 1$. Hence DTMCs' behaviors are infinite sequences of states. We call such sequences *paths*.

**Definition 2.1.2** (path)**.** A *path* in a DTMC is an infinite state sequence $s_0 s_1 s_2 \ldots \in S^\omega$ such that $P(s_i, s_{i+1}) > 0$ for all $i \in \mathbb{N}$. A *path fragment* in a DTMC is a finite state sequence $s_0 s_1 \ldots s_n$ such that $n \geq 0$ and $P(s_i, s_{i+1}) > 0$ for all $i \in \{0, 1, \ldots, n-1\}$.

The set of all paths that start from a state $s$ is denoted by

$$\mathrm{Path}(s) = \{s_0 s_1 \ldots \text{ is a path} \mid s_0 = s\}$$

and the set of all paths that start from initial states of a DTMC $D = (S, P, \iota_{\text{init}}, AP, L)$ is denoted by

$$\mathrm{Path}(D) = \left\{\xi \in S^\omega \,\middle|\, \exists s \in S.\, (\xi \in \mathrm{Path}(s) \wedge \iota_{\text{init}}(s) > 0)\right\} .$$

**Definition 2.1.3.** For each path fragment $\widehat{\xi} = s_0 s_1 \dots s_n$ in a DTMC $D$ let

$$P(\widehat{\xi}) = \prod_{i=0}^{n-1} P(s_i, s_{i+1}) \ .$$

**Definition 2.1.4** (prefix)**.** Let $\xi = s_0 s_1 s_2 \dots$ be a path. Then the path fragment $s_0 s_1 \dots s_n$ is denoted by $\xi[0, n]$. A path fragment $\widehat{\xi}$ is called a *prefix* of $\xi$ if there exists $n \in \mathbb{N}$ such that $\widehat{\xi} = \xi[0, n]$.

**Definition 2.1.5** (suffix)**.** Let $\xi = s_0 s_1 s_2 \dots$ be a path. Then the path $s_n s_{n+1} \dots$ is denoted by $\xi[n \dots]$. A path $\xi'$ is called a *suffix* of $\xi$ if there exists $n \in \mathbb{N}$ such that $\xi' = \xi[n \dots]$.

**Definition 2.1.6** (element)**.** Let $\xi = s_0 s_1 s_2 \dots$ be a path. The state $s_n$ is called an $n$-th *element* and is denoted by $\xi[n]$.

The set of successors and predecessors for a state are defined by follows.

**Definition 2.1.7.** Let $s$ be a state in a DTMC with a state space $S$ and transition probability $P$. Then we define

- $\mathrm{Post}(s) := \{s' \in S \mid P(s, s') > 0\}$,
- $\mathrm{Pre}(s) := \{s' \in S \mid P(s', s) > 0\}$,
- $\mathrm{Post}^*(s) := \{s' \in S \mid \text{there is a path fragment } s = s_0 s_1 \dots s_n = s'\}$ and
- $\mathrm{Pre}^*(s) := \{s' \in S \mid s \in \mathrm{Post}^*(s')\}$.

In order to define probabilities of events in a DTMC $D$, we associate with $D$ a $\sigma$-algebra over $\mathrm{Path}(D)$. The $\sigma$-algebra associated with $D$ is generated by cylinder sets.

**Definition 2.1.8** (cylinder set)**.** A *cylinder set* of a path fragment is defined by

$$\mathrm{Cyl}(\widehat{\xi}) = \{\xi \in \mathrm{Path}(D) \mid \widehat{\xi} \text{ is a prefix of } \xi\} \ .$$

The cylinder sets are basis events of the $\sigma$-algebra associated with $D$.

**Definition 2.1.9** ($\sigma$-algebra of a DTMC)**.** A $\sigma$-algebra $(\mathrm{Path}(D), \mathcal{F})$ associated with a DTMC $D$ is the smallest $\sigma$-algebra that contains the cylinder set $\mathrm{Cyl}(\widehat{\xi})$ for each prefix $\widehat{\xi}$ of each path $\xi \in \mathrm{Path}(D)$.

**Definition 2.1.10** (measurability)**.** Let $(\mathrm{Path}(D), \mathcal{F})$ be a $\sigma$-algebra associated with a DTMC $D$. A event $E \subseteq \mathrm{Path}(D)$ is *measurable* in $D$ if $E \in \mathcal{F}$.

**Definition 2.1.11** (probability measure of a DTMC)**.** A *probability measure* $Pr^M$ of a DTMC $D$ is a function $Pr^M \colon \mathcal{F} \to [0, 1]$ that satisfies the following conditions.

- The probability of the whole event is 1. That is, $Pr^M(\mathrm{Path}(D)) = 1$.
- If $(E_n)_{n \in \mathbb{N}}$ is a family of events $E_n \in \mathcal{F}$, then

$$(\forall i, j \geq 0.\, i \neq j \Rightarrow E_i \cap E_j = \emptyset) \Rightarrow Pr^M\left(\bigcup_{n \geq 0} E_n\right) = \sum_{n \geq 0} Pr^M(E_n) \ .$$

- For each path fragment $s_0 \dots s_n$ in D,

$$Pr^M(\mathrm{Cyl}(s_0 \dots s_n)) = \iota_{\mathrm{init}}(s_0) \prod_{0 \le i < n} P(s_i, s_{i+1}) \enspace .$$

It is an easy measure-theoretic fact that a probability measure of a DTMC is well-defined. There exists a unique probability measure for each DTMC.

We now describe $\mathcal{F}$ in concrete terms (Proposition 2.1.16). The following theorem is from [20].

**Theorem 2.1.12.** *Let $A$ be a set. Suppose that a function $f\colon 2^A \to 2^A$ satisfies that $a \subseteq b \Rightarrow f(a) \subseteq f(b)$ for each $a, b \in 2^A$. Then*

- *there exists the smallest set $X_f \in 2^A$ in $\{X \in 2^A \mid f(X) = X\}$, and*
- *a set $P \in 2^A$ satisfies that $X_f \subseteq P$ if $P$ satisfies that $Y \subseteq P \Rightarrow f(Y) \subseteq P$ for each $Y \in 2^A$.*

**Proof.** Suppose that $F = \bigcap \{X \in 2^A \mid f(X) \subseteq X\}$. Then

$$\forall X \in 2^A.\, f(F) \subseteq f(X) \subseteq X \enspace ,$$

since $F \subseteq X$. Hence $f(F) \subseteq F$. Since $f(f(F)) \subseteq f(F)$, we have $F \subseteq f(F)$. Hence $f(F) = F$. If a set $F' \in 2^A$ satisfies that $f(F') = F'$, we have $F' \in \{X \in 2^A \mid f(X) \subseteq X\}$. Since $F \subseteq F'$, the set $F$ is the smallest. That is, $X_f = F$.

Suppose that a set $P \in 2^A$ satisfies that $Y \subseteq P \Rightarrow f(Y) \subseteq P$. Then $f(P) \subseteq P$, since $P \subseteq P$. Hence $X_f \subseteq P$. □

**Definition 2.1.13.** Let $D$ be a DTMC. A set $\mathcal{F}'$ of events in $D$ is defined by the smallest among the sets $X$ that satisfy the following conditions.

- $\{\mathrm{Cyl}(\widehat{\xi}) \mid \widehat{\xi}$ is a prefix of $\xi \in \mathrm{Path}(D)\} \cup \{\emptyset\} \subseteq X$ and
- if $(E_n)_{n\in\mathbb{N}}$ is a family of pairwise disjoint events $E_n \subseteq \mathrm{Path}(D)$, then

$$(\forall n \in \mathbb{N}.\, E_n \in X) \Rightarrow \left( \bigcup_{n\in\mathbb{N}} E_n \in X \ \wedge\ \mathrm{Path}(D) \setminus \bigcup_{n\in\mathbb{N}} E_n \in X \right) \enspace .$$

**Proposition 2.1.14.** *Let $E_1$ and $E_2$ be subsets of* $\mathrm{Path}(D)$ *such that $E_1 \in \mathcal{F}'$ and $E_2 \in \mathcal{F}'$. Then $E_1 \cap E_2 \in \mathcal{F}'$.*

**Proof.** For each path fragments $\widehat{\xi}_1, \widehat{\xi}_2$,

$$\mathrm{Cyl}(\widehat{\xi}_1) \cap \mathrm{Cyl}(\widehat{\xi}_2) = \begin{cases} \mathrm{Cyl}(\widehat{\xi}_1) & \text{if } \mathrm{Cyl}(\widehat{\xi}_1) \subseteq \mathrm{Cyl}(\widehat{\xi}_2) \\ \mathrm{Cyl}(\widehat{\xi}_2) & \text{if } \mathrm{Cyl}(\widehat{\xi}_2) \subseteq \mathrm{Cyl}(\widehat{\xi}_1) \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $(E_n)_{n\in\mathbb{N}}$ be a family of pairwise disjoint sets $E_n \subseteq \mathrm{Path}(D)$ and $\widehat{\xi}$ be a path fragment in $D$. Suppose that $E_n \cap \mathrm{Cyl}(\widehat{\xi}) \in \mathcal{F}'$ for each $n \in \mathbb{N}$. Then

$$\left( \bigcup_{n\in\mathbb{N}} E_n \right) \cap \mathrm{Cyl}(\widehat{\xi}) = \bigcup_{n\in\mathbb{N}} (E_n \cap \mathrm{Cyl}(\widehat{\xi})) \in \mathcal{F}'$$

and

$$\left( \Omega \setminus \bigcup_{n\in\mathbb{N}} E_n \right) \cap \mathrm{Cyl}(\widehat{\xi}) = \Omega \setminus \left( \bigcup_{n\in\mathbb{N}} (E_n \cap \mathrm{Cyl}(\widehat{\xi})) \cup (\Omega \setminus \mathrm{Cyl}(\widehat{\xi})) \right) \in \mathcal{F}' \enspace .$$

Hence for each $E \in \mathcal{F}'$ and for each $\mathrm{Cyl}(\widehat{\xi}) \in \mathcal{F}'$, we have $E \cap \mathrm{Cyl}(\widehat{\xi}) \in \mathcal{F}'$.

Let $(E_n)_{n\in\mathbb{N}}$ be a family of pairwise disjoint sets $E_n \subseteq \mathrm{Path}(D)$ and $E \in \mathcal{F}'$ be a set. Suppose that $E_n \cap \mathrm{Cyl}(\widehat{\xi}) \in \mathcal{F}'$ for each $n \in \mathbb{N}$. Then

$$\left(\bigcup_{n\in\mathbb{N}} E_n\right) \cap E = \bigcup_{n\in\mathbb{N}} (E_n \cap E) \in \mathcal{F}'$$

and

$$\left(\Omega \setminus \bigcup_{n\in\mathbb{N}} E_n\right) \cap E = \Omega \setminus \left(\bigcup_{n\in\mathbb{N}} (E_n \cap E) \cup (\Omega \setminus E))\right) \in \mathcal{F}' \ .$$

$\square$

**Proposition 2.1.15.** *Let $(E_n)_{n\in\mathbb{N}}$ be a family of subsets $E_n \subseteq \mathrm{Path}(D)$. Suppose that $E_n \in \mathcal{F}'$ for each $n \in \mathbb{N}$. Then we have $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{F}'$.*

**Proof.** Suppose that

$$E'_n = E_n \cap \left(\bigcap_{i=0}^{n-1} (\mathrm{Path}(D) \setminus E_i)\right) \ .$$

Then $(E'_n)_{n\in\mathbb{N}}$ is a family of pairwise disjoint events and for each $n \in \mathbb{N}$ and we have $E'_n \in \mathcal{F}'$ by Proposition 2.1.14. Hence

$$\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} E'_n \in \mathcal{F}' \ .$$

$\square$

**Proposition 2.1.16.**

$$\mathcal{F} = \mathcal{F}' \ .$$

**Proof.** Since $\mathcal{F}'$ is the smallest, $\mathcal{F}' \subseteq \mathcal{F}$. By Proposition 2.1.15, $\mathcal{F} \subseteq \mathcal{F}'$. $\square$

In order to generate a probability space over the set of paths starting in a certain state, we construct a DTMC with a unique initial state.

**Definition 2.1.17.** Let $D = (S, P, \iota_{\mathrm{init}}, AP, L)$ be a DTMC and $s \in S$ be a state in $D$. Then the DTMC $D^s$ is defined by $D^s = (S, P, \iota_s, AP, L)$ where

$$\iota_s(t) := \begin{cases} 1 & \text{if } s = t \\ 0 & \text{otherwise.} \end{cases}$$

The probability of $D^s$ is denoted by $Pr_s^M$. That is, $Pr_s^M$ is defined by

$$Pr_s^M(\mathrm{Cyl}(s_0 \dots s_n)) = \prod_{0 \le i < n} P(s_i, s_{i+1})$$

where $s_0 = s$. Since there is the only initial state $s$ in $D^s$, it is not necessary to define $Pr_s^M$ where $s_0 \neq s$.

