Categories of Filters as Fibered Completions

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Butz '04, “Saturated models of intuitionistic theories”

- Filter logics
  - $\mathbb{B} \hookrightarrow \mathbf{F}\mathbb{B}$
    satisfies a saturation principle

Models with saturation principles

- ultrafilter construction (classical)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, tail, "\prod_{\mathcal{U}}"] & \mathbf{Set} \\
S \arrow[r, mapsto] & \prod_{\mathcal{U}} S
\end{tikzcd}$$

- filter construction [Pitts '83][Palmgren '97] (intuitionistic)

$$\begin{tikzcd}
\mathbb{B} \arrow[r, hook] & \mathbf{F}\mathbb{B} \arrow[r, hook] & \mathbf{Sh}(\mathbf{F}\mathbb{B}) \\
B \arrow[rr, mapsto] & & \prod_{(-)} B
\end{tikzcd}$$

- [Butz '04]

$$\begin{tikzcd}
\mathbb{B} \arrow[r, hook] & \mathbf{F}\mathbb{B}
\end{tikzcd}$$

Blass '74, “Two closed categories of filters”

- $\mathbf{Filt}(\mathbb{B})$, $\mathbf{F}\mathbb{B}$

- $\mathbf{Filt}(\mathbb{B})$, $\mathbf{F}\mathbb{B}$

Question

Why $\mathbf{F}\mathbb{B}$ has good properties?

Why not $\mathbf{Filt}(\mathbb{B})$?

Answer

categorical models	$\subseteq$	fibrational models
$\mathbb{B}$		$\begin{array}{c}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{array}$
$\mathbb{B} \hookrightarrow \mathbf{F}\mathbb{B}$ [Butz '04]		$\begin{array}{ccc}\mathbf{Sub}(\mathbb{B}) & & \mathbf{Filt}(\mathbb{B})\\ \downarrow & \hookrightarrow & \downarrow\\ \mathbb{B} & & \mathbb{B}\end{array}$

- $\mathbf{F}\mathbb{B} \cong \mathbf{Filt}(\mathbb{B})[W^{-1}]$ (localization)

Overview

0. Introduction
1. Categories of filters [Koubek & Reiterman '70][Blass '74]
2. Categorical logic for filters [Butz '04]
3. Categorical models vs. fibrational models

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Filters of a semilattice

- L: (bounded) (meet-)semilattice

Definition

$\mathscr{F}$: **filter** of L

$\overset{\text{def.}}{\Longleftrightarrow}$ $\mathscr{F} \subseteq L$: upward closed subset s.t.

$$\begin{cases} \bullet\ \top \in \mathscr{F} \\ \bullet\ x \wedge y \in \mathscr{F} \text{ if } x \in \mathscr{F} \text{ and } y \in \mathscr{F} \end{cases}$$

$\mathscr{F} \leq \mathscr{G}$ $\overset{\text{def.}}{\Longleftrightarrow}$ $\mathscr{F} \supseteq \mathscr{G}$

Definition

$$\mathbf{F}L := \{\ \mathscr{F}\text{: filter of } L\ \} \cong \mathbf{SLat}(L, 2)^{\mathrm{op}}$$

cat. of semilattices

cat. of **complete** semilattices

Theorem

$$\begin{tikzcd}
\mathbf{SLat} \arrow[r, shift left=1.5, "\mathbf{F}"] \arrow[r, phantom, "\perp" description] & \bigwedge\text{-}\mathbf{SLat} \arrow[l, shift left=1.5]
\end{tikzcd}$$

$$\begin{tikzcd}
L \arrow[r, mapsto] & \mathbf{F}L \\
M & M \arrow[l, mapsto]
\end{tikzcd}$$

Filters on an object

- $\mathbb{B}$: category with pullbacks

Definition

$\mathscr{F}$: **filter** on $I \in \mathbb{B}$

$\overset{\text{def.}}{\iff}$ $\mathscr{F}$: filter of $\mathbf{Sub}_{\mathbb{B}}(I)$

Definition

$$\mathrm{Filt}_{\mathbb{B}}(I) := \{\ \mathscr{F}\text{: filter on } I\ \}$$

$$= \mathbf{F}(\mathrm{Sub}_{\mathbb{B}}(I))$$

$$\begin{tikzcd}
\mathbb{B}^{\mathrm{op}} \arrow[r, "\mathrm{Sub}_{\mathbb{B}}"] \arrow[rr, bend left, "\mathrm{Filt}_{\mathbb{B}}"] & \mathbf{SLat} \arrow[r, shift left, "\mathbf{F}"] \arrow[r, phantom, "\perp"] & \bigwedge\text{-}\mathbf{SLat} \arrow[l, shift left, hook]
\end{tikzcd}$$

Two categories of filters

[Blass, '74]

