Codensity Games for Bisimilarity

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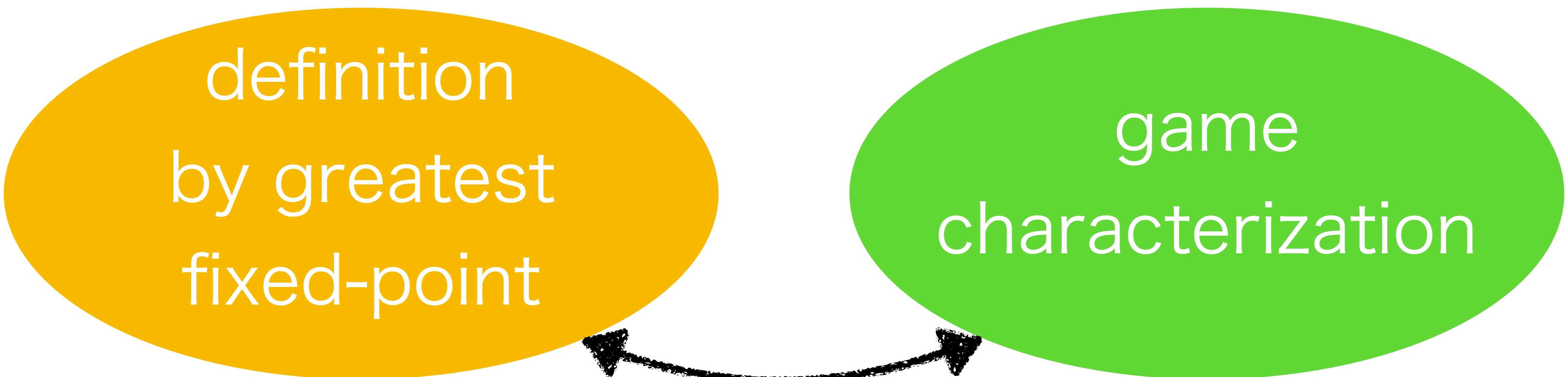
Background

Each <u>bisimilarity-like notion</u> for <u>coalgebras</u> separately has

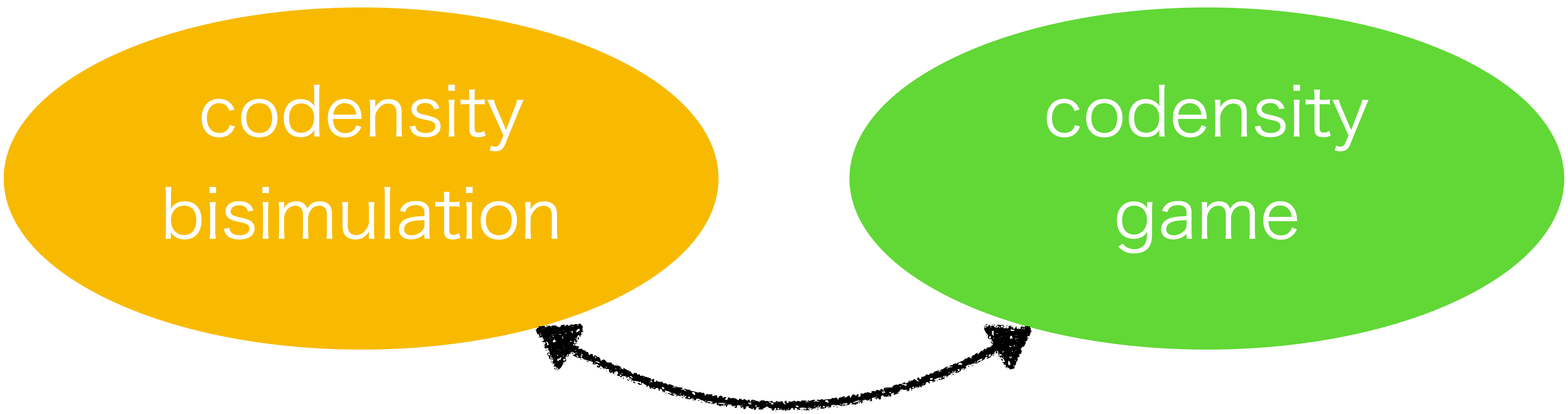


“GFP=game” theorem

- Bisimilarity for LTS [Park 1981][Milner 1989]
- Bisimilarity for Markov chains [Larsen & Skou 1991][Fijalkow+ ICALP2017]
- Bisimulation metric for Markov chains [Desharnais+ 2004][Desharnais+ 2008]