**Definition 2.1.18.** Let $s$ be a state in a DTMC with a state space $S$ and $E \subseteq S^\omega$ be a set of paths. For each path fragment $\widehat{\xi}$ starting from the state $s$, we define $\widehat{\xi} + E$ by

$$\widehat{\xi} + E = \{\widehat{\xi}\sigma \mid \sigma \in E\} \cap \mathrm{Path}(s) \ .$$

**Proposition 2.1.19.** *Let $s$ be a state in a DTMC $D$ and $E$ be a measurable event in a DTMC $D'$ with the same state space and the same transition probability as $D$. Then $\widehat{\xi} + E$ is a measurable event in $D^s$, for each path fragment $\widehat{\xi}$ starting from the state $s$.*

**Proof.** For each path fragment $\widehat{\xi'}$, an event $\widehat{\xi} + \mathrm{Cyl}(\widehat{\xi'}) = \mathrm{Cyl}(\widehat{\widehat{\xi}\widehat{\xi'}}) \cap \mathrm{Path}(s)$ is measurable in $D^s$, and $\widehat{\xi} + \emptyset = \emptyset$ is measurable in $D^s$. Suppose that $(E_n)_{n \in \mathbb{N}}$ is a family of pairwise disjoint events that are measurable in a DTMC $D'$ and $\widehat{\xi} + E_n$ is measurable in $D^s$ for each $n \in \mathbb{N}$. Then $\widehat{\xi} + \bigcup_{n \in \mathbb{N}} E_n = \bigcup_{n \in \mathbb{N}} (\widehat{\xi} + E_n)$ and

$$\widehat{\xi} + \left( \mathrm{Path}(D') \setminus \bigcup_{n \in \mathbb{N}} E_n \right) = \left( \left( \bigcup_{s' \in \mathit{initial}(D')} \mathrm{Cyl}(\widehat{\xi} s') \right) \setminus \bigcup_{n \in \mathbb{N}} \left( \widehat{\xi} + E_n \right) \right) \cap \mathrm{Path}(s)$$

are measurable in $D^s$ where $\mathit{initial}(D') = \{\xi[0] \mid \xi \in \mathrm{Path}(D')\}$. $\square$

## 2.2 Probabilistic Computation Tree Logic

We define probabilistic computation tree logic (PCTL). The logic PCTL is a branching-time temporal logic with the probabilistic operator. The logic PCTL can express a LTL formula as a PCTL path formula.

**Definition 2.2.1** (syntax of PCTL)**.** *PCTL state formulas* over the set $AP$ are defined by the following grammar:

$$\Phi ::= \mathrm{false} \mid \mathrm{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2 \mid \neg\Phi \mid \mathbb{P}_{\geq p}(\phi) \mid \mathbb{P}_{>p}(\phi)$$

where $a \in AP$, $p \in [0, 1]$ and $\phi$ is a PCTL path formula.

*PCTL path formulas* are defined by the following grammar:

$$\phi ::= \Phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \neg\phi \mid \bigcirc\phi \mid \phi_1 \,\mathrm{U}\, \phi_2 \mid \phi_1 \,\mathrm{R}\, \phi_2$$

where $\Phi$ is a PCTL state formula.

The implication operator $\Rightarrow$, the eventually operator $\Diamond$ and the always operator $\Box$ are defined by

$$\Phi_1 \Rightarrow \Phi_2 := \neg\Phi_1 \vee \Phi_2, \quad \Diamond\Phi := \mathrm{true}\,\mathrm{U}\,\Phi \quad \text{and} \quad \Box\Phi := \mathrm{false}\,\mathrm{R}\,\Phi \ .$$

**Definition 2.2.2** (semantics of PCTL over states and paths)**.** Let $D$ be a DTMC. Let $\Phi$ be a PCTL state formula and $\phi$ be a PCTL path formula. Then we define $\mathrm{Sat}^D(\Phi) \subseteq S$ and $\mathrm{Sat}^D_s(\phi) \subseteq \mathrm{Path}(s)$ for each state $s \in S$ inductively by the following. Here $s \in \mathrm{Sat}^D(\Phi)$ and $\xi \in \mathrm{Sat}^D_{\xi[0]}(\phi)$ are denoted by $s \vDash \Phi$ and $\xi \vDash \phi$, respectively:

- $\mathrm{Sat}^D(\mathrm{false}) := \emptyset$ and $\mathrm{Sat}^D(\mathrm{true}) := S$,
- $\mathrm{Sat}^D(a) := \{s \in S \mid a \in L(s)\}$,
- $\mathrm{Sat}^D(\Phi_1 \wedge \Phi_2) := \mathrm{Sat}^D(\Phi_1) \cap \mathrm{Sat}^D(\Phi_2)$,
- $\mathrm{Sat}^D(\Phi_1 \vee \Phi_2) := \mathrm{Sat}^D(\Phi_1) \cup \mathrm{Sat}^D(\Phi_2)$,
- $\mathrm{Sat}^D(\neg\Phi) := S \setminus \mathrm{Sat}^D(\Phi)$,
- $\mathrm{Sat}^D(\mathbb{P}_{\geq p}(\phi)) := \{s \in S \mid Pr_s(\mathrm{Sat}^D_s(\phi)) \geq p\}$,
- $\mathrm{Sat}^D(\mathbb{P}_{>p}(\phi)) := \{s \in S \mid Pr_s(\mathrm{Sat}^D_s(\phi)) > p\}$,

- $\mathrm{Sat}_s^D(\Phi) := \mathrm{Path}(s)$ (if $s \models \Phi$),
- $\mathrm{Sat}_s^D(\Phi) := \emptyset$ (otherwise),
- $\mathrm{Sat}_s^D(\phi_1 \wedge \phi_2) := \mathrm{Sat}_s^D(\phi_1) \cap \mathrm{Sat}_s^D(\phi_2)$,
- $\mathrm{Sat}_s^D(\phi_1 \vee \phi_2) := \mathrm{Sat}_s^D(\phi_1) \cup \mathrm{Sat}_s^D(\phi_2)$,
- $\mathrm{Sat}_s^D(\neg\phi) := \mathrm{Path}(s) \setminus \mathrm{Sat}_s^D(\phi)$,
- $\mathrm{Sat}_s^D(\bigcirc\phi) := \{\xi \in \mathrm{Path}(s) \mid \xi[1\ldots] \models \phi\}$,
- $\mathrm{Sat}_s^D(\phi_1 \,\mathrm{U}\, \phi_2) :=$
  $\{\xi \in \mathrm{Path}(s) \mid \exists j \geq 0.\, (\xi[j\ldots] \models \phi_2 \wedge \forall i \geq 0.\, (i < j \Rightarrow \xi[i\ldots] \models \phi_1))\}$ and
- $\mathrm{Sat}_s^D(\phi_1 \,\mathrm{R}\, \phi_2) := \mathrm{Sat}_s^D(\neg(\neg\phi_1 \,\mathrm{U}\, \neg\phi_2))$.

**Proposition 2.2.3.** *Let $\phi$ be a PCTL path formula. For each state $s$ in a DTMC $D$, $\mathrm{Sat}_s^D$ is measurable in $D^s$.*

**Proof.** We proceed by induction on the structure of $\phi$.

*Case $\phi = \bigcirc\phi'$.* By Definition 2.2.2,

$$\mathrm{Sat}_s^D(\phi) = \bigcup_{s' \in \mathrm{Post}(s)} (s + \mathrm{Sat}_{s'}^D(\phi')) \ .$$

For each $s' \in \mathrm{Post}(s)$, the event $\mathrm{Sat}_{s'}^D(\phi')$ is measurable in $D^{s'}$ by induction hypothesis. Hence the event $\mathrm{Sat}_s^D(\phi)$ is measurable in $D^s$ by Proposition 2.1.19.

*Case $\phi = \phi_1 \,\mathrm{U}\, \phi_2$.* For each state $s$ in $D$, we define $U_n^s$ inductively by

$$U_n^s := \begin{cases} \mathrm{Sat}_s^D(\phi_2) & \text{if } n = 0 \\ \bigcup_{s' \in \mathrm{Post}(s)} (s + U_{n-1}^{s'}) \cap \mathrm{Sat}_s^D(\phi_1) & \text{otherwise.} \end{cases}$$

Then $U_n^s$ is a measurable in $D^s$ for each $n \in \mathbb{N}$ by induction hypothesis and Proposition 2.1.19. Hence $\mathrm{Sat}_s^D(\phi)$ is a measurable in $D^s$, since $\mathrm{Sat}_s^D(\phi) = \bigcup_{n \in \mathbb{N}} U_n^s$.

The proofs of the other cases are obvious. □

**Proposition 2.2.4.** *Let $\xi$ be a path and $s$ be a state in a DTMC $D$. Let $\Phi_1$ and $\Phi_2$ be PCTL path formulas. Let $\phi_1$ and $\phi_2$ be PCTL path formulas. Then the following equivalence is satisfied:*

- *$s \models \neg\neg\Phi_1$ if and only if $s \models \Phi_1$,*
- *$s \models \Phi_1 \vee \Phi_2$ if and only if $s \models \neg(\neg\Phi_1 \wedge \neg\Phi_2)$,*
- *$\xi \models \neg\neg\phi_1$ if and only if $\xi \models \phi_1$,*
- *$\xi \models \bigcirc\phi_1$ if and only if $\xi \models \neg \bigcirc \neg\phi_1$,*
- *$\xi \models \phi_1 \vee \phi_2$ if and only if $\xi \models \neg(\neg\phi_1 \wedge \neg\phi_2)$,*
- *$\xi \models \phi_1 \,\mathrm{R}\, \phi_2$ if and only if $\xi \models \neg(\neg\phi_1 \,\mathrm{U}\, \neg\phi_2)$.*

**Proof.** Suppose that $\xi \models \bigcirc\phi$. Then $\xi[1\ldots] \models \phi$. Since $\xi[1\ldots] \not\models \neg\phi$, we have $\xi \not\models \bigcirc\neg\phi$. Hence $\xi \models \neg \bigcirc \neg\phi$. Suppose that $\xi \models \neg \bigcirc \neg\phi$. Then $\xi \not\models \bigcirc\neg\phi$. Since $\xi[1\ldots] \not\models \neg\phi$, we have $\xi[1\ldots] \models \phi$. Hence $\xi \models \bigcirc\phi$.

The proofs of the others are obvious. □

**Proposition 2.2.5.** *Let $\xi$ be a path in a DTMC $D$. Let $\phi_1$ and $\phi_2$ be PCTL path formulas. Then*

$$\xi \vDash \phi_1 \mathrm{U}\, \phi_2 \textit{ if and only if } \xi \vDash \phi_2 \vee (\phi_1 \wedge \bigcirc(\phi_1 \mathrm{U}\, \phi_2)) \ .$$

**Proof.** We define $U_n^s$ inductively by the same way in the proof of Proposition 2.2.3. For each state $s$,

$$\begin{aligned}
\mathrm{Sat}_s^D(\phi_1 \mathrm{U}\, \phi_2)) &= \bigcup_{n \in \mathbb{N}} U_n^s \\
&= \mathrm{Sat}_s^D(\phi_2) \cup \bigcup_{n \in \mathbb{N}} \left( \bigcup_{s' \in \mathrm{Post}(s)} (s + U_n^{s'}) \cap \mathrm{Sat}_s^D(\phi_1) \right) \\
&= \mathrm{Sat}_s^D(\phi_2) \cup \left( \mathrm{Sat}_s^D(\phi_1) \cap \left( \bigcup_{s' \in \mathrm{Post}(s)} \left( s + \bigcup_{n \in \mathbb{N}} U_n^{s'} \right) \right) \right) \\
&= \mathrm{Sat}_s^D(\phi_2) \cup \left( \mathrm{Sat}_s^D(\phi_1) \cap \mathrm{Sat}_s^D\left( \bigcirc(\phi_1 \mathrm{U}\, \phi_2) \right) \right) \ .
\end{aligned}$$

□

When we simulate a die by a fair coin, for example, we sometimes want to know how many times on average we will flip the fair coin to decide a roll. For that purpose, we extend DTMCs.

**Definition 2.2.6** (DTMRC)**.** Let $D$ be a DTMC and *rew* be a function $\mathit{rew}\colon S \times S \to \mathbb{N}$. Then the tuple $DR = (D, \mathit{rew})$ is a *DTMRC*. The function *rew* is called a *reward function* of $DR$.

The definition of a reward function in this thesis is different from that of a reward function in [4]. However, the reward in [4] can be expressed as the reward in this thesis.

**Definition 2.2.7.** For a path fragment $\widehat{\xi} = s_0 s_1 \dots s_n$ in a DTMC $D$ let

$$\mathit{rew}(\widehat{\xi}) = \sum_{i=0}^{n-1} \mathit{rew}(s_i, s_{i+1}) \ .$$

We describe the expected reward until reaching a set of states. First we describe the reward until reaching a set of state along a path.