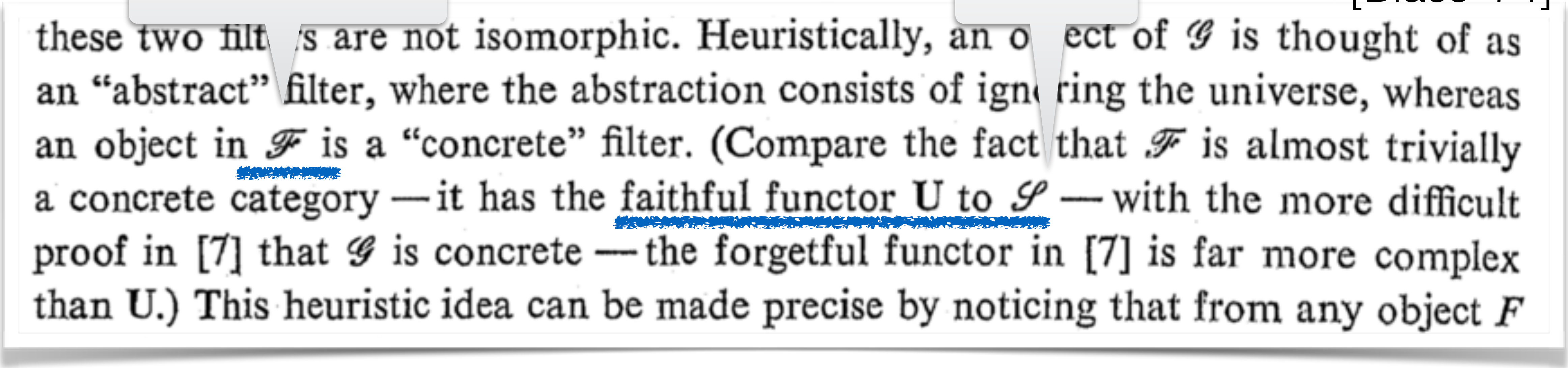
- The category of **concrete** filters $\mathbf{Filt}(\mathbb{B})$
- The category of **abstract** filters $\mathbf{F}\mathbb{B}$

Cat. of concrete filters

Filt$(\mathbb{B})$

- object $(I, \mathscr{F})$ $(I \in \mathbb{B}, \mathscr{F}$: filter on $I)$
- morphism

$$\frac{u\colon (I,\mathcal{F}) \to (J,\mathcal{G}) \quad \textbf{in Filt}(\mathbb{B})}{\begin{array}{c} u\colon I \to J \quad \textbf{in } \mathbb{B} \\ \forall Y \in \mathcal{G}.\, u^{-1}Y \in \mathcal{F} \end{array}}$$



Filt(Set)

Set

[Blass '74]

these two filt s are not isomorphic. Heuristically, an o ect of $\mathscr{G}$ is thought of as an "abstract" filter, where the abstraction consists of ign ring the universe, whereas an object in $\mathscr{F}$ is a "concrete" filter. (Compare the fact that $\mathscr{F}$ is almost trivially a concrete category — it has the faithful functor U to $\mathscr{S}$ — with the more difficult proof in [7] that $\mathscr{G}$ is concrete — the forgetful functor in [7] is far more complex than U.) This heuristic idea can be made precise by noticing that from any object F

Lemma

$$\begin{tikzcd} \mathbf{Filt}(\mathbb{B}) \arrow[d] \\ \mathbb{B} \end{tikzcd}$$ is the Grothendieck construction

from the functor $\mathbf{Filt}_{\mathbb{B}} : \mathbb{B}^{\mathrm{op}} \to \bigwedge\text{-}\mathbf{SLat}$.

Cat. of abstract filters

$\mathbf{F}\mathbb{B}$

- object $(I, \mathscr{F})$ $(I \in \mathbb{B}$, $\mathscr{F}$: filter on $I)$
- morphism $[v]\colon (I, \mathcal{F}) \to (J, \mathcal{G})$ in $\mathbf{F}\mathbb{B}$

  is defined as

  $\mathcal{F} \ni X \overset{v}{\to} J$ in $\mathbb{B}$

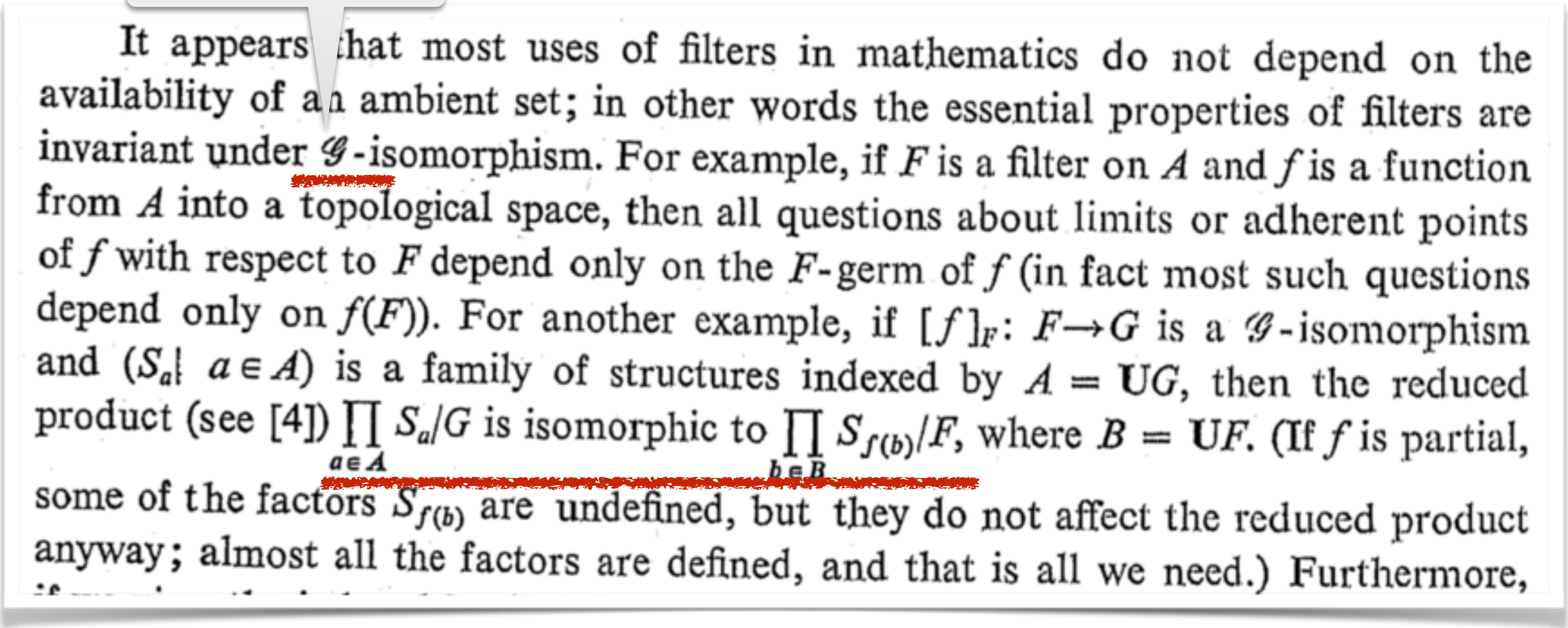
  $\forall Y \in \mathcal{G}.\, v^{-1}Y \in \mathcal{F}$ (under $v^{-1}Y \subseteq X \subseteq I$)