Contribution

- We give a general template of this picture:



general “GFP=game” theorem

- We use
  - fibrational coinduction [Hermida & Jacobs 1998]
  - codensity lifting [Katsumata & Sato CALCO15] [Sprunger+ CMCS18]

Background

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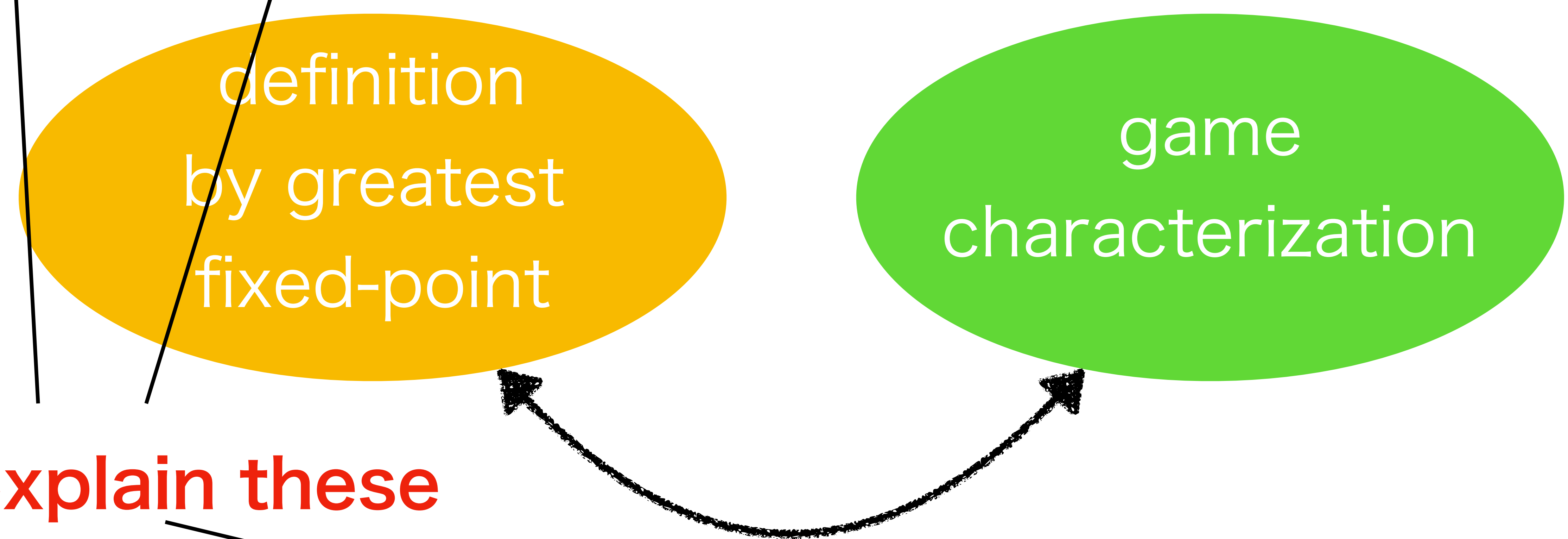

I'll explain these

"GFP=game" theorem

Coalgebra

$\mathbb{C}$: category F: $\mathbb{C} \to \mathbb{C}$

An F-coalgebra is a pair

$$(X \in \mathbb{C},\ t{:}X \to FX)$$

How states behave

We'll mainly consider $\mathbb{C}$=**Set**.

- $\mathcal{P}$-coalgebras = Kripke frames
- $\mathcal{D}$-coalgebras = Markov chains
- LTS, (non-deterministic/deterministic/weighted) automata, and many others

Bisimilarity-like notions

- Bisimilarity relation

  Equivalence rel. representing which states behave the same

- Bisimulation metric [Desharnais+,TCS318(3),2004]

  Pseudometric refining bisimilarity, used mainly for probabilistic systems

GFP definition: example

Let $t\colon X \to \mathcal{D}X$

Define $\Phi\colon 2^{X\times X} \to 2^{X\times X}$ (predicate transformer) by

$(x, y) \in \Phi(R)$

$\Leftrightarrow$ For any R-closed $Y \subseteq X$,

$t(x)(Y) = t(y)(Y)$

use "observations" $Y \subseteq X$ to distinguish states

Bisimilarity relation is $\nu\Phi$.