**Definition 2.2.8.** Let $(D, \mathit{rew})$ is a DTMRC with a state space $S$ and $B \subseteq S$ a set of state. For each path $\xi$ in $D$ let

$$\mathit{rew}(\xi, \Diamond B) := \begin{cases} \displaystyle\sum_{i=0}^{n-1} \mathit{rew}(\xi[i], \xi[i+1]) & \text{if there exists } n \in \mathbb{N} \text{ such that } \xi[i] \notin B \\ & \text{for each } i < n \text{ and } \xi[n] \in B \\ \infty & \text{otherwise,} \end{cases}$$

and

$$\mathit{rew}^a(\xi, \Diamond B) := \begin{cases} a & \text{if } \mathit{rew}(\xi, \Diamond B) = \infty \text{ or } \mathit{rew}(\xi, \Diamond B) \geq a \\ \mathit{rew}(\xi, \Diamond B) & \text{otherwise,} \end{cases}$$

where $a \in \mathbb{R}$.

Then we define an expected reward until reaching a set $B$ of states by the expected value of $\lambda\xi.rew(\xi, \Diamond B)$.

**Definition 2.2.9.** For state $s$ and $B \subseteq S$, an *expected reward* until reaching $B$ from $s$ is defined by

$$\mathit{ExpRew}(s \vDash \Diamond B) := \begin{cases} \sum_{\widehat{\xi}} P(\widehat{\xi})\mathit{rew}(\widehat{\xi}) & \text{if } \mathit{Pr}_s\{\xi \in \mathrm{Path}(s) \mid \exists n \in \mathbb{N}.\, \xi[n] \in B\} = 1 \\ \infty & \text{otherwise,} \end{cases}$$

where $\widehat{\xi}$ ranges over all path fragments $s_0 \dots s_n$ such that $s_0 = s$, $s_n \in B$ and $s_i \notin B$ for each $i < n$.

We add the operator about an expected reward and the operator about a bounded reward and we describe the logic for DTMRCs, called PRCTL [4].

**Definition 2.2.10** (syntax of PRCTL)**.** *PRCTL state formulas* over the set $AP$ are defined by the following grammar:

$$\Phi ::= \mathrm{false} \mid \mathrm{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2 \mid \neg\Phi \mid \mathbb{P}_{\geq p}(\phi) \mid \mathbb{P}_{>p}(\phi) \mid \mathbb{E}_{\leq r}(\Phi) \mid \mathbb{E}_{<r}(\Phi)$$

where $a \in AP$, $p \in [0,1]$, $r \in \mathbb{R}$ and $\phi$ is a PRCTL path formula.

*PRCTL path formulas* are defined by the following grammar:

$$\phi ::= \Phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \neg\phi \mid \bigcirc\phi \mid \phi_1 \,\mathrm{U}\, \phi_2 \mid \phi_1 \,\mathrm{R}\, \phi_2 \mid \phi_1 \,\mathrm{U}_{\leq r}\, \phi_2$$

where $r \in \mathbb{R}$ and $\Phi$ is a PRCTL state formula.

The implication operator $\Rightarrow$, the eventually operator $\Diamond$ and the always operator $\Box$ are defined by

$$\Phi_1 \Rightarrow \Phi_2 := \neg\Phi_1 \vee \Phi_2, \quad \Diamond\Phi := \mathrm{true}\,\mathrm{U}\,\Phi \quad \text{and} \quad \Box\Phi := \mathrm{false}\,\mathrm{R}\,\Phi \ .$$

**Definition 2.2.11** (semantics of PRCTL over states and paths)**.** Let $DR$ be a DTMRC. Let $\Phi$ be a PRCTL state formula and $\phi$ be a PRCTL path formula. Then we define $\mathrm{Sat}^{DR}(\Phi) \subseteq S$ and $\mathrm{Sat}^{DR}_s(\phi) \subseteq \mathrm{Path}(s)$ for each state $s \in S$ inductively by the following. Here $s \in \mathrm{Sat}^{DR}(\Phi)$ and $\xi \in \mathrm{Sat}^{DR}_{\xi[0]}(\phi)$ are denoted by $s \vDash \Phi$ and $\xi \vDash \phi$, respectively:

- $\mathrm{Sat}^{DR}(\mathrm{false}) := \emptyset$ and $\mathrm{Sat}^{DR}(\mathrm{true}) := S$,
- $\mathrm{Sat}^{DR}(a) := \{s \in S \mid a \in L(s)\}$,
- $\mathrm{Sat}^{DR}(\Phi_1 \wedge \Phi_2) := \mathrm{Sat}^{DR}(\Phi_1) \cap \mathrm{Sat}^{DR}(\Phi_2)$,
- $\mathrm{Sat}^{DR}(\Phi_1 \vee \Phi_2) := \mathrm{Sat}^{DR}(\Phi_1) \cup \mathrm{Sat}^{DR}(\Phi_2)$,
- $\mathrm{Sat}^{DR}(\neg\Phi) := S \setminus \mathrm{Sat}^{DR}(\Phi)$,
- $\mathrm{Sat}^{DR}(\mathbb{P}_{\geq p}(\phi)) := \{s \in S \mid \mathit{Pr}_s(\mathrm{Sat}^{DR}_s(\phi)) \geq p\}$,
- $\mathrm{Sat}^{DR}(\mathbb{P}_{>p}(\phi)) := \{s \in S \mid \mathit{Pr}_s(\mathrm{Sat}^{DR}_s(\phi)) > p\}$,
- $\mathrm{Sat}^{DR}(\mathbb{E}_{\leq r}(\Phi)) := \{s \in S \mid \mathit{ExpRew}(s \vDash \Diamond\, \mathrm{Sat}^{DR}(\Phi)) \leq r\}$,
- $\mathrm{Sat}^{DR}(\mathbb{E}_{<r}(\Phi)) := \{s \in S \mid \mathit{ExpRew}(s \vDash \Diamond\, \mathrm{Sat}^{DR}(\Phi)) < r\}$,
- $\mathrm{Sat}^{DR}_s(\Phi) := \mathrm{Path}(s)$ (if $s \vDash \Phi$),

- $\mathrm{Sat}_s^{DR}(\Phi) := \emptyset$ (otherwise),
- $\mathrm{Sat}_s^{DR}(\phi_1 \wedge \phi_2) := \mathrm{Sat}_s^{DR}(\phi_1) \cap \mathrm{Sat}_s^{DR}(\phi_2)$,
- $\mathrm{Sat}_s^{DR}(\phi_1 \vee \phi_2) := \mathrm{Sat}_s^{DR}(\phi_1) \cup \mathrm{Sat}_s^{DR}(\phi_2)$,
- $\mathrm{Sat}_s^{DR}(\neg\phi) := \mathrm{Path}(s) \setminus \mathrm{Sat}_s^{DR}(\phi)$,
- $\mathrm{Sat}_s^{DR}(\bigcirc\phi) := \{\xi \in \mathrm{Path}(s) \mid \xi[1\ldots] \vDash \phi\}$,
- $\mathrm{Sat}_s^{DR}(\phi_1 \mathbin{\mathrm{U}} \phi_2) :=$ $\{\xi \in \mathrm{Path}(s) \mid \exists j \geq 0.\, (\xi[j\ldots] \vDash \phi_2 \wedge \forall i \geq 0.\, (i < j \Rightarrow \xi[i\ldots] \vDash \phi_1))\}$,
- $\mathrm{Sat}_s^{DR}(\phi_1 \mathbin{\mathrm{R}} \phi_2) := \mathrm{Sat}_s^{DR}(\neg(\neg\phi_1 \mathbin{\mathrm{U}} \neg\phi_2))$ and
- $\mathrm{Sat}_s^{DR}(\phi_1 \mathbin{\mathrm{U}_{\leq r}} \phi_2) :=$ $\{\xi \in \mathrm{Path}(s) \mid \exists j \geq 0.\, (\mathit{rew}(\xi[0,j]) \leq r \wedge \xi[j\ldots] \vDash \phi_2 \wedge \forall i \geq 0.\, (i < j \Rightarrow \xi[i\ldots] \vDash \phi_1))\}$.

**Proposition 2.2.12.** *Let $\phi$ be a PRCTL path formula. For each state $s$ in a DTMRC $DR = (D, \mathit{rew})$, $\mathrm{Sat}_s^{DR}$ is measurable in $D^s$.*

**Proof.** We proceed by induction on the structure of $\phi$.

*Case* $\phi = \phi_1 \mathbin{\mathrm{U}_{\leq r}} \phi_2$. For each path fragment $\widehat{\xi}$ starting from the state $s$ in $D$, we define $U_{\widehat{\xi}}$ inductively by

$$U_{\widehat{\xi}} := \begin{cases} \mathrm{Sat}_s^{DR}(\phi_2) & \text{if } \widehat{\xi} = s \\ (\widehat{\xi}' + \mathrm{Sat}_{s'}^{DR}(\phi_2)) \cap \mathrm{Sat}_s^{DR}(\phi_1 \mathbin{\mathrm{U}} \phi_2) & \text{if } \widehat{\xi} = \widehat{\xi}' s' \wedge \mathit{rew}(\widehat{\xi}) \leq r \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $U_{\widehat{\xi}}$ is a measurable in $D^s$ for each $\widehat{\xi}$ by induction hypothesis and Proposition 2.1.19. Hence $\mathrm{Sat}_s^{DR}(\phi)$ is a measurable in $D^s$, since $\mathrm{Sat}_s^{D}(\phi) = \bigcup_{\widehat{\xi}} U_{\widehat{\xi}}$ where $\widehat{\xi}$ ranges over all path fragments starting from the state $s$.

The proofs of the other cases are similar to those of Proposition 2.2.3. $\square$

# Chapter 3

# Game-Theoretic Probability

## 3.1 Basic Definitions in Game-Theoretic Probability

We present basics of the game-theoretic probability following [18]. In this thesis we will be using *gambling protocols* (of games) in the following format. The format is more restricted than the general one commonly used in the game-theoretic (see [18, Chapter 8]) but we will not need the generality.

**Definition 3.1.1** (gambling protocol)**.**
*Parameter:* $\mathbf{S}_t, \mathbf{R}_t, \lambda_t$
*Players:* Skeptic, Reality
*Protocol:*
$K_0 := 1$.
FOR $n = 1, 2, \ldots$:
Skeptic announces $\mathbf{s}_n \in \mathbf{S}_t$.
Reality announces $\mathbf{r}_n \in \mathbf{R}_t$.
$K_n := K_{n-1} + \lambda_t(\mathbf{s}_n, \mathbf{r}_n)$.
Here

- the *sample space* $\Omega$ is the set of all infinite sequences $\mathbf{r}_1\mathbf{r}_2\ldots$ of moves Reality can make;
- the set $\Omega^\diamond := \{\widehat{\xi} \mid \widehat{\xi}$ is a prefix of $\xi \in \Omega\}$ is the set of *situations*;
- for each situation $t \in \Omega^\diamond$, $\mathbf{S}_t$ is the nonempty set of *moves* of Skeptic;
- for each situation $t \in \Omega^\diamond$, $\mathbf{R}_t := \{\mathbf{r} \mid t\mathbf{r} \in \Omega^\diamond\}$ is the nonempty set of *moves* of Reality; and
- for each situation $t \in \Omega^\diamond$,

$$\lambda_t \colon \mathbf{S}_t \times \mathbf{R}_t \longrightarrow \mathbb{R}$$

  is a *gain function.*

Note that $\mathbf{R}_t$ is defined inductively: once $\mathbf{R}_t$ is defined for $t$ of length up to $n$, $\mathbf{R}_{t\mathbf{r}}$ is defined for each $\mathbf{r} \in \mathbf{R}_t$.

**Definition 3.1.2** (probability protocol)**.** A gambling protocol is called a *probability protocol* if it satisfies the following conditions.

1. The set $\mathbf{S}_t$ of Skeptic's moves in a situation $t$ is a convex cone in some linear space. That is, if $\mathbf{s}_1$ and $\mathbf{s}_2$ are in $\mathbf{S}_t$ and $a_1$ and $a_2$ are nonnegative numbers, then $a_1\mathbf{s}_1 + a_2\mathbf{s}_2$ is in $\mathbf{S}_t$.

2. The gain function $\lambda_t$ in the situation $t$ has the following linearity property: if $\mathbf{s}_1$ and $\mathbf{s}_2$ are in $\mathbf{S}_t$ and $a_1$ and $a_2$ are nonnegative numbers, then $\lambda_t(a_1\mathbf{s}_1 + a_2\mathbf{s}_2, \mathbf{r}) = a_1\lambda_t(\mathbf{s}_1, \mathbf{r}) + a_2\lambda_t(\mathbf{s}_2, \mathbf{r})$ for every $\mathbf{r} \in \mathbf{R}_t$.