  under

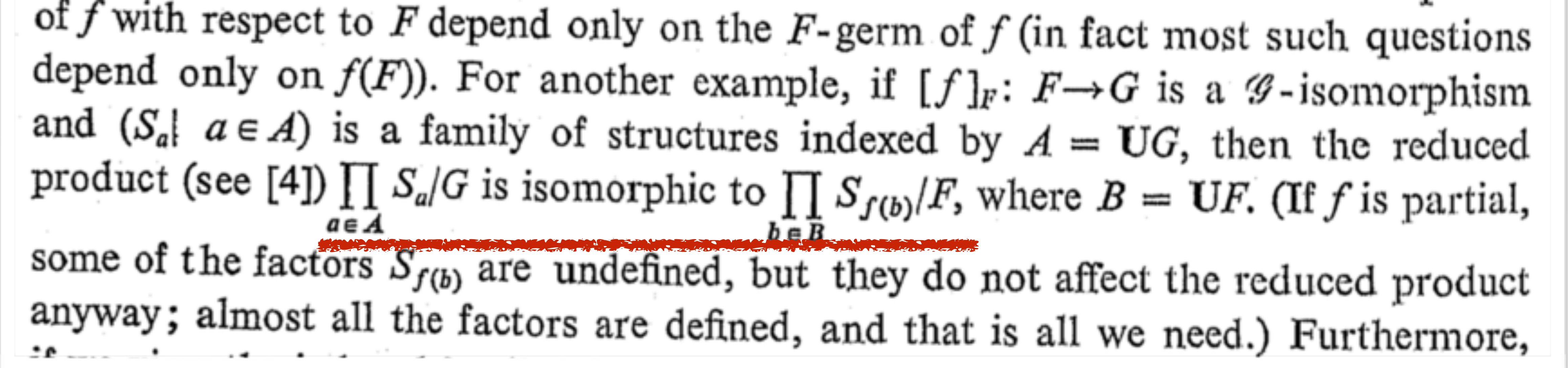
  $[v] = [v'] \overset{\text{def.}}{\iff} v|_{X''} = v'|_{X''}\ (X'' \subseteq X \cap X')$

F(Set)

[Blass '74]



It appears that most uses of filters in mathematics do not depend on the availability of an ambient set; in other words the essential properties of filters are invariant under $\mathscr{G}$-isomorphism. For example, if F is a filter on A and f is a function from A into a topological space, then all questions about limits or adherent points of f with respect to F depend only on the F-germ of f (in fact most such questions depend only on $f(F)$). For another example, if $[f]_F\colon F\to G$ is a $\mathscr{G}$-isomorphism and $(S_a \mid a\in A)$ is a family of structures indexed by $A = \mathrm{U}G$, then the reduced product (see [4]) $\prod_{a\in A} S_a/G$ is isomorphic to $\prod_{b\in B} S_{f(b)}/F$, where $B = \mathrm{U}F$. (If f is partial, some of the factors $S_{f(b)}$ are undefined, but they do not affect the reduced product anyway; almost all the factors are defined, and that is all we need.) Furthermore,



of f with respect to F depend only on the F-germ of f (in fact most such questions depend only on $f(F)$). For another example, if $[f]_F \colon F \to G$ is a $\mathcal{G}$-isomorphism and $(S_a \mid a \in A)$ is a family of structures indexed by $A = UG$, then the reduced product (see [4]) $\prod_{a \in A} S_a/G$ is isomorphic to $\prod_{b \in B} S_{f(b)}/F$, where $B = UF$. (If f is partial, some of the factors $S_{f(b)}$ are undefined, but they do not affect the reduced product anyway; almost all the factors are defined, and that is all we need.) Furthermore,

Definition

$$\prod_{\mathcal{F}} = \operatorname*{Colim}_{X \in \mathcal{F}} \mathbb{B}(X, -) \qquad : \mathbb{B} \to \mathbf{Set}$$

is called the **reduced product**

Lemma

[K. ?]