Game characterization

2 players 😇 👿;   😇 wins any infinite play

😇: I think $(x, y) \in \nu\Phi$.

👿: Is it true? If Y is $\nu\Phi$-closed, then $(x, y) \notin \nu\Phi$.

- From $(x, y) \in X \times X$

  👿 chooses $Y \subseteq X$ s.t. $t(x)(Y) \neq t(y)(Y)$

- From $Y \subseteq X$

  😇 chooses (x', y') s.t. $x' \in Y$ and $y' \notin Y$

😇: Don't worry. If $(x', y') \in \nu\Phi$, then Y is not $\nu\Phi$-closed.

GFP=game theorem: example

<u>Theorem</u> [Fijalkow+ ICALP2017]

$(x, y) \in \nu\Phi$

GFP definition

if and only if

😇 has a winning strategy

starting from $(x, y) \in X \times X$.

game characterization

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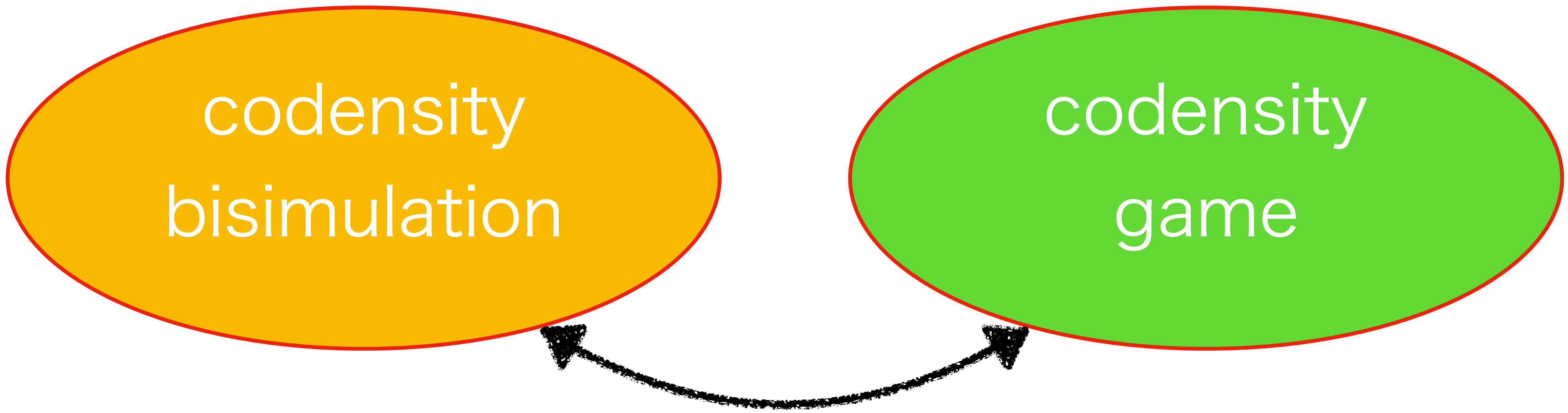


general "GFP=game" theorem

- We use
  - fibrational coinduction [Hermida & Jacobs 1998]
  - codensity lifting [Katsumata & Sato CALCO15] [Sprunger+ CMCS18]