**Definition 3.1.3** (coherent protocol)**.** A gambling protocol is *coherent* if

$$\forall t \in \Omega^\Diamond. \forall \mathbf{s} \in \mathbf{S}_t. \exists \mathbf{r} \in \mathbf{R}_t. \lambda_t(\mathbf{s}, \mathbf{r}) \leq 0 \ .$$

**Definition 3.1.4** (symmetric protocol)**.** A gambling protocol is *symmetric* if

$$\forall t \in \Omega^\Diamond. \forall \mathbf{s} \in \mathbf{S}_t. (-\mathbf{s} \in \mathbf{S}_t \wedge \forall \mathbf{r} \in \mathbf{R}_t. \lambda_t(-\mathbf{s}, \mathbf{r}) = -\lambda_t(\mathbf{s}, \mathbf{r})) \ .$$

**Definition 3.1.5** (strategy)**.** A *strategy* is a strategy for Skeptic. That is, a strategy $\mathcal{P}$ is a function $\mathcal{P} \colon \Omega^\Diamond \to \mathbf{S}_t$.

**Definition 3.1.6** (capital process)**.** Let $\mathcal{P}$ be a strategy and $\epsilon \in \Omega^\Diamond$ be the empty sequence. Then a function $K^{\mathcal{P}} \colon \Omega^\Diamond \to \mathbb{R}$ is defined inductively by: $K^{\mathcal{P}}(\epsilon) = 0$ and

$$K^{\mathcal{P}}(t\mathbf{r}) = K^{\mathcal{P}}(t) + \lambda_t(\mathcal{P}(t), \mathbf{r}) \ .$$

we call $K^{\mathcal{P}}$ a *capital process.*

**Proposition 3.1.7.** *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be strategies. Let $a_1$ and $a_2$ be nonnegative numbers. In a probability protocol,*

$$K^{a_1\mathcal{P}_1 + a_2\mathcal{P}_2} = a_1 K^{\mathcal{P}_1} + a_2 K^{\mathcal{P}_2} \ .$$

**Proof.** For the empty sequence $\epsilon$,

$$K^{a_1\mathcal{P}_1 + a_2\mathcal{P}_2}(\epsilon) = 0 = a_1 K^{\mathcal{P}_1}(\epsilon) + a_2 K^{\mathcal{P}_2}(\epsilon) \ .$$

Suppose that for a situation $t$

$$K^{a_1\mathcal{P}_1 + a_2\mathcal{P}_2}(t) = a_1 K^{\mathcal{P}_1}(t) + a_2 K^{\mathcal{P}_2}(t) \ .$$

Then for each $\mathbf{r} \in \mathbf{R}_t$

$$\begin{aligned} K^{a_1\mathcal{P}_1 + a_2\mathcal{P}_2}(t\mathbf{r}) &= K^{a_1\mathcal{P}_1 + a_2\mathcal{P}_2}(t) + \lambda_t(a_1\mathcal{P}_1(t) + a_2\mathcal{P}_2(t), \mathbf{r}) \\ &= a_1 K^{\mathcal{P}_1}(t) + a_2 K^{\mathcal{P}_2}(t) + a_1\lambda_t(\mathcal{P}_1(t), \mathbf{r}) + a_2\lambda_t(\mathcal{P}_2(t), \mathbf{r}) \\ &= a_1 K^{\mathcal{P}_1}(t\mathbf{r}) + a_2 K^{\mathcal{P}_2}(t\mathbf{r}) \ . \end{aligned}$$

□

By the linearity of the gain function $\lambda_t$ of a symmetric probability protocol, we can prove similarly the following proposition.

**Proposition 3.1.8.** *Let $\mathcal{P}$ be a strategy. In a symmetric probability protocol,*

$$K^{-\mathcal{P}} = -K^{\mathcal{P}} \ .$$

**Proof.** For the empty sequence $\epsilon$,

$$K^{-\mathcal{P}}(\epsilon) = 0 = -K^{\mathcal{P}}(\epsilon) \ .$$

Suppose that for a situation $t$

$$K^{-\mathcal{P}}(t) = -K^{\mathcal{P}}(t) \ .$$

Then for each $\mathbf{r} \in \mathbf{R}_t$

$$\begin{aligned} K^{-\mathcal{P}}(t\mathbf{r}) &= K^{-\mathcal{P}}(t) + \lambda_t(-\mathcal{P}(t), \mathbf{r}) \\ &= -(K^{\mathcal{P}}(t) + \lambda_t(\mathcal{P}(t), \mathbf{r})) \\ &= -K^{\mathcal{P}}(t\mathbf{r}) \ . \end{aligned}$$

□

**Definition 3.1.9** (variable)**.** A *variable* $x$ is a function $x\colon \Omega \to \mathbb{R} \cup \{-\infty, \infty\}$.

**Definition 3.1.10** (price)**.** Let $x$ be a variable. Then we define $\overline{\mathbb{E}}x$ and $\underline{\mathbb{E}}x$ by

$$\overline{\mathbb{E}}x := \inf\{\alpha \mid \text{there is a strategy } \mathcal{P} \text{ such that } \forall\xi \in \Omega.\ \liminf_{n\to\infty} K^{\mathcal{P}}(\xi[0,n]) + \alpha \geq x(\xi)\}$$

and

$$\underline{\mathbb{E}}x := -\overline{\mathbb{E}}[-x] \ .$$

We call $\overline{\mathbb{E}}x$ and $\underline{\mathbb{E}}x$ the *upper price* of $x$ and the *lower price* of $x$, respectively.

If $\overline{\mathbb{E}}x = \underline{\mathbb{E}}x = \alpha$, we define $\mathbb{E}x = \alpha$ and call $\mathbb{E}x$ the *price* of $x$.

In a probability protocol, the price satisfies the following propositions.

**Proposition 3.1.11.** *In a probability protocol, the upper price of a variable $x$ satisfy that*

$$\overline{\mathbb{E}}[ax] \leq a\overline{\mathbb{E}}x$$

*for each $a > 0$.*

**Proof.** There exists a strategy $\mathcal{P}$ such that

$$\forall\xi \in \Omega.\ \overline{\mathbb{E}}x + \liminf_{n\to\infty} K^{\mathcal{P}} \geq x(\xi) \ .$$

Then the strategy $a\mathcal{P}$ satisfies that

$$\forall\xi \in \Omega.\ a\overline{\mathbb{E}}x + \liminf_{n\to\infty} K^{a\mathcal{P}} \geq ax(\xi) \ .$$

□

**Proposition 3.1.12.** *Let $x_1$ and $x_2$ be variables in a probability protocol. Suppose that $x_1 \leq x_2$. Then*

$$\overline{\mathbb{E}}x_1 \leq \overline{\mathbb{E}}x_2 \ .$$

**Proof.** There exists a strategy $\mathcal{P}$ such that

$$\forall\xi \in \Omega.\ \overline{\mathbb{E}}x_2 + \liminf_{n\to\infty} K^{\mathcal{P}} \geq x_2(\xi) \ .$$

Since $x_1 \leq x_2$, we have

$$\forall\xi \in \Omega.\ \overline{\mathbb{E}}x_1 + \liminf_{n\to\infty} K^{\mathcal{P}} \geq x_1(\xi) \ .$$

□

**Proposition 3.1.13.** *Let $x_1$ and $x_2$ be variables in a probability protocol. If $\overline{\mathbb{E}}x_1, \overline{\mathbb{E}}x_2 \in \mathbb{R}$ and $x_1(\xi), x_2(\xi) \in \mathbb{R}$ for each path $\xi$, then*

$$\overline{\mathbb{E}}[x_1 + x_2] \leq \overline{\mathbb{E}}x_1 + \overline{\mathbb{E}}x_2 \ .$$

**Proof.** There exists strategies $\mathcal{P}_1, \mathcal{P}_2$ such that

$$\forall\xi \in \Omega.\ \overline{\mathbb{E}}x_1 + \liminf_{n\to\infty} K^{\mathcal{P}_1} \geq x_1(\xi)$$

and

$$\forall\xi \in \Omega.\ \overline{\mathbb{E}}x_2 + \liminf_{n\to\infty} K^{\mathcal{P}_2} \geq x_2(\xi) \ .$$

Then the strategy $\mathcal{P}_1 + \mathcal{P}_2$ satisfies that

$$\forall\xi \in \Omega.\ \overline{\mathbb{E}}x_1 + \overline{\mathbb{E}}x_2 + \liminf_{n\to\infty} K^{\mathcal{P}_1+\mathcal{P}_2} \geq (x_1 + x_2)(\xi) \ .$$

□

In a coherent probability protocol, the price satisfies the following propositions.

**Proposition 3.1.14.** *In a coherent probability protocol, the price of a constant variable $a \in \mathbb{R}$ exists and satisfies that*

$$\mathbb{E}a = a \ \ .$$

**Proof.** By the definition of probability protocol, there exists the strategy $0\mathcal{P}$ that satisfies $a + \liminf_{n\to\infty} K^{0\mathcal{P}}(\xi[0,n]) = a$ for each $\xi \in \Omega$. Hence $\overline{\mathbb{E}}a \leq a$. Similarly, we can prove that $\underline{\mathbb{E}}a \geq a$. By Proposition 3.1.15, $\underline{\mathbb{E}}a = \overline{\mathbb{E}}a = a$. $\square$

**Proposition 3.1.15.** *In a coherent probability protocol, the upper price and the lower price of a variable $x$ satisfy that*

$$\underline{\mathbb{E}}x \leq \overline{\mathbb{E}}x \ \ .$$

**Proof.** If $\overline{\mathbb{E}}x < \underline{\mathbb{E}}x$, there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\overline{\mathbb{E}}x < \alpha_1 < \alpha_2 < \underline{\mathbb{E}}x \ \ .$$

Since $\overline{\mathbb{E}}x < \alpha_1$ and $\alpha_2 < \underline{\mathbb{E}}x$, there exist strategies $\mathcal{P}_1$ and $\mathcal{P}_2$ such that

$$\forall \xi \in \Omega.\, K^{\mathcal{P}_1}(\xi) \geq x(\xi) - \alpha_1 \text{ and } \forall \xi \in \Omega.\, K^{\mathcal{P}_2}(\xi) \geq \alpha_2 - x(\xi) \ \ .$$

Hence the strategy $\mathcal{P}_1 + \mathcal{P}_2$ satisfies

$$\forall \xi \in \Omega.\, K^{\mathcal{P}_1+\mathcal{P}_2}(\xi) \geq \alpha_2 - \alpha_1 > 0 \ \ .$$

This contradicts coherence. $\square$

**Proposition 3.1.16.** *Let $x_1$ and $x_2$ be variables in a coherent probability protocol. Suppose that the price of $x_1$ and the price of $x_2$ exist. If $\mathbb{E}x_1, \mathbb{E}x_2 \in \mathbb{R}$ and $x_1(\xi), x_2(\xi) \in \mathbb{R}$ for each path $\xi$, then*

$$\mathbb{E}[x_1 + x_2] = \mathbb{E}x_1 + \mathbb{E}x_2 \ \ .$$

**Proof.** By Proposition 3.1.13,

$$\overline{\mathbb{E}}[x_1 + x_2] \leq \mathbb{E}x_1 + \mathbb{E}x_2$$

and

$$\underline{\mathbb{E}}[x_1 + x_2] = -\overline{\mathbb{E}}[(-x_1) + (-x_2)] \geq -\overline{\mathbb{E}}[(-x_1)] - \overline{\mathbb{E}}[(-x_2)] = \mathbb{E}x_1 + \mathbb{E}x_2 \ \ .$$

By Proposition 3.1.15,

$$\overline{\mathbb{E}}[x_1 + x_2] = \underline{\mathbb{E}}[x_1 + x_2] = \mathbb{E}x_1 + \mathbb{E}x_2 \ \ .$$

$\square$

**Proposition 3.1.17.** *Let $x$ be a variable in a coherent probability protocol. Suppose that the price of $x$ exists. Then for each $a \in \mathbb{R}$*

$$\mathbb{E}[ax] = a\mathbb{E}x \ \ .$$

**Proof.** Suppose that $a > 0$. Then $\overline{\mathbb{E}}[ax] \leq a\mathbb{E}x$ and $\underline{\mathbb{E}}[ax] = -\overline{\mathbb{E}}[a(-x)] \geq a\mathbb{E}x$ by Proposition 3.1.11. By Proposition 3.1.15, $\underline{\mathbb{E}}[ax] = \overline{\mathbb{E}}[ax] = a\mathbb{E}x$.

Suppose that $a < 0$. Then $\overline{\mathbb{E}}[ax] \leq -a\overline{\mathbb{E}}[-x] = a\mathbb{E}x$ and $\underline{\mathbb{E}}[ax] = -\overline{\mathbb{E}}[-ax] \geq a\mathbb{E}x$ by Proposition 3.1.11. By Proposition 3.1.15, $\underline{\mathbb{E}}[ax] = \overline{\mathbb{E}}[ax] = a\mathbb{E}x$.