$\prod_{(\_)} \colon \mathbf{F}\mathbb{B} \to (\mathbf{Set}^{\mathbb{B}})^{\mathrm{op}}$: fully faithful

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2. Categorical logic for filters [Butz '04]
3. Categorical models vs. fibrational models

First-order logic and its fragments

terms $t ::= x \mid f(t_1, \ldots, t_{|f|})$

formulas $\varphi ::= R(t_1, \ldots, t_{|R|}) \mid t_1 = t_2$

$\mid \top \mid \varphi_1 \wedge \varphi_2$

“left exact logic”

$\mid \exists x. \varphi$

regular logic

$\mid \bot \mid \varphi_1 \vee \varphi_2$

coherent logic

$\mid \varphi_1 \Rightarrow \varphi_2 \mid \forall x. \varphi$

Categorical models

- $\mathbb{B}$: category with finite products
- interpretation in $\mathbb{B}$
    - types $[\![\sigma]\!] \in \mathbb{B}$
    - function symbols $[\![f]\!] \in \mathbb{B}([\![\sigma]\!]_1 \times \ldots \times [\![\sigma]\!]_{|f|})$
    - relation symbols $[\![R]\!] \in \mathbf{Sub}_{\mathbb{B}}([\![\sigma]\!]_1 \times \ldots \times [\![\sigma]\!]_{|R|})$

- $\mathbb{B}$: left exact category ⇒ $\mathbb{B}$ models left exact logics
(has finite limits)
- $\mathbb{B}$: regular category ⇒ $\mathbb{B}$ models regular logics
(lex. & has p.b.-stable
coequalizers of kernel pairs)
- $\mathbb{B}$: coherent category ⇒ $\mathbb{B}$ models coherent logics
(reg. & has
p.b.-stable finite unions)
- $\mathbb{B}$: Heyting category ⇒ $\mathbb{B}$ models first-order logics

Filter logics [Butz '04]

- $\mathbf{F}\mathbb{B}$ models filter logics

$$\begin{aligned} t &::= \; x \mid f(t_1, \ldots, t_{|f|}) \\ \varphi &::= \; R(t_1, \ldots, t_{|R|}) \mid t_1 = t_2 \\ &\qquad \mid \top \mid \varphi_1 \wedge \varphi_2 \mid \bigwedge_{\varphi \in \Phi} \varphi \end{aligned}$$

“left exact **filter** logic”

$$\mid \exists x.\, \varphi$$

regular **filter** logic

$$\mid \bot \mid \varphi_1 \vee \varphi_2$$

coherent **filter** logic

$$\mid \varphi_1 \Rightarrow \varphi_2 \mid \forall x.\, \varphi$$

Filter logics [Butz '04]

$$\frac{}{\bigwedge_{\varphi\in\Phi}\varphi \vdash \varphi_0} \qquad \frac{\psi \vdash \varphi \quad \text{for each } \varphi \in \Phi}{\psi \vdash \bigwedge_{\varphi\in\Phi}\varphi}$$

$$\frac{\psi \vdash \exists x.\, \varphi_1 \wedge \cdots \wedge \varphi_n \quad \text{for each } \{\varphi_1, \ldots, \varphi_n\} \underset{\text{fin.}}{\subseteq} \Phi}{\psi \vdash \exists x.\, \bigwedge_{\varphi\in\Phi}\varphi}$$

saturation

$$\frac{}{\bigwedge_{\varphi\in\Phi}(\psi \vee \varphi) \vdash \psi \vee \bigwedge_{\varphi\in\Phi}\varphi}$$

Characterization of $\mathbf{F}\mathbb{B}$

Lemma

[Blass '74]

$\mathrm{Sub}_{\mathbf{F}\mathbb{B}}(I, \mathscr{F}) \cong \{ (I, \mathscr{G}) \mid \mathscr{G} \leq \mathscr{F} \}$

: complete (meet-)semilattice

Definition

[Butz '04]

$\mathbb{A}$: **filtered meet lex category**

$\overset{\text{def.}}{\iff}$ $\mathbb{A}$: lex category,

$\mathrm{Sub}_{\mathbb{A}} : \mathbb{A}^{\mathrm{op}} \to \bigwedge\text{-}\mathbf{SLat}$

Definition

[Butz '04]

$\mathbb{A}$: **filtered meet lex category**

$\overset{\text{def.}}{\Longleftrightarrow}$ $\mathbb{A}$: lex category,

$\mathrm{Sub}_{\mathbb{A}} \colon \mathbb{A}^{\mathrm{op}} \to \bigwedge\text{-}\mathbf{SLat}$

Theorem

[Butz '04]

$$\begin{tikzcd}
\mathbf{Lex} \arrow[r, shift left=1.5, "\mathbf{F}"] \arrow[r, phantom, "\perp"] & \bigwedge^{\mathrm{filt}}\text{-}\mathbf{Lex} \arrow[l, shift left=1.5] \\
\mathbb{B} \arrow[r, mapsto] & \mathbf{F}\mathbb{B} \\
\mathbb{A} & \mathbb{A} \arrow[l, mapsto]
\end{tikzcd}$$

$\mathbf{Lex}$: category of lex categories

$\bigwedge^{\mathrm{filt}}\text{-}\mathbf{Lex}$: category of filtered meet lex categories