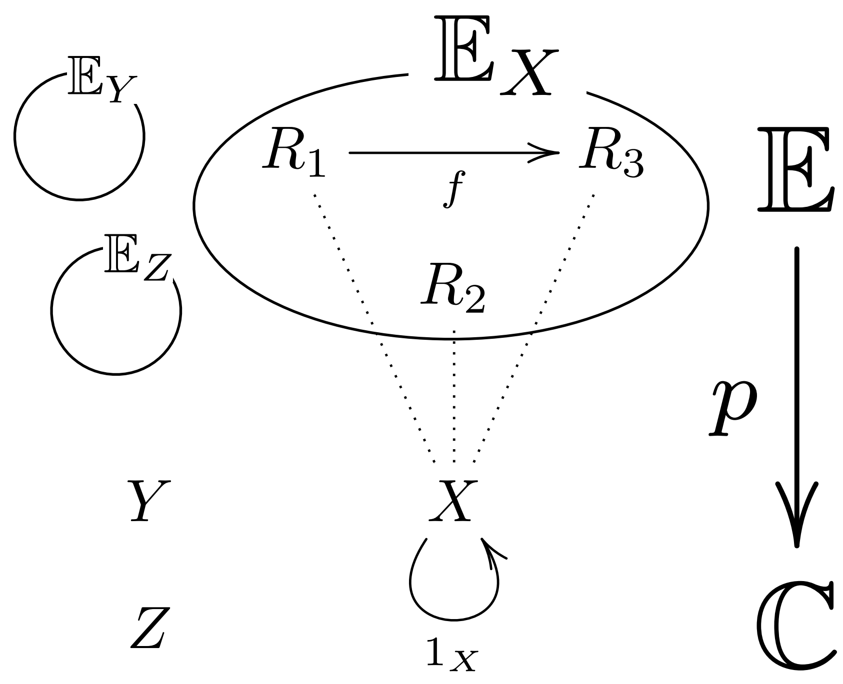
Fibrations

I'll explain later

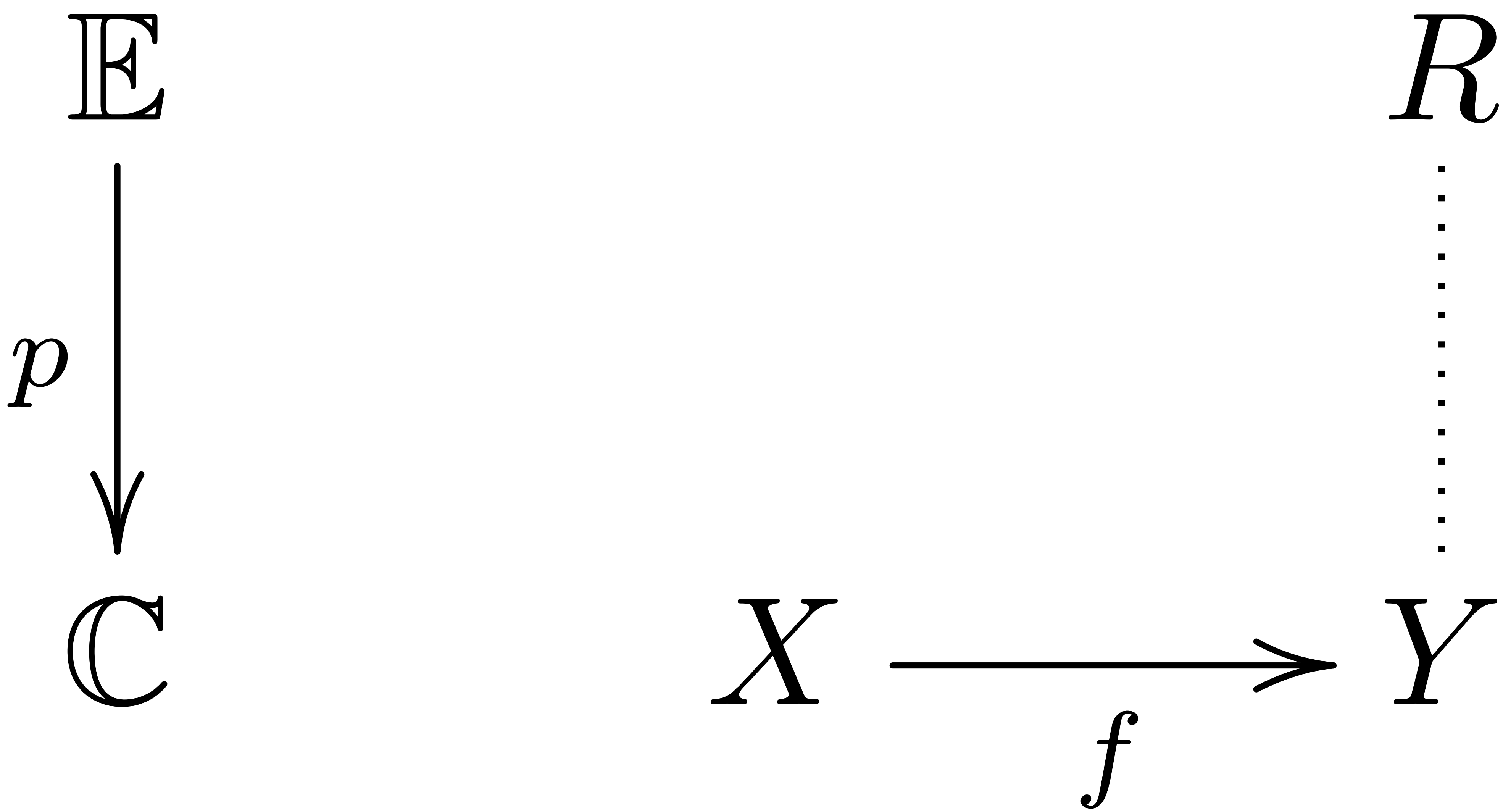
- Fibration: functor p: $\mathbb{E} \to \mathbb{C}$ satisfying <u>cartesian lifting property</u>.
- $R \in \mathbb{E}$ is above $X \in \mathbb{C} \Leftrightarrow pR = X$
- Fiber $\mathbb{E}_X$ over $X \in \mathbb{C}$