Suppose that $a = 0$. Then $\mathbb{E}[ax] = 0$ by Proposition 3.1.14. $\square$

**Proposition 3.1.18.** *Let $(x_n)_{n\in\mathbb{N}}$ be variables in a coherent probability protocol. Suppose that $x_n \le x_{n+1}$ for each $n \in \mathbb{N}$ and that the price of $x_n$ exists for each $n \in \mathbb{N}$. If $\mathbb{E}x_n > -\infty$ and $x_n(\xi) \in \mathbb{R}$ for each path $\xi$ for each $n \in \mathbb{N}$, then the price of $\lim_{n\to\infty} x_n$ exists and*

$$\mathbb{E}[\lim_{n\to\infty} x_n] = \lim_{n\to\infty} \mathbb{E}x_n \ .$$

**Proof.** Suppose that there exists $i \in \mathbb{N}$ such that $\mathbb{E}x_i = \infty$. Since $x_i \le \lim_{n\to\infty} x_n$, we have $\mathbb{E}x_i \le \mathbb{E}[\lim_{n\to\infty} x_n]$. Hence $\mathbb{E}[\lim_{n\to\infty} x_n] = \infty$. Moreover, $\mathbb{E}x_n = \infty$ for each $n \ge i$, since $x_n \ge x_i$ for each $n \ge i$. Hence $\mathbb{E}[\lim_{n\to\infty} x_n] = \lim_{n\to\infty} \mathbb{E}x_n$.

Suppose that $\mathbb{E}x_n \in \mathbb{R}$ for each $n \in \mathbb{N}$. Let $(x'_n)_{n\in\mathbb{N}}$ be variables such that $x'_n = x_{n+1} - x_n$ for each $n \in \mathbb{N}$. Then

$$\overline{\mathbb{E}}[\lim_{n\to\infty} x_n] = \overline{\mathbb{E}}[x_0 + \sum_{n\in\mathbb{N}} x'_n] \text{ and } \mathbb{E}x_n = \mathbb{E}x_0 + \sum_{i=0}^{n-1} \mathbb{E}x'_i \ .$$

Since $x_i \le \lim_{n\to\infty} x_n$ for each $i \in \mathbb{N}$,

$$\mathbb{E}x_0 + \sum_{n\in\mathbb{N}} \mathbb{E}x'_n = \lim_{n\to\infty} \mathbb{E}x_n \le \overline{\mathbb{E}}[\lim_{n\to\infty} x_n] \ .$$

By Proposition 3.1.13,

$$\mathbb{E}x_0 + \sum_{n\in\mathbb{N}} \mathbb{E}x'_n \ge \overline{\mathbb{E}}[x_0 + \sum_{n\in\mathbb{N}} x'_n] \ .$$

Hence $\overline{\mathbb{E}}[\lim_{n\to\infty} x_n] = \lim_{n\to\infty} \mathbb{E}x_n$. Since $\underline{\mathbb{E}}[\lim_{n\to\infty} x_n] = -\overline{E}[(-x_0)+\sum_{n\in\mathbb{N}}(-x'_n)]$, we have

$$\underline{\mathbb{E}}[\lim_{n\to\infty} x_n] \ge -\mathbb{E}[-x_0] - \sum_{n\in\mathbb{N}} \mathbb{E}[-x'_n] = \lim_{n\to\infty} \mathbb{E}x_n \ .$$

By Proposition 3.1.15,

$$\overline{\mathbb{E}}[\lim_{n\to\infty} x_n] = \underline{\mathbb{E}}[\lim_{n\to\infty} x_n] = \lim_{n\to\infty} \mathbb{E}x_n \ .$$

□

**Definition 3.1.19** (event)**.** An *event* $E$ is a subset of $\Omega$.

**Definition 3.1.20** (indicator variable)**.** Given an event $E \subseteq \Omega$, we define the *indicator variable* $\mathbb{I}_E$ by

$$\mathbb{I}_E(\xi) := \begin{cases} 1 & \text{if } \xi \in E \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 3.1.21** (probability)**.** Given an event $E \subseteq \Omega$, we define an *upper probability* of $E$ by

$$\overline{Pr^G}(E) := \overline{\mathbb{E}}\mathbb{I}_E$$

and a *lower probability* of $E$ by

$$\underline{Pr^G}(E) := \underline{\mathbb{E}}\mathbb{I}_E \ .$$

If $\overline{Pr^G}(E) = \underline{Pr^G}(E) = p$, we define a *probability* of $E$ by $Pr^G(E) = p$.

**Proposition 3.1.22.** *Let $E$ be an event in a probability protocol. Then*

$$0 \leq \underline{Pr^G}(E) \text{ and } \overline{Pr^G}(E) \leq 1 \ .$$

**Proof.** By the definition of probability protocol, there exists the strategy $0\mathcal{P}$ that satisfies $K^{0\mathcal{P}}(\xi[0,n]) = 0$ for each $n \in \mathbb{N}$ and for each $\xi \in \Omega$. Since

$$\liminf_{n\to\infty} K^{0\mathcal{P}} + 0 \geq -\mathbb{I}_E \text{ and } \liminf_{n\to\infty} K^{0\mathcal{P}} + 1 \geq \mathbb{I}_E$$

for each $\xi \in \Omega$. Hence $\overline{\mathbb{E}}[-\mathbb{I}_E] \leq 0$ and $\overline{\mathbb{E}}[\mathbb{I}_E] \leq 1$. □

**Proposition 3.1.23.** *Let $E$ be an event in a symmetric probability protocol. Suppose that the probability of $E$ exists. Then the probability of $\Omega \setminus E$ exists and*

$$Pr^G(\Omega \setminus E) = 1 - Pr^G(E) \ .$$

**Proof.** Since $\mathbb{I}_{\Omega\setminus E} = 1 - \mathbb{I}_E$, we have

$$Pr^G(\Omega \setminus E) = \mathbb{E}[1 - \mathbb{I}_E] = 1 - Pr^G(E) \ .$$

□

**Definition 3.1.24** (force)**.** Let $E$ be an event in a coherent probability protocol. We say Skeptic can *force* $E$ if there exists a strategy $\mathcal{P}$ such that

$$\exists a \in \mathbb{R}. \forall \xi \in \Omega. \liminf_{n\to\infty} K^{\mathcal{P}}(\xi[0,n]) > a \ \wedge \ \forall \xi \in \Omega \setminus E. \lim_{n\to\infty} K^{\mathcal{P}}(\xi[0,n]) = \infty \ .$$

**Proposition 3.1.25.** *Let $E$ be an event in a coherent probability protocol. Suppose that Skeptic can force $E$. Then $Pr^G(E) = 1$.*

**Proof.** Since Skeptic can force $E$, there exist a strategy $\mathcal{P}$ and a number $a \in \mathbb{R}$ such that

$$\forall \xi \in \Omega. \liminf_{n\to\infty} K^{\mathcal{P}}(\xi[0,n]) > a \wedge \xi \in \Omega \setminus E. \lim_{n\to\infty} K^{\mathcal{P}}(\xi[0,n]) = \infty \ .$$

For each positive number $r > 0$, the strategy $r\mathcal{P}/|a|$ satisfies that

$$\forall \xi \in E. \liminf_{n\to\infty} K^{r\mathcal{P}/|a|}(\xi[0,n]) + r > -r + r = 0 = \mathbb{I}_{\Omega\setminus E}(\xi)$$

and

$$\forall \xi \in \Omega \setminus E. \liminf_{n\to\infty} K^{r\mathcal{P}/|a|}(\xi[0,n]) + r = \infty > \mathbb{I}_{\Omega\setminus E}(\xi) \ .$$

Hence $\overline{Pr^G}(\Omega \setminus E) \leq 0$. By Proposition 3.1.15 and 3.1.22, $0 \leq \underline{Pr^G}(\Omega \setminus E) \leq \overline{Pr^G}(\Omega \setminus E) \leq 0$. Hence $Pr^G(\Omega \setminus E) = 0$. Then

$$\overline{Pr^G}(E) = \overline{\mathbb{E}}[1 - \mathbb{I}_{\Omega\setminus E}] = 1 + \overline{\mathbb{E}}[-\mathbb{I}_{\Omega\setminus E}] = 1 - \underline{Pr^G}(\Omega \setminus E) = 1$$

and

$$\underline{Pr^G}(E) = \underline{\mathbb{E}}[1 - \mathbb{I}_{\Omega\setminus E}] = 1 - \overline{\mathbb{E}}[\mathbb{I}_{\Omega\setminus E}] = 1 - \overline{Pr^G}(\Omega \setminus E) = 1 \ .$$

□

The notion of replicating, and the result in Proposition 3.1.27 are presented in [19] for finite games. We do the same for infinite games.

**Definition 3.1.26** (replicating strategy)**.** Let $x$ be a variable. We call a strategy $\mathcal{P}$ a *replicating strategy* for $x$ with the *replicating initial capital* $\alpha \in \mathbb{R}$ if

$$\forall \xi \in \Omega.\, \alpha + \liminf_{n\to\infty} K^{\mathcal{P}}(\xi[0,n]) = x(\xi) \ .$$

**Proposition 3.1.27.** *In a coherent symmetric protocol, if $\mathcal{P}$ is a replicating strategy for $x$ with the replicating initial capital $\alpha$, then*

$$\underline{\mathbb{E}}x = \overline{\mathbb{E}}x = \alpha \ .$$

**Proof.** By Definition 3.1.10 and 3.1.26, $\overline{\mathbb{E}}x \leq \alpha$. By symmetry, $-\mathcal{P}$ is a replicating strategy for $-x$ with replicating initial capital $-\alpha$. Hence $\overline{\mathbb{E}}[-x] \leq -\alpha$. Since $\alpha \geq \underline{\mathbb{E}}x$, we have $\overline{\mathbb{E}}x \geq \underline{\mathbb{E}}x$. By Proposition 3.1.15, $\underline{\mathbb{E}}x \geq \overline{\mathbb{E}}x$ $\square$

By Definition 3.1.10, if there exists a replicating strategy for $x$ with the replicating initial capital $\alpha$ in a coherent symmetric gambling protocol, then the price of $x$ exists and $\mathbb{E}x = \alpha$.

## 3.2 Game for DTMC

We define the game for a DTMC. This seems to be new.

**Definition 3.2.1** (game for a DTMC)**.** A protocol of the *game for a DTMC* $D = (S, P, \iota_{\text{init}}, AP, L)$ is described as follows. Here $P' \colon (\{*\} \cup S) \times S \to [0,1]$ satisfies that

$$P'(s,s') = \begin{cases} \iota_{\text{init}}(s') & \text{if } s = * \\ P(s,s') & \text{otherwise.} \end{cases}$$

*Parameter:* $S, P', x_0 = *$
*Protocol:*
  $K_0 := 1$.
  FOR $n = 1, 2, \ldots$:
    Skeptic announces a bounded function $f_n \colon S \to \mathbb{R}$.
    Reality announces $x_n \in \{s \in S \mid P'(x_{n-1}, s) > 0\}$.
    $K_n := K_{n-1} + f_n(x_n) - \sum_{s\in S} f_n(s) P'(x_{n-1}, s)$.

The protocol of the game for a DTMC is a coherent symmetric probabilistic protocol. In this game, Skeptic bets some money on each state and Reality decides the next state. Then Skeptic gets the money bet on this next state and loses the money that equal to the total expected value of bet money. If we translate Skeptic's move $f_n$ to $f'_n(s) = f_n(s)P'(s', s)$ where $s'$ is the predecessor state, the meaning of this game may be more natural. In this case $K_n = K_{n-1} + f'_n(x_n)/P'(x_{n-1}, s) - \sum_{s\in S} f'_n(s)$. Hence Skeptic gets the money that is the money bet on the next state multiplied the odds following the transition probability, loses the total bet money.

In the game for a DTMC $D$, the set $\mathrm{Path}(D)$ is the sample space $\Omega$ and a prefix $\widehat{\xi}$ of a path $\xi \in \mathrm{Path}(D)$ is a situation. To check sanity of our definition of games, we show that the probabilities of measurable events in terms of this game are equal to their counterparts in terms of the measure-theoretic probability.