Filtered meet vs. arbitrary meet

arbitrary meets $=$ finite meets + filtered meets

$$\bigwedge_{x \in X} x = \bigwedge_{\{x_1, \ldots, x_n\} \underset{\text{fin.}}{\subseteq} X} (x_1 \wedge \cdots \wedge x_n)$$

$$\exists x.\varphi(x) \wedge \psi(x) \vdash \big(\exists x.\varphi(x)\big) \wedge \big(\exists x.\psi(x)\big)$$

✗ ✗ ✓

arbitrary meets $=$ finite meets + filtered meets

$$\bigwedge_{x\in X} x = \bigwedge_{\{x_1,\dots,x_n\}\underset{\text{fin.}}{\subseteq} X} (x_1 \wedge \cdots \wedge x_n)$$

Definition

[Butz '04]

a **filtered meet regular** (resp. **coherent**) **category** is a regular (resp. coherent) category
with filtered meets s.t.
∃ (and ∨) distributes over filtered meets

Theorem

[Butz '04]

$$\begin{tikzcd} \mathbf{Lex} \arrow[r, shift left, "\mathbf{F}"] \arrow[r, phantom, "\perp"] & \bigwedge^{\text{filt}}\text{-}\mathbf{Lex} \arrow[l, shift left] \end{tikzcd}$$

restricts to

$$\begin{tikzcd} \mathbf{Reg} \arrow[r, shift left, "\mathbf{F}"] \arrow[r, phantom, "\perp"] & \bigwedge^{\text{filt}}\text{-}\mathbf{Reg} \arrow[l, shift left] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbf{Coh} \arrow[r, shift left, "\mathbf{F}"] \arrow[r, phantom, "\perp"] & \bigwedge^{\text{filt}}\text{-}\mathbf{Coh} \arrow[l, shift left] \end{tikzcd}$$

Filter logics [Butz '04]

$$\frac{}{\bigwedge_{\varphi \in \Phi} \varphi \vdash \varphi_0} \qquad \frac{\psi \vdash \varphi \quad \text{for each } \varphi \in \Phi}{\psi \vdash \bigwedge_{\varphi \in \Phi} \varphi}$$

$$\frac{\psi \vdash \exists x.\, \varphi_1 \wedge \cdots \wedge \varphi_n \quad \text{for each } \{\varphi_1, \ldots, \varphi_n\} \underset{\text{fin.}}{\subseteq} \Phi}{\psi \vdash \exists x.\, \bigwedge_{\varphi \in \Phi} \varphi}$$

$$\frac{}{\bigwedge_{\varphi \in \Phi} (\psi \vee \varphi) \vdash \psi \vee \bigwedge_{\varphi \in \Phi} \varphi}$$

distributive laws

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Fibrational models

- $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$: fibration, $\mathbb{B}$ has finite products
  - $[\![R]\!] \in \mathbb{E}_{[\![\sigma]\!]_1 \times \ldots \times [\![\sigma]\!]_{|R|}}$

- Generalization of <u>categorical model</u> $\mathbb{B}$

(subobject model)

$\begin{smallmatrix}\mathbf{Sub}(\mathbb{B})\\ \downarrow\\ \mathbb{B}\end{smallmatrix}$ for $\mathbb{B}$: lex category

$$[\![R]\!] \in \big(\mathbf{Sub}(\mathbb{B})\big)_{[\![\sigma_1]\!]\times\cdots\times[\![\sigma_{|R|}]\!]} = \mathbf{Sub}_{\mathbb{B}}([\![\sigma_1]\!] \times \cdots \times [\![\sigma_{|R|}]\!])$$

Fibered completion

- $\begin{tikzcd}\mathbf{Filt}(\mathbb{B}) \arrow[d] \\ \mathbb{B}\end{tikzcd}$ is the fibered completion of $\begin{tikzcd}\mathbf{Sub}(\mathbb{B}) \arrow[d] \\ \mathbb{B}\end{tikzcd}$

recall

$$\begin{tikzcd}
\mathbb{B}^{\mathrm{op}} \arrow[r, "\mathrm{Sub}_{\mathbb{B}}"] \arrow[rr, bend left, "\mathrm{Filt}_{\mathbb{B}}"] & \mathbf{SLat} \arrow[r, shift left, "\mathbf{F}"] \arrow[r, phantom, "\perp"] & \bigwedge\text{-}\mathbf{SLat} \arrow[l, shift left]
\end{tikzcd}$$

Given

$\mathbb{B}$: categorical model of
a (lex/regular/coherent) logic

[Butz '04]

- $\mathbf{F}\mathbb{B}$ is the “free” (categorical) model of its filter logic

[K.]

- $\begin{array}{c}\mathbf{Filt}(\mathbb{B})\\ \downarrow\\ \mathbb{B}\end{array}$ is the “free” fibrational model its filter logic

Fibrations vs. categories

(General preicates vs. subobjects)

$$\begin{tikzcd}
\mathbf{Sub}(\mathbb{B}) \arrow[r] \arrow[d] \arrow[rr, bend left] & \mathbf{Filt}(\mathbb{B}) \arrow[r, dashed] \arrow[d] & \mathbf{Sub}(\mathbf{F}\mathbb{B}) \arrow[d] \\
\mathbb{B} \arrow[r, "\mathrm{id}"] \arrow[rr, bend right] & \mathbb{B} \arrow[r, dashed] & \mathbf{F}\mathbb{B}
\end{tikzcd}$$