  object: $R \in \mathbb{E}$ above X

  arrow: f in $\mathbb{E}$ s.t. $pf = 1_X$



Cartesian lifting property

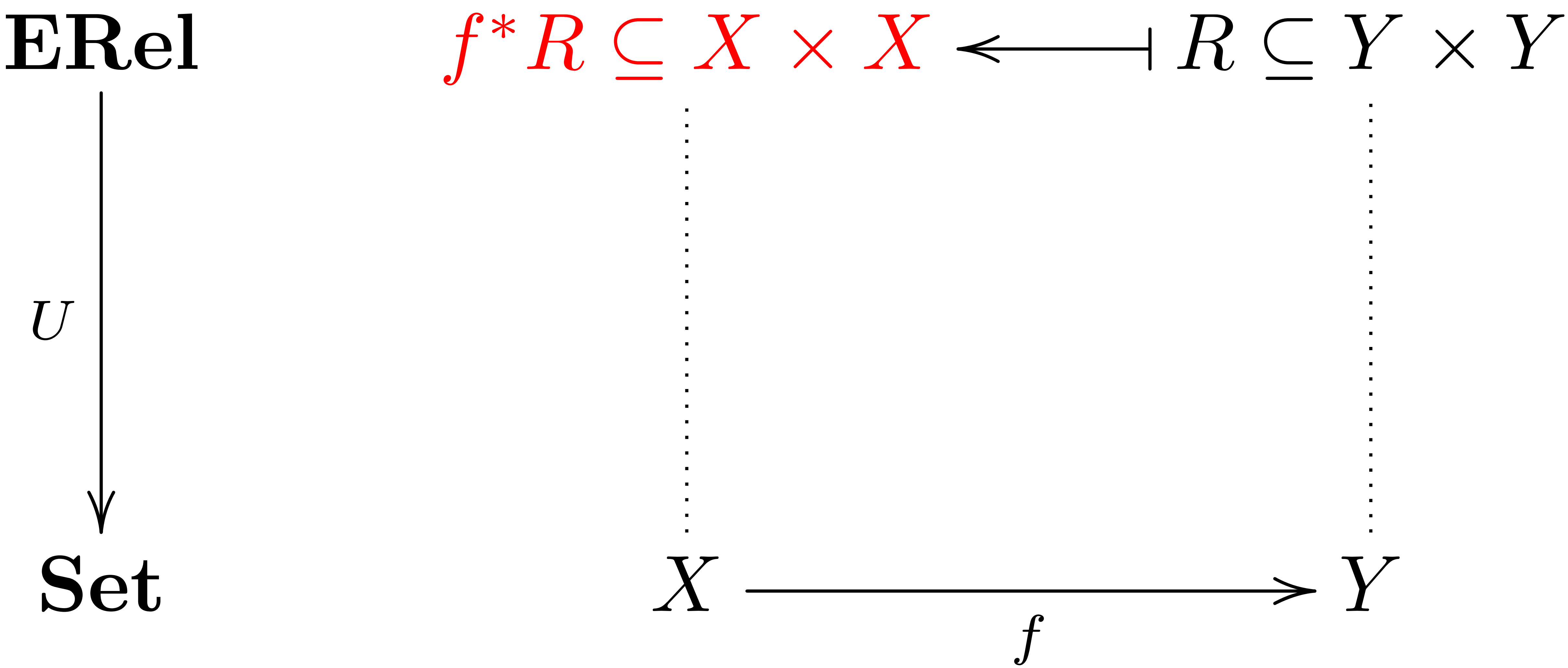


$$f^* : \mathbb{E}_Y \to \mathbb{E}_X$$

Fibrations: example

- Category $\mathbf{ERel}$
  - object: set with binary rel.
  - arrow: relation-preserving map
- Forgetful func. U: $\mathbf{ERel} \to \mathbf{Set}$ is a fibration.
- Fiber $\mathbf{ERel}_X = 2^{X \times X}$

Fibrations: example



$$f^*R = \{(x, x') \in X \times X \mid (f(x), f(x')) \in Y \times Y\}$$

Fibration: other examples

- $\mathbf{Pred} \to \mathbf{Set}$