**Proposition 3.2.2.** *Let $D = (S, P, \iota_{\text{init}}, AP, L)$ be a DTMC with a state space $S$ and $(\mathrm{Path}(D), \mathcal{F})$ be the $\sigma$-algebra associated with $D$. In the game for $D$, there exists the probability of $E$ and $Pr^G(E) = Pr^M(E)$ for each event $E \in \mathcal{F}$.*

**Proof.** Let $\mathrm{Cyl}(t)$ be the cylinder set of a path fragment $t$ in $D$ and $\widehat{\xi} = s_0 \dots s_n \in \Omega^{\Diamond}$ be a path fragment in $D$. We define a strategy $\mathcal{P}$ by

$$\mathcal{P}(\widehat{\xi'})(s') = \begin{cases} \frac{Pr^M(\mathrm{Cyl}(t))}{P'(\widehat{\xi'}s')} & \text{if } \widehat{\xi'}s' \text{ is a prefix of } t \\ 0 & \text{otherwise} \end{cases}$$

where $P'(\widehat{\xi}) = \iota_{\mathrm{init}}(\widehat{\xi}[0])P(\widehat{\xi})$. It is to see that the strategy $\mathcal{P}$ is a replicating strategy for $\mathbb{I}_{\mathrm{Cyl}(t)}$ with the replicating initial capital $Pr^M(\mathrm{Cyl}(t))$. Hence $Pr^G(\mathrm{Cyl}(t)) = Pr^M(\mathrm{Cyl}(t))$.

A strategy $\mathcal{P}$ such that $\mathcal{P}(t)(s) = 0$ for each situation $t$ and state $s$ is a replicating strategy for $\mathbb{I}_{\emptyset}$ with the replicating initial capital 0.

Suppose that $(E_n)_{n\in\mathbb{N}}$ is a family of pairwise disjoint events $E_n$ and that there exists the probability of $E_n$ such that $Pr^G(E_n) = Pr^G(E_n)$ for each $n \in \mathbb{N}$. Since $\mathbb{I}_{\bigcup_{n\in\mathbb{N}} E_n} = \sum_{n\in\mathbb{N}} \mathbb{I}_{E_n}$, there exist the probability of $\bigcup_{n\in\mathbb{N}} E_n$ and the probability of $\Omega \setminus \bigcup_{n\in\mathbb{N}} E_n$ and

$$Pr^G(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n\in\mathbb{N}} Pr^G(E_n) = \sum_{n\in\mathbb{N}} Pr^M(E_n) = Pr^M(\bigcup_{n\in\mathbb{N}} E_n)$$

and

$$Pr^G(\Omega \setminus \bigcup_{n\in\mathbb{N}} E_n) = 1 - Pr^G(\bigcup_{n\in\mathbb{N}} E_n) = 1 - Pr^M(\bigcup_{n\in\mathbb{N}} E_n) = Pr^G(\Omega \setminus \bigcup_{n\in\mathbb{N}} E_n)$$

by Proposition 3.1.18 and 3.1.23. $\square$

By Proposition 3.2.2, $Pr^G(E) = Pr^M(E)$ for each measurable event $E$ in a DTMC. Therefore the probability of a measurable event $E$ in a DTMC is denoted by $Pr(E) = Pr^G(E) = Pr^M(E)$. The probability $Pr_s(E)$ is defined similarly.

**Proposition 3.2.3.** *Let $(D, rew)$ be a DTMRC. In the game for $D$, for each $a \in \mathbb{R}$ and for each set $B$ of states, there exists the price of $\lambda\xi.rew^a(\xi, \Diamond B)$ and*

$$\mathbb{E}[\lambda\xi.rew^a(\xi, \Diamond B)] = aPr\{\xi \in \mathrm{Path}(D) \mid rew(\xi, \Diamond B) \geq a\} + \sum_{\widehat{\xi}} \iota_{\mathrm{init}}(\widehat{\xi}[0])P(\widehat{\xi})rew(\widehat{\xi})$$

*where $\widehat{\xi}$ ranges over all path fragments $s_0 \dots s_n$ such that $rew(s_0 \dots s_n) < a$, $s_0$ is an initial state of $D$, $s_n \in B$ and $s_i \notin B$ for each $i < n$.*

**Proof.** Let $E = \{\xi \in \mathrm{Path}(D) \mid rew(\xi, \Diamond B) \geq a\}$ be an event. The event $E$ is measurable in $D$. By the definition of $rew^a$,

$$rew^a = a\mathbb{I}_E + \sum_{\widehat{\xi}} rew(\widehat{\xi})\mathbb{I}_{\mathrm{Cyl}(\widehat{\xi})}$$

where $\widehat{\xi}$ ranges over all path fragments $s_0 \dots s_n$ such that $rew(s_0 \dots s_n) < a$, $s_0$ is the initial state of $D$, $s_n \in B$ and $s_i \notin B$ for each $i < n$. By Proposition 3.1.18 and 3.2.2, there exists the price of $\lambda\xi.rew^a(\xi, \Diamond B)$ and

$$\mathbb{E}[\lambda\xi.rew^a(\xi, \Diamond B)] = aPr(E) + \sum_{\widehat{\xi}} \iota_{\mathrm{init}}(\widehat{\xi}[0])P(\widehat{\xi})rew(\widehat{\xi})$$

where $\widehat{\xi}$ ranges over all path fragments $s_0 \dots s_n$ such that $rew(s_0 \dots s_n) < a$, $s_0$ is the initial state of $D$, $s_n \in B$ and $s_i \notin B$ for each $i < n$. $\square$

**Definition 3.2.4.** Let $D$ be a DTMC and let $s$ be a state in $D$. The price of a variable $x$ in a game for $D^s$ is denoted by $\mathbb{E}_s[x]$.

**Proposition 3.2.5.** *Let $(D, rew)$ be a DTMRC with a state space $S$ and $s$ be a state in $D$. Then for each $B \subseteq S$,*

$$\mathit{ExpRew}(s \vDash \Diamond B) = \mathbb{E}_s[\lambda\xi.\mathit{rew}(\xi, \Diamond B)] = \lim_{a\to\infty} \mathbb{E}_s[\lambda\xi.\mathit{rew}^a(\xi, \Diamond B)] \ .$$

**Proof.** By Proposition 3.2.3, for each $a \in \mathbb{R}$

$$\mathbb{E}[\lambda\xi.\mathit{rew}^a(\xi, \Diamond B)] = a\mathit{Pr}_s\{\xi \in \mathrm{Path}(s) \mid \mathit{rew}(\xi, \Diamond B) \geq a\} + \sum_{\widehat{\xi}} P(\widehat{\xi})\mathit{rew}(\widehat{\xi})$$

where $\widehat{\xi}$ ranges over all path fragments $s_0 \dots s_n$ such that $\mathit{rew}(s_0 \dots s_n) < a$, $s_0 = s$, $s_n \in B$ and $s_i \notin B$ for each $i < n$. Since $\mathit{rew}(\xi, \Diamond B) = \lim_{a\to\infty} \mathit{rew}^a(\xi, \Diamond B)$ for each path $\xi \in \mathrm{Path}(s)$,

$$\begin{aligned}\mathbb{E}_s[\lambda\xi.\mathit{rew}(\xi, \Diamond B)] &= \lim_{a\to\infty} \mathbb{E}_s[\lambda\xi.\mathit{rew}^a(\xi, \Diamond B)] \\ &= \begin{cases} \sum_{\widehat{\xi}} P(\widehat{\xi})\mathit{rew}(\widehat{\xi}) & \text{if } \mathit{Pr}_s\{\xi \in \mathrm{Path}(s) \mid \mathit{rew}(\xi, \Diamond B) = \infty\} = 0 \\ \infty & \text{otherwise} \end{cases}\end{aligned}$$

where $\widehat{\xi}$ ranges over all path fragments $s_0 \dots s_n$ such that $s_0 = s$, $s_n \in B$ and $s_i \notin B$ for each $i < n$. Hence

$$\mathit{ExpRew}(s \vDash \Diamond B) = \mathbb{E}_s[\lambda\xi.\mathit{rew}(\xi, \Diamond B)] = \lim_{a\to\infty} \mathbb{E}_s[\lambda\xi.\mathit{rew}^a(\xi, \Diamond B)] \ .$$

$\square$

# Chapter 4

# Results on DTMCs in Game-Theoretic Probability

A state of a DTMC changes step by step. Therefore one transition of states in DTMCs is regarded as one round of games. It seems to be natural that DTMCs are expressed as games. In this chapter, we show the application of this idea by using the game for a DTMC defined in the previous chapter.

## 4.1 Fairness Theorem

In this section we prove the fairness theorem for DTMCs [4]. The fairness theorem means that an event $E$ happens almost surely if $E$ is such that if a certain state is visited infinitely often, then the each successor of the state is visited infinitely often. Since we want to check whether or not a certain state is visited, we use states as atomic propositions in this section. That is, we assume that $AP = S$ and $L(s) = \{s\}$ for each $s \in S$. Then the PCTL path formula $\Box\Diamond t$ denotes the property that the state $t$ is visited infinitely often. For a set $B$ of states, the state formula $B$ denotes $\bigvee_{t\in B} t$. In order to prove the fairness theorem, we prove the following key lemma.

**Lemma 4.1.1.** *Let $D$ be a DTMC and $s$, $u$ be states be in $D$. For any $T \subseteq \mathrm{Pre}(u)$, Skeptic can force the following event:*

$$\{\xi \in \mathrm{Path}(s) \mid \xi \models \Box\Diamond T \Rightarrow \Box\Diamond(T \wedge \bigcirc u)\} \ .$$

**Proof.** Take Skeptic's strategy such that

$$f_n(v) = \begin{cases} 0 & \text{if } x_{n-1} \in T \wedge v = u \\ K_{n-1} & \text{otherwise.} \end{cases}$$

At the $n$-th step of the game, situations are classified into two groups.

(I) $x_{n-1} \notin T$. In this case, $K_n = K_{n-1}$.
(II) $x_{n-1} \in T$. In this case:
 (i) If Reality announces $x_n \neq u$, $K_n = (1 + P(x_{n-1}, u))K_{n-1}$.
 (ii) If Reality announces $x_n = u$,
  $K_n = K_n - (1 - P(x_{n-1}, u)))K_{n-1} = P(x_{n-1}, u)K_{n-1}$.

Hence $K_0 > 0 \Rightarrow \forall n.\, K_n > 0$. Suppose that a path $\xi' \in \mathrm{Path}(s)$ satisfies

$$\xi' \models \Box\Diamond T \wedge \neg\Box\Diamond(T \wedge \bigcirc u) \ .$$

The situations classified as (II) happen infinitely often. Hence, along $\xi'$, capital is multiplied by $P(x_{n-1}, u) > 0$ finitely often and capital is multiplied by $(1 + P(x_{n-1}, v)) > 1$ infinitely often. Hence Skeptic can force the following event:

$$\{\xi \in \mathrm{Path}(s) \mid \xi \vDash \neg(\Box\Diamond T \wedge \neg\Box\Diamond(T \wedge \bigcirc u))\} \ .$$

That is, Skeptic can force the following event:

$$\{\xi \in \mathrm{Path}(s) \mid \xi \vDash \Box\Diamond T \Rightarrow \Box\Diamond(T \wedge \bigcirc u)\} \ .$$

$\Box$

**Lemma 4.1.2.** *Let $D$ be a DTMC and $s$, $t$ be states in $D$. For each state $u \in \mathrm{Post}^*(t)$:*

$$Pr(s \vDash \Box\Diamond t \Rightarrow \Box\Diamond u) = 1 \ .$$

**Proof.** Since $u \in \mathrm{Post}^*(t)$, there exists a path fragment $s_0 s_1 \dots s_{m+1} \in \mathrm{Path}^*(t)$ such that $s_0 = t \wedge s_{m+1} = u$. Then let $E_i := \{\xi \in \mathrm{Path}(s) \mid \xi \vDash \Box\Diamond s_i \Rightarrow \Box\Diamond s_{i+1}\}$ and $E := \{\xi \in \mathrm{Path}(s) \mid \xi \vDash \Box\Diamond t \Rightarrow \Box\Diamond u\}$. By Lemma 4.1.1, for each $i \in \{0, 1, \dots, m\}$, Skeptic can force $E_i$. That is, there exists a strategy $\mathcal{P}_i$ such that

$$\exists a \in \mathbb{R}. \forall \xi \in \Omega. \liminf_{n\to\infty} K^{\mathcal{P}_i}(\xi[0,n]) > a \ \wedge \ \forall \xi \in \Omega \setminus E_i. \lim_{n\to\infty} K^{\mathcal{P}_i}(\xi[0,n]) = \infty$$

for each $i \in \{0, 1, \dots, m\}$. Since $(\Omega \setminus E) \subseteq \bigcup_{0 \le i \le m}(\Omega \setminus E_i)$, Skeptic (who follows the strategy $\sum_{0 \le i \le m} \mathcal{P}_i$) can force $E$. By Proposition 3.1.25,

$$Pr(s \vDash \Box\Diamond t \Rightarrow \Box\Diamond u) = 1 \ .$$

$\Box$

**Theorem 4.1.3** (fairness theorem)**.** *For DTMC $D$ and $s$, $t$ be states in $D$:*

$$Pr(s \vDash \Box\Diamond t) = Pr(s \vDash \bigwedge_{u \in \mathrm{Post}^*(t)} \Box\Diamond u) \ .$$

**Proof.** Let

$$\begin{aligned}
E_1 &:= \{\xi \in \mathrm{Path}(s) \mid \xi \vDash \Box\Diamond t\}, \\
E_2 &:= \{\xi \in \mathrm{Path}(s) \mid \xi \vDash \bigwedge_{u \in \mathrm{Post}^*(t)} \Box\Diamond u\}, \text{ and} \\
E_3 &:= \{\xi \in \mathrm{Path}(s) \mid \xi \vDash \Box\Diamond t \wedge \neg(\bigwedge_{u \in \mathrm{Post}^*(t)} \Box\Diamond u)\}.
\end{aligned}$$

Then $E_1 = E_2 \cup E_3$. By Lemma 4.1.2, $Pr_s(E_3) = 0$. Since $Pr_s(E_1) = Pr_s(E_2 \cup E_3) = Pr_s(E_2)$,

$$Pr(s \vDash \Box\Diamond t) = Pr(s \vDash \bigwedge_{u \in \mathrm{Post}^*(t)} \Box\Diamond u) \ .$$

$\Box$

## 4.2 Probabilistic Simulation

Here we define a *strong probabilistic simulation* between two DTMCs based on Segala's [16]. We also prove in terms of the game-theoretic probability that this simulation preserves which states a restricted PCTL holds in.