Coproducts over monomorphisms

$$\begin{tikzcd} X \arrow[r, two heads, "\cong"] \arrow[d, tail] & (\exists m)X \arrow[d, tail] \\ I \arrow[r, tail, "m"] & J \end{tikzcd}$$

$$\begin{array}{c} \mathbf{Sub}(\mathbb{B}) \\ \downarrow \\ \mathbb{B} \end{array}$$ has

$$\left.\begin{array}{l} \exists m \dashv m^* \text{ for each monomorphism } m \text{ in } \mathbb{B} \\ \text{satisfying the Beck-Chevalley condition} \end{array}\right\}$$

$\exists m$: **coproduct**

and the Frobenius reciprocity

Coproducts over monomorphisms

Lemma

[K.]

$$\begin{pmatrix} \mathbf{Sub}(\mathbb{B}) \\ \downarrow \\ \mathbb{B} \end{pmatrix} \xrightarrow{\begin{pmatrix} K \\ H \end{pmatrix}} \begin{pmatrix} \mathbf{Sub}(\mathbb{A}) \\ \downarrow \\ \mathbb{A} \end{pmatrix}$$

: morphism of fibrations preserving (1, ×, ⊤, ∧)

$K = \mathbf{Sub}(H) \;\Leftrightarrow\; \begin{pmatrix} K \\ H \end{pmatrix}$ preserves $\exists m$ for m: mono

Left exact fibrations

[K.]

Definition

A **left exact fibration**

is a fibered poset $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ s.t.

- $\mathbb{B}$ has finite limits
- p has fibered finite meets
- p has coproducts over monomorphisms satisfying Frobenius

Theorem

[K.]

$$\begin{tikzcd} \mathbf{Lex} \arrow[r, hook, "\perp", "\perp"'] & \mathbf{LexFib} \arrow[l, shift right=3ex, "\mathbf{L}"'] \arrow[l, shift left=3ex, "\mathbf{R}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{E}[W^{-1}] & \begin{array}{c}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{array} \arrow[l, mapsto, "\mathbf{L}"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{A} \arrow[r, mapsto] & \begin{array}{c}\mathbf{Sub}(\mathbb{A})\\ \downarrow\\ \mathbb{A}\end{array} \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{B} & \begin{array}{c}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{array} \arrow[l, mapsto, "\mathbf{R}"'] \end{tikzcd}$$

where $W = \{(X \to (\exists m)X \mid m\colon I \rightarrowtail J, X \in \mathbb{E}_I\}$

Localization of a category

- Q_W: $\mathbb{E} \to \mathbb{E}[W^{-1}]$ is universal among
  F: $\mathbb{E} \to \mathbb{D}$ s.t. $F(w)$: isom. for $w \in W$

Proof of the theorem (1)

$$\frac{H\colon \mathbb{A}\to\mathbb{B}\text{: lex}}{\begin{pmatrix}K\\ H\end{pmatrix} : \begin{pmatrix}\mathbf{Sub}(\mathbb{A})\\ \downarrow\\ \mathbb{A}\end{pmatrix} \to \begin{pmatrix}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{pmatrix}\text{: mor. of lex fibrations}}$$

$K(A') := (\exists i)\top_{A'}$ for $A' \in \mathbf{Sub}_{\mathbb{A}}(A)$

where $i\colon A' \rightarrowtail A$

$$\begin{tikzcd}
\mathbf{Lex} \arrow[r, hook] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathbf{LexFib} \arrow[l, shift right=4, "\mathbf{L}"'] \arrow[l, shift left=4, "\mathbf{R}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{E}[W^{-1}] & \begin{pmatrix}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{pmatrix} \arrow[l, mapsto, "\mathbf{L}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{A} \arrow[r, mapsto] & \begin{pmatrix}\mathbf{Sub}(\mathbb{A})\\ \downarrow\\ \mathbb{A}\end{pmatrix} \\
\mathbb{B} & \begin{pmatrix}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{pmatrix} \arrow[l, mapsto, "\mathbf{R}"']
\end{tikzcd}$$

Proof of the theorem (2)

$$\dfrac{\begin{pmatrix}K\\ H\end{pmatrix} : \begin{pmatrix}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{pmatrix} \to \begin{pmatrix}\mathbf{Sub}(\mathbb{A})\\ \downarrow{\color{blue}\mathrm{cod}}\\ \mathbb{A}\end{pmatrix} : \text{ mor. of lex fibrations}}{H\colon \mathbb{E}[W^{-1}] \to \mathbb{A}\text{: lex}}$$

where $W = \{(X \to (\exists m)X \mid m\colon I \rightarrowtail J, X \in \mathbb{E}_I\}$

$$\begin{tikzcd}
\mathbf{Lex} \arrow[r, hook, shift left=0] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathbf{LexFib} \arrow[l, shift right=4, "\mathbf{L}"'] \arrow[l, shift left=4, "\mathbf{R}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{E}[W^{-1}] & \begin{pmatrix}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{pmatrix} \arrow[l, mapsto, "\mathbf{L}"'] \\
\mathbb{A} \arrow[r, mapsto] & \begin{pmatrix}\mathbf{Sub}(\mathbb{A})\\ \downarrow\\ \mathbb{A}\end{pmatrix}
\end{tikzcd}$$

$$M' := (\mathbb{E} \xrightarrow{K} \mathbf{Sub}(\mathbb{A}) \xrightarrow{{\color{red}\mathrm{dom}}} \mathbb{A})$$