- $\mathbf{Set}^{\to} \to \mathbf{Set}$
- $\mathbf{PMet} \to \mathbf{Set}$
- $\mathbf{Top} \to \mathbf{Set}$, $\mathbf{Meas} \to \mathbf{Set}$

Lifting

$\dot{F}$ is called a lifting of F along p if $\cdots$

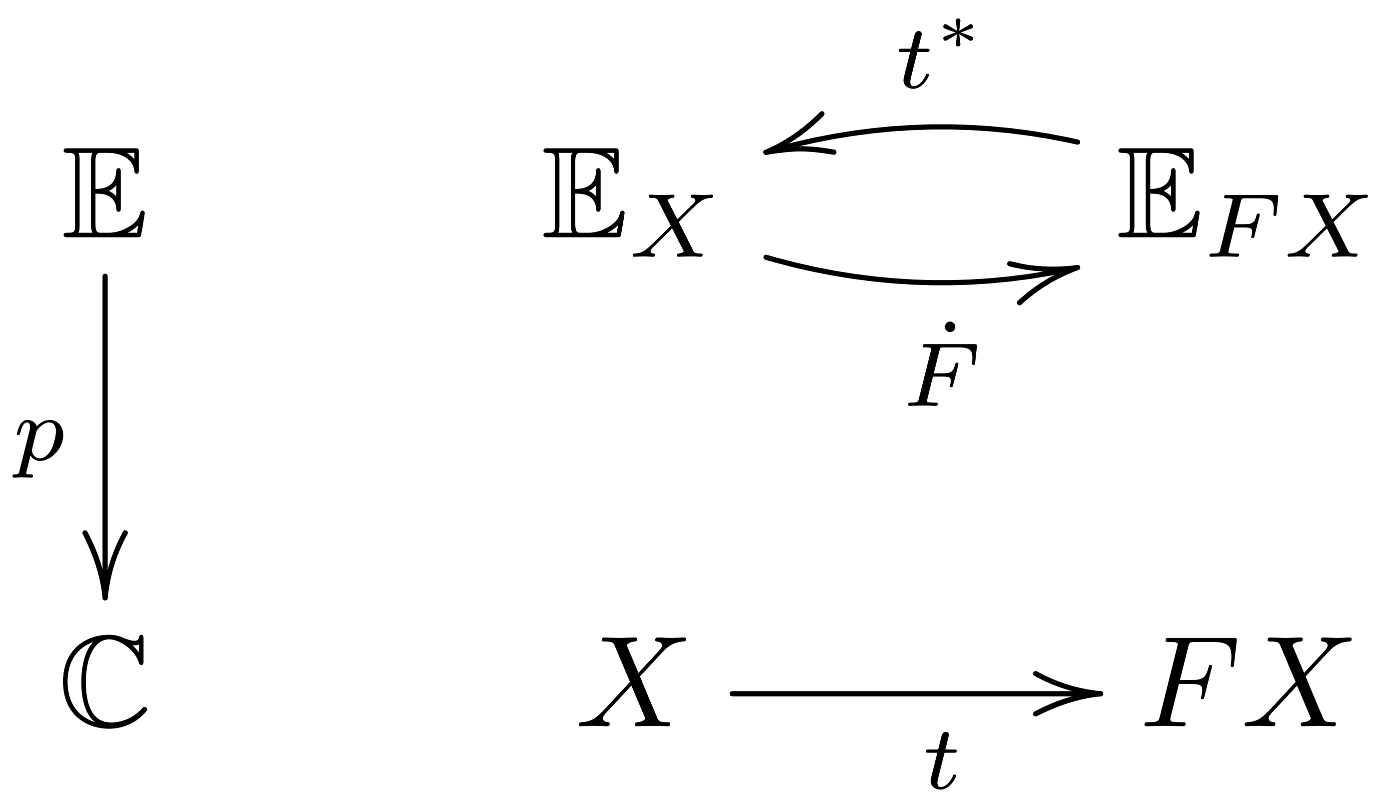
$$
\begin{array}{ccc}
\mathbb{E} & \xrightarrow[\dot{F}]{} & \mathbb{E} \\
{\scriptstyle p}\downarrow & \circlearrowleft & {\scriptstyle p}\downarrow \\
\mathbb{C} & \xrightarrow[F]{} & \mathbb{C}
\end{array}
\qquad\qquad
\begin{array}{ccc}
R & \longmapsto & \dot{F}R \\
\vdots & & \vdots \\
X & \longmapsto & FX
\end{array}
$$