**Definition 4.2.1.** Let $X$ be a set. A function $\mu\colon X \to [0,1]$ is called a *distribution* on $X$ if

$$\sum_{x\in X} \mu(x) = 1 \ .$$

**Definition 4.2.2** (weight function)**.** Let $R \subseteq X \times Y$ be a relation between two sets $X, Y$, and let $\mu$ and $\mu'$ be distributions on $X$ and $Y$, respectively. A function $\delta\colon X \times Y \to [0,1]$ is a *weight function* for $\mu$ and $\mu'$ with respect to $R$ if:

- for each $x \in X$, $\sum_{y\in Y} \delta(x,y) = \mu(x)$,
- for each $y \in Y$. $\sum_{x\in X} \delta(x,y) = \mu'(y)$, and
- for each $(x,y) \in X \times Y$, if $\delta(x,y) > 0$ then $xRy$.

**Definition 4.2.3.** Let $R \subseteq X \times Y$ be a relation between two sets $X, Y$, and let $\mu$ and $\mu'$ be distributions on $X$ and $Y$, respectively. Then $\mu$ and $\mu'$ are in relation $\sqsubseteq_R$, written $\mu \sqsubseteq_R \mu'$, if there exists a weight function for $\mu$ and $\mu'$ with respect to $R$.

**Definition 4.2.4** (strong probabilistic simulation)**.** Let $D_1 = (S_1, P_1, \iota^1_{\text{init}}, AP, L_1)$ and $D_2 = (S_2, P_2, \iota^2_{\text{init}}, AP, L_2)$ be DTMCs. A *strong probabilistic simulation* between $D_1$ and $D_2$ is a relation $R \subseteq S_1 \times S_2$ such that

1. for each $(s,s') \in S_1 \times S_2$, if $sRs'$ then $L_1(s) \supseteq L_2(s')$, and
2. for each $(s,s') \in S_1 \times S_2$, if $sRs'$ then $\lambda t.P_1(s,t) \sqsubseteq_R \lambda t.P_2(s',t)$.

**Definition 4.2.5.** Let $R$ be a strong probabilistic simulation between two DTMCs $D_1, D_2$. Let $s_1$ and $s_2$ be states in $D_1$ and $D_2$, respectively. Assume that $s_1Rs_2$, $E_1 \subseteq \mathrm{Path}(s_1)$ and $E_2 \subseteq \mathrm{Path}(s_2)$, we define

- $s_1 \uparrow_R := \{s' \in S_2 \mid s_1Rs'\}$,
- $s_2 \downarrow_R := \{s \in S_1 \mid sRs_2\}$,
- $E_1 \uparrow_R^{s_2} := \{\xi^2 \in \mathrm{Path}(s_2) \mid \exists \xi^1 \in E_1. \forall n \in \mathbb{N}. \xi^1[n]R\xi^2[n]\}$ and
- $E_2 \downarrow_R^{s_1} := \{\xi^1 \in \mathrm{Path}(s_1) \mid \exists \xi^2 \in E_2. \forall n \in \mathbb{N}. \xi^1[n]R\xi^2[n]\}$.

For $B \subseteq S_1$ and $C \subseteq S_2$, we define

$$B \uparrow_R = \bigcup_{s\in B} s \uparrow_R \text{ and } C \downarrow_R = \bigcup_{s'\in C} s' \downarrow_R \ .$$

The game-theoretic proof of Lemma 4.2.6 is a main contribution in this section.

**Lemma 4.2.6.** *Let $R$ be a strong probabilistic simulation between two DTMCs $D_1, D_2$. Let $s_1$ and $s_2$ be states in $D_1$ and $D_2$, respectively. If $s_1Rs_2$, events $E \subseteq \mathrm{Path}(s_1)$ and $E \uparrow_R^{s_2} \subseteq \mathrm{Path}(s_2)$ satisfy that*

$$\overline{Pr_{s_1}}(E) \le \overline{Pr_{s_2}}(E \uparrow_R^{s_2}) \ .$$

**Proof.** Let $\mathcal{P}_2$ be a strategy in game for $D_2^{s_2}$ and $\xi^1 \in \mathrm{Path}(s_1)$ be a path. We shall construct a path $\xi^2 \in \mathrm{Path}(s_2)$, and Skeptic's strategy $\mathcal{P}_1$ in the game for $D_1^{s_1}$, in the following mutually inductive way. Assume that the path fragment

$\xi^2[0,n]$ has been already defined and that $f_n^{\mathcal{P}_2}$ is the function that Skeptic (who follows $\mathcal{P}_2$) announces in the situation $\xi^2[0,n]$. We define

$$\mathcal{P}_1(\xi^1[0,n])(s) := \begin{cases} \dfrac{\sum_{s' \in s\uparrow_R} \delta(s,s') f_n^{\mathcal{P}_2}(s')}{P_1(\xi^1[n],s)} & \text{if } P_1(\xi^1[n],s) > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta$ is a weight function for $\lambda t.P_1(\xi^1[n],t)$ and $\lambda t.P_2(\xi^2[n],t)$ with respect to $R$ (that exists due to Definition 4.2.4). Such $\mathcal{P}_1(\xi^1[0,n])$ is bounded. Then $f_n^{\mathcal{P}_1}$ denotes the function $f_n^{\mathcal{P}_1} = \mathcal{P}_1(\xi^1[0,n])$. Then we define the element $\xi^2[n+1]$ by:

$$\xi^2[n+1] := s' \text{ such that } s' \in (\xi^1[n+1]) \uparrow_R \text{ and } f_n^{\mathcal{P}_1}(\xi^1[n+1]) \geq f_n^{\mathcal{P}_2}(s') \ .$$

Such $s'$ exists: indeed, if $f_n^{\mathcal{P}_1}(\xi^1[n+1]) < f_n^{\mathcal{P}_2}(s')$ for each $s' \in (\xi^1[n+1]) \uparrow_R$, $f_n^{\mathcal{P}_1}(\xi^1[n+1]) < f_n^{\mathcal{P}_1}(\xi^1[n+1])$. Then $f_n^{\mathcal{P}_1}(\xi^1[n+1]) < f_n^{\mathcal{P}_2}(\xi^2[n+1])$ and

$$\sum_{s \in S_1} P_1(\xi^1[n], \xi^1[n+1]) f_n^{\mathcal{P}_1}(s) = \sum_{s' \in S_2} P_2(\xi^2[n], \xi^2[n+1]) f_n^{\mathcal{P}_2}(s') \ .$$

Since $\forall n \in \mathbb{N}.\, K^{\mathcal{P}_1}(\xi^1[0,n]) \geq K^{\mathcal{P}_2}(\xi^2[0,n])$,

$$\liminf_{n\to\infty} K^{\mathcal{P}_1}(\xi^1[0,n]) \geq \liminf_{n\to\infty} K_n^{\mathcal{P}_2}(\xi^2[0,n]) \ .$$

Moreover if $\xi^1 \in E$, then $\xi^2 \in E \uparrow_R^{s_2}$. □

The definition of $f_n^{\mathcal{P}_1}$ in this proof is regarded as the simulation following the ratio of the values of the weight function. It is also regarded as the division of the total bet money by translating $f_n^{\mathcal{P}_1}$ and $f_n^{\mathcal{P}_2}$ to $f_n^{\mathcal{P}_1}(s)/P_1(\xi^1[n],s)$ and $f_n^{\mathcal{P}_2}(s)/P_2(\xi^2[n],s)$, respectively.

**Lemma 4.2.7.** *Let $R$ be a strong probabilistic simulation between two DTMCs $D_1, D_2$. Let $s_1$ and $s_2$ be states in $D_1$ and $D_2$, respectively. If $s_1 R s_2$, events $E \subseteq \mathrm{Path}(s_2)$ and $E \downarrow_R^{s_1} \subseteq \mathrm{Path}(s_1)$ satisfy that*

$$\overline{Pr_{s_1}}(E \downarrow_R^{s_1}) \geq \overline{Pr_{s_2}}(E) \ .$$

**Proof.** Condition 1 of Definition 4.2.4 is not used in proof of Lemma 4.2.6. The relation $R^{-1}$ does not satisfy Condition 1 of Definition 4.2.4 in general but satisfies Condition 2 of Definition 4.2.4. Hence

$$\overline{Pr_{s_2}}(E) \leq \overline{Pr_{s_1}}(E \uparrow_{R^{-1}}^{s_2})$$

by Lemma 4.2.6. Since $E \uparrow_{R^{-1}}^{s_1} = E \downarrow_R^{s_1}$,

$$\overline{Pr_{s_1}}(E \downarrow_R^{s_1}) \geq \overline{Pr_{s_2}}(E) \ .$$

□

**Theorem 4.2.8.** *Let $R$ be a strong probabilistic simulation between two DTMCs $D_1, D_2$. Let $s_1$ and $s_2$ be states in $D_1$ and $D_2$, respectively. Assume that a PTCL state formula $\Phi$ and a PTCL path formula $\phi$ do not contain any occurrence of $\neg$. If $s_1 R s_2$ then*

$$s_2 \in \mathrm{Sat}^{D_2}(\Phi) \Rightarrow s_1 \in \mathrm{Sat}^{D_1}(\Phi) \ \textit{and}\ (\mathrm{Sat}_{s_2}^{D_2}(\phi)) \downarrow_R^{s_1} \subseteq \mathrm{Sat}_{s_1}^{D_1}(\phi) \ .$$

**Proof.** We proceed by induction on the structure of $\Phi$ and $\phi$.

*Case* $\Phi = \mathbb{P}_{\geq p}(\phi)$. If $s_2 \in \mathrm{Sat}^{D_2}(\Phi)$,

$$\overline{Pr_{s_2}}((\mathrm{Sat}^{D_2}_{s_2}(\phi)) = Pr_{s_2}((\mathrm{Sat}^{D_2}_{s_2}(\phi)) \geq p \ .$$

By Lemma 4.2.7

$$\overline{Pr_{s_1}}((\mathrm{Sat}^{D_2}_{s_2}(\phi)) \downarrow^{s_1}_{R}) \geq \overline{Pr_{s_2}}((\mathrm{Sat}^{D_2}_{s_2}(\phi)) \geq p \ .$$

Since $(\mathrm{Sat}^{D_2}_{s_2}(\phi)) \downarrow^{s_1}_{R} \subseteq \mathrm{Sat}^{D_1}_{s_1}(\phi)$ by induction hypothesis,

$$Pr_{s_1}(\mathrm{Sat}^{D_1}_{s_1}(\phi)) = \overline{Pr_{s_1}}(\mathrm{Sat}^{D_1}_{s_1}(\phi)) \geq \overline{Pr_{s_1}}((\mathrm{Sat}^{D_2}_{s_2}(\phi)) \downarrow^{s_1}_{R}) \geq p \ .$$

Hence $s_1 \in \mathrm{Sat}^{D_1}(\Phi)$.

*Case* $\Phi = \mathbb{P}_{>p}(\phi)$. Similar to the previous case.

The proofs of the other cases are obvious. $\square$

**Definition 4.2.9** (reward probabilistic simulation)**.** Let $D_1 = (S_1, P_1, \iota^1_{\mathrm{init}}, AP, L_1)$ and $D_2 = (S_2, P_2, \iota^2_{\mathrm{init}}, AP, L_2)$ be DTMCs. Let $rew_1$ and $rew_2$ be reward functions on $D_1$ and $D_2$, respectively. A strong probabilistic simulation $R$ between $D_1$ and $D_2$ is called a *reward probabilistic simulation* between two DTMRCs $(D_1, rew_1)$ and $(D_2, rew_2)$ if it satisfies the following condition.