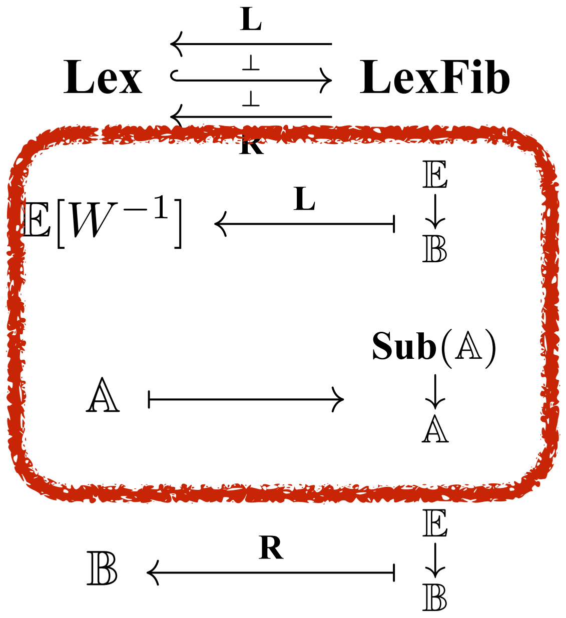
M' induces M by the universality

- For $w \in W$, $M'w$ is isom. (because Kw is opCartesian lifting, too)

Proof of the theorem (3)

$$\dfrac{H\colon \mathbb{E}[W^{-1}] \to \mathbb{A}\colon \text{lex}}{\begin{pmatrix}K\\ H\end{pmatrix}\colon \begin{pmatrix}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{pmatrix} \to \begin{pmatrix}\mathbf{Sub}(\mathbb{A})\\ \downarrow\\ \mathbb{A}\end{pmatrix}\colon \text{mor. of lex fibrations}}$$

where $W = \{(X \to (\exists m)X \mid m\colon I \rightarrowtail J, X \in \mathbb{E}_I\}$



$$\begin{tikzcd}
\mathbf{Lex} \arrow[r, hook] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathbf{LexFib} \arrow[l, shift right=6, "\mathbf{L}"'] \arrow[l, shift left=6, "\mathbf{R}"] \\
\mathbb{E}[W^{-1}] & \begin{matrix}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{matrix} \arrow[l, mapsto, "\mathbf{L}"'] \\
\mathbb{A} \arrow[r, mapsto] & \begin{matrix}\mathbf{Sub}(\mathbb{A})\\ \downarrow\\ \mathbb{A}\end{matrix} \\
\mathbb{B} & \begin{matrix}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{matrix} \arrow[l, mapsto, "\mathbf{R}"']
\end{tikzcd}$$

$M' := (\mathbb{E} \overset{Q_W}{\to} \mathbb{E}[W^{-1}] \overset{M}{\to} \mathbb{A})$

$H(I) := M'(\top_I)$ for $I \in \mathbb{B}$

$K(X) := M'X \in \mathbf{Sub}_{\mathbb{B}}(H(I))$ for $X \in \mathbb{E}_I$

fibered completion

Thm. [K.]

Cor. [K.]

$$\begin{tikzcd}
& \mathbf{LexFib} \arrow[r, shift left] \arrow[r, phantom, "\perp"] \arrow[dl, shift left, "\mathbf{R}"] \arrow[dl, phantom, "\perp"] & \bigwedge^{\text{filt}}\text{-}\mathbf{LexFib} \arrow[l, hook', shift left] \arrow[dr, shift left, "\mathbf{L}"] \arrow[dr, phantom, "\perp"] & \\
\mathbf{Lex} \arrow[ur, hook, shift left] \arrow[rrr, shift left, "\mathbf{F}"] \arrow[rrr, phantom, "\perp"] & & & \bigwedge^{\text{filt}}\text{-}\mathbf{Lex} \arrow[lll, hook', shift left] \arrow[ul, hook, shift left]
\end{tikzcd}$$

[Butz '04]

$$\begin{tikzcd}
& \begin{pmatrix} \mathbf{Sub}(\mathbb{B}) \\ \downarrow \\ \mathbb{B} \end{pmatrix} \arrow[r, mapsto] & \begin{pmatrix} \mathbf{Filt}(\mathbb{B}) \\ \downarrow \\ \mathbb{B} \end{pmatrix} \arrow[dr, mapsto] & \\
\mathbb{B} \arrow[ur, mapsto] & & & \mathbf{F}\mathbb{B}
\end{tikzcd}$$

Technical condition (*)

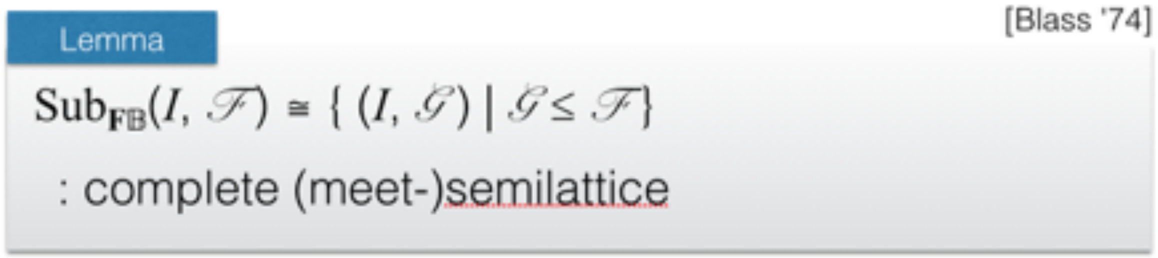
[K.]

Lemma

$$\mathbb{E}_I \hookrightarrow \mathbf{Sub}_{\mathbb{E}[W^{-1}]}(\top_I)$$

Condition (*)

The embedding is an isomorphism



[Blass '74]

Lemma

$\mathrm{Sub}_{\mathbb{FB}}(I, \mathscr{F}) \cong \{ (I, \mathscr{G}) \mid \mathscr{G} \leq \mathscr{F} \}$

: complete (meet-)semilattice

Theorem

[K.]

$$\begin{tikzcd} \mathbf{Lex} \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathbf{LexFib} \arrow[l, shift right, "\mathbf{L}"'] \end{tikzcd}$$

restricts to

$$\begin{tikzcd} \bigwedge^{\text{filt}}\text{-}\mathbf{Lex} \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \bigwedge^{\text{filt}}\text{-}\mathbf{LexFib} \arrow[l, hook, shift right, "\mathbf{L}"'] \end{tikzcd}$$

defined on filtered meet lex fibrations <u>satisfying (*)</u>

$$\begin{tikzcd} & \mathbf{LexFib} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] \arrow[dl, shift left, "\mathbf{R}"] & \bigwedge^{\text{filt}}\text{-}\mathbf{LexFib} \arrow[l, shift left] \arrow[dr, shift left, "\mathbf{L}"] & \\ \mathbf{Lex} \arrow[ur, hook, shift left] \arrow[ur, phantom, "\perp"] \arrow[rrr, shift left, "\mathbf{F}"] \arrow[rrr, phantom, "\perp"] & & & \bigwedge^{\text{filt}}\text{-}\mathbf{Lex} \arrow[lll, hook, shift left] \arrow[ul, hook, shift left] \arrow[ul, phantom, "\perp"] \end{tikzcd}$$

- Define **regular fibrations** and **coherent fibrations** as usual (e.g. [Jacobs '99])
  but we require base categories $\mathbb{B}$ have finite limits
  (not only finite products)

Theorem

[K.]

$\mathbf{L}$ restricts to $[\bigwedge^{\mathrm{filt}}\text{-}](\mathbf{Reg/Coh})\mathbf{Fib} \to [\bigwedge^{\mathrm{filt}}\text{-}](\mathbf{Reg/Coh})$

defined on fibrations satisfying (*)

$\mathbf{R}$ restricts to $(\mathbf{Reg/Coh})\mathbf{Fib} \to (\mathbf{Reg/Coh})$
defined on fibrations satisfying (*)

$$\begin{tikzcd}
& \mathbf{LexFib} \arrow[r, shift left] \arrow[r, phantom, "\perp"] \arrow[dl, shift left, "\mathbf{R}"] \arrow[dl, phantom, "\perp"] & \bigwedge^{\mathrm{filt}}\text{-}\mathbf{LexFib} \arrow[l, hook, shift left] \arrow[dr, shift left, "\mathbf{L}"] \arrow[dr, phantom, "\perp"] & \\
\mathbf{Lex} \arrow[ur, hook, shift left] \arrow[rrr, shift left, "\mathbf{F}"] \arrow[rrr, phantom, "\perp"] & & & \bigwedge^{\mathrm{filt}}\text{-}\mathbf{Lex} \arrow[ul, hook, shift left] \arrow[lll, hook, shift left]
\end{tikzcd}$$

Related work (1)

- $\mathrm{FRel}(p)$: category of functional relations
  FRel: $\mathbf{RegFib} \to \mathbf{Reg}$
  - objects $X \in \mathbb{E}$ i.e. a pair ($I \in \mathbb{B}$, $X \in \mathbb{E}_I$)
  - morphisms $R \in \mathbb{E}_{I \times J}$ s.t. “functional relation”

- The tripos-topos construction [Pitts '81]
  - objects ($I \in \mathbb{B}$, $X \in \mathbb{E}_{I \times I}$: “partial equivalence”)

Related work (2)

- Logical characterization of subobject fibrations

(e.g. [Jacobs '99])

  - Eq-fibration with strong equality
  - has full subset types
  - has unique-choice

Summary

$$\begin{tikzcd}
\text{categorical models} \arrow[r, hook, ""{name=M}] & \text{fibrational models} \arrow[l, bend right, "\mathbf{L}"'] \arrow[l, bend left, "\mathbf{R}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\text{categorical models} & \text{fibrational models} \arrow[l, shift right=6, "\mathbf{L}"'] \arrow[l, phantom, shift right=3, "\perp"] \\[-2.5em]
\end{tikzcd}$$

$$\mathbb{B} \hookrightarrow \mathbf{F}\mathbb{B}$$

[Butz '04]

$$\begin{tikzcd}
\mathbf{Sub}(\mathbb{B}) \arrow[d] & & \mathbf{Filt}(\mathbb{B}) \arrow[d] \\
\mathbb{B} & \hookrightarrow & \mathbb{B}
\end{tikzcd}$$

$\mathbf{F}\mathbb{B} \cong \mathbf{L}\left(\begin{array}{c}\mathbf{Filt}(\mathbb{B}) \\ \downarrow \\ \mathbb{B}\end{array}\right)$, which is a localization of $\mathbf{Filt}(\mathbb{B})$