- For each $(s, s') \in S_1 \times S_2$, if $sRs'$ then there exists a weight function $\delta$ for $\lambda t.P_1(s,t)$ and $\lambda t.P_2(s',t)$ with respect to $R$ such that

$$\forall t \in S_1.\, rew_1(s,t) \leq \frac{\sum_{t' \in S_2} \delta(t,t') rew_2(s',t')}{P_1(s,t)} \ .$$

**Lemma 4.2.10.** *Let $R$ be a reward probabilistic simulation between two DTMRCs $(D_1, rew_1), (D_2, rew_2)$. Let $s_1$ and $s_2$ be states in $D_1$ and $D_2$, respectively. If $s_1 R s_2$, a set $C \subseteq S_1$ of states in $D_1$ satisfies that*

$$\mathit{ExpRew}(s_1 \vDash \Diamond C) \leq \mathit{ExpRew}(s_2 \vDash \Diamond (C \uparrow_R)) \ .$$

**Proof.** This proof is similar to the proof of Lemma 4.2.6. By Proposition 3.2.5, for each $a \in \mathbb{R}$, there exists a strategy $\mathcal{P}^a_2$ for Skeptic in the game for $D^{s_2}_2$ such that for each path $\xi^2 \in \mathrm{Path}(s_2)$ in $D^{s_2}_2$,

$$\mathbb{E}_{s_2}[\lambda\xi.rew^a_2(\xi, \Diamond(C \uparrow_R))] \geq rew^a(\xi^2, \Diamond(C \uparrow_R)) - \liminf_{n\to\infty} K^{\mathcal{P}^a_2}(\xi^2[0,n]) \ .$$

Let $\xi^1 \in \mathrm{Path}(s_1)$ be a path in $D^{s_1}_1$. We shall construct a path $\xi^2 \in \mathrm{Path}(s_2)$, and Skeptic's strategy $\mathcal{P}^a_1$ in the game for $D^{s_1}_1$, in the following mutually inductive way. Assume that the path fragment $\xi^2[0,n]$ has been already defined and that $f^{\mathcal{P}^a_2}_n$ is the function that Skeptic who follows $\mathcal{P}^a_2$ announces in the situation of $\xi^2[0,n]$. We define

$$\mathcal{P}^a_1(\xi^1[0,n])(s) := \begin{cases} \dfrac{\sum_{s' \in s\uparrow_R} \delta(s,s') f^{\mathcal{P}^a_2}_n(s')}{P_1(\xi^1[n], s)} & \text{if } P_1(\xi^1[n], s) > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta$ is a weight function for $\lambda t.P_1(\xi^1[n], t)$ and $\lambda t.P_2(\xi^2[n], t)$ with respect to $R$ such that satisfies the condition of Definition 4.2.9 (that exists due to the definition of reward probabilistic simulation). Such $\mathcal{P}^a_1(\xi^1[0,n])$ is bounded. Then

$f_n^{\mathcal{P}_1^a}$ denotes the function $f_n^{\mathcal{P}_1^a} = \mathcal{P}_1^a(\xi^1[0,n])$. Then we define the element $\xi^2[n+1] := s'$ such that

$s' \in (\xi^1[n+1]) \uparrow_R$ and $rew_1(\xi^1[n], \xi^1[n+1]) - f_n^{\mathcal{P}_1^a}(\xi^1[n+1]) \leq rew_2(\xi^2[n], s') - f_n^{\mathcal{P}_2^a}(s')$

if $\xi^1[i] \notin C$ and for each $i < n$ and $rew_2(\xi^2[0,n], \Diamond(C \downarrow_R)) < a$, and

$$s' \in (\xi^1[n+1]) \uparrow_R \text{ and } f_n^{\mathcal{P}_1^a}(\xi^1[n+1]) \geq f_n^{\mathcal{P}_2^a}(s')$$

otherwise. Such $s$ exists by Definition 4.2.9 and the definition of $f_n^{\mathcal{P}_2^a}$. Then we defined $rew_1(a, \xi^1[0,n])$ inductively by

$$rew_1(a, \xi^1[0,n]) := \begin{cases} 0 & \text{if } n = 0 \\ rew_1(a, \xi^1[0,n-1]) + rew_1(\xi^1[n-1], \xi^1[n]) & \text{if } n > 0,\ \forall i < n.\, \xi^1[i] \notin C \text{ and} \\ & rew_2(\xi^2[0,n], \Diamond(C \downarrow_R)) < a \\ rew_1(a, \xi^1[0,n-1]) & \text{otherwise.} \end{cases}$$

Let $rew_1(a, \xi^1) = \lim_{n\to\infty} rew_1(a, \xi^1[0,n])$. Since

$$\sum_{s \in S_1} P_1(\xi^1[n], \xi^1[n+1]) f_n^{\mathcal{P}_1^a}(s) = \sum_{s' \in S_2} P_2(\xi^2[n], \xi^2[n+1]) f_n^{\mathcal{P}_2^a}(s') \ ,$$

we can prove that for each $\xi^1 \in \mathrm{Path}(s_1)$

$$\begin{aligned} rew_1(a, \xi^1) - \liminf_{n\to\infty} K^{\mathcal{P}_1^a}(\xi^1[0,n]) &\leq rew_2^a(\xi^2, \Diamond(C \downarrow_R)) - \liminf_{n\to\infty} K^{\mathcal{P}_2^a}(\xi^2[0,n]) \\ &\leq \mathbb{E}_{s_2}[\lambda\xi. rew_2^a(\xi, \Diamond(C \downarrow_R))] \ . \end{aligned}$$

Hence

$$\mathbb{E}_{s_1}[\lambda\xi. rew_1(a, \xi)] \leq \mathbb{E}_{s_2}[\lambda\xi. rew_2^a(\xi, \Diamond(C \downarrow_R))] \ .$$

Then

$$\lim_{a\to\infty} \mathbb{E}_{s_1}[\lambda\xi. rew_1(a, \xi)] = \mathit{ExpRew}(s_1 \vDash \Diamond C)$$

and

$$\lim_{a\to\infty} \mathbb{E}_{s_2}[\lambda\xi. rew_2^a(\xi, \Diamond(C \downarrow_R))] = \mathit{ExpRew}(s_2 \vDash \Diamond(C \downarrow_R)) \ .$$

Hence

$$\mathit{ExpRew}(s_1 \vDash \Diamond C) \leq \mathit{ExpRew}(s_2 \vDash \Diamond(C \downarrow_R)) \ .$$

□

**Lemma 4.2.11.** *Let $R$ be a reward probabilistic simulation between two DTMRCs $(D_1, rew_1), (D_2, rew_2)$. Let $s_1$ and $s_2$ be states in $D_1$ and $D_2$, respectively. If $s_1 R s_2$, a set $C \subseteq S_2$ of states in $D_2$ satisfies that*

$$\mathit{ExpRew}(s_1 \vDash \Diamond(C \downarrow_R)) \leq \mathit{ExpRew}(s_2 \vDash \Diamond C) \ .$$

**Proof.** By Lemma 4.2.10,

$$\mathit{ExpRew}(s_1 \vDash \Diamond(C \downarrow_R)) \leq \mathit{ExpRew}(s_2 \vDash \Diamond((C \downarrow_R) \uparrow_R)) \ .$$

Since $C \subseteq (C \downarrow_R) \uparrow_R$,

$$\mathit{ExpRew}(s_2 \vDash \Diamond((C \downarrow_R) \uparrow_R)) \leq \mathit{ExpRew}(s_2 \vDash \Diamond C) \ .$$

□

**Theorem 4.2.12.** *Let $R$ be a reward probabilistic simulation between two DTMRCs $DR_1, DR_2$. Let $s_1$ and $s_2$ be states in $DR_1$ and $DR_2$, respectively. Assume that a PRCTL state formula $\Phi$ and a PRCTL path formula $\phi$ do not contain any occurrence of $\neg$ and $\mathrm{U}_{\leq r}$. If $s_1 R s_2$ then*

$$s_2 \in \mathrm{Sat}^{DR_2}(\Phi) \Rightarrow s_1 \in \mathrm{Sat}^{DR_1}(\Phi) \text{ and } (\mathrm{Sat}^{DR_2}_{s_2}(\phi)) \downarrow^{s_1}_{R} \subseteq \mathrm{Sat}^{DR_1}_{s_1}(\phi) \ .$$

**Proof.** We proceed by induction on the structure of $\Phi$ and $\phi$. Since $R$ is a strong probabilistic simulation, the proofs of the cace $\Phi = \mathbb{E}_{\leq r}(\Phi')$ and the cace $\Phi = \mathbb{E}_{<r}(\Phi')$ suffice for the proof of this theorem.

*Case* $\Phi = \mathbb{E}_{\leq r}(\Phi')$. If $s_2 \in \mathrm{Sat}^{D_2}(\Phi)$,

$$\mathit{ExpRew}(s_2 \vDash \Diamond(\mathrm{Sat}^{DR_2}(\Phi')) \leq r \ .$$

By Lemma 4.2.11

$$\mathit{ExpRew}(s_1 \vDash \Diamond(\mathrm{Sat}^{DR_2}(\Phi') \downarrow_R)) \leq \mathit{ExpRew}(s_2 \vDash \Diamond(\mathrm{Sat}^{DR_2}(\Phi')) \leq r \ .$$

Since $\mathrm{Sat}^{DR_2}(\Phi') \downarrow_R \subseteq \mathrm{Sat}^{DR_1}(\Phi')$ by induction hypothesis,

$$\mathit{ExpRew}(s_1 \vDash \Diamond(\mathrm{Sat}^{DR_1}(\Phi')) \leq \mathit{ExpRew}(s_1 \vDash \Diamond(\mathrm{Sat}^{DR_2}(\Phi') \downarrow_R)) \leq r \ .$$

Hence $s_1 \in \mathrm{Sat}^{D_1}(\Phi)$.

*Case* $\Phi = \mathbb{E}_{<r}(\Phi')$. Similar to the previous case. □

**Theorem 4.2.13.** *Let $R$ be a probabilistic simulation between two DTMRCs $D_1, D_2$. Let $s_1$ and $s_2$ be states in $D_1$ and $D_2$, respectively. Suppose that $DR_1 = (D_1, \mathit{rew}_1)$ and $DR_2 = (D_2, \mathit{rew}_2)$ are DTMRCs and that $sRs' \wedge tRt'\mathit{rew}_1 \Rightarrow (s,t) \leq \mathit{rew}_2(s',t')$ for each $s,t \in S_1$ and for each $s',t' \in S_2$. Assume that a PRCTL state formula $\Phi$ and a PRCTL path formula $\phi$ do not contain any occurrence of $\neg$. If $s_1 R s_2$ then*

$$s_2 \in \mathrm{Sat}^{DR_2}(\Phi) \Rightarrow s_1 \in \mathrm{Sat}^{DR_1}(\Phi) \text{ and } (\mathrm{Sat}^{DR_2}_{s_2}(\phi)) \downarrow^{s_1}_{R} \subseteq \mathrm{Sat}^{DR_1}_{s_1}(\phi) \ .$$

**Proof.** We proceed by induction on the structure of $\Phi$ and $\phi$. Since $R$ is a reward probabilistic simulation, the proof of the cace $\phi = \phi_1 \,\mathrm{U}_{\leq r}\, \phi_2$ suffices for the proof of this theorem and it is obvious. □

# Chapter 5

# Conclusions and Future Work

## 5.1 Conclusions

We have translated DTMCs in terms of the game-theoretic probability. In this game, Skeptic bets some money for each state, gets the bet money multiplied by the odds and loses the total bet money. Since the state of a DTMC changes step by step, the translation of DTMCs to the game seems to be natural.

In this thesis, we have proved Lemma 4.1.1 in terms of the game for DTMCs in order to prove the fairness theorem. In the proof of Lemma 4.1.1, we have only used the simple strategy that Skeptic does not bet on the state that is not visited infinitely often.

We have also described the probabilistic simulation based on Segala's. In terms of the game for DTMCs, weight functions have intuitive meaning. That is, the ratio of the values of a weight function is regarded as the ratio of division of bet money. Following this idea, we have defined a reward probabilistic simulation.

## 5.2 Future Work

In this thesis, DTMCs are translated to the games. Hence we can perhaps translate other variations of Markov chains to games. For example, translation of Markov decision processes (MDPs) [4] — transition systems that have both non-deterministic and probabilistic choices — or continuous-time Markov chains (CTMCs) [2] — whose time is expressed by real numbers — to games is included in future work.

In this thesis, we have only focused on probabilities or expected values. We can focus on other values in probabilistic systems, for example, variance. Future work also includes the application of the game-theoretic probability to making models to check properties about variance.

The game-theoretic probability may be used to model open systems or quantum systems. We believe that the game-theoretic probability can be applied to verification of open systems or quantum systems.

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