Fibrational coinduction

p: $\mathbb{E} \to \mathbb{C}$: fibration, F: $\mathbb{C} \to \mathbb{C}$,

$\dot{F}$: $\mathbb{E} \to \mathbb{E}$ lifting of F along p,

t:$X \to FX$ F-coalgebra



$$\Phi = t^* \circ \dot{F}$$

predicate transformer

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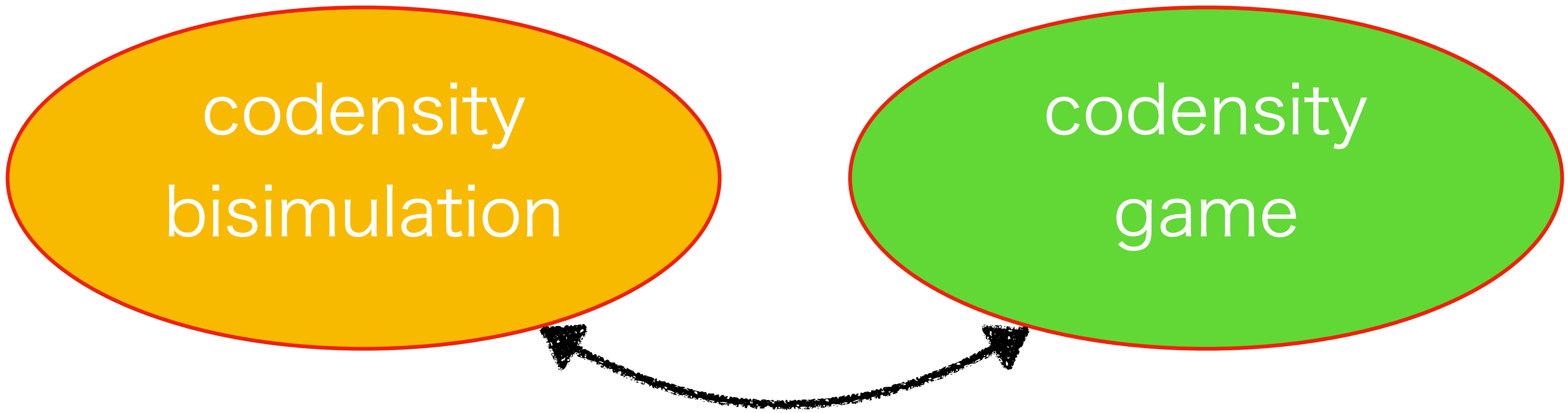

general "GFP=game" theorem

- We use
    - fibrational coinduction ✔ [Hermida & Jacobs 1998]
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$\mathbf{CLat}_\sqcap$-fibration

$\cdots$ is a fibration where

- each fiber $\mathbb{E}_X$ is a complete lattice
- each pullback functor $f^* : \mathbb{E}_Y \to \mathbb{E}_X$ preserves meets

Examples: $\mathbf{ERel} \to \mathbf{Set}$, $\mathbf{PMet}_1 \to \mathbf{Set}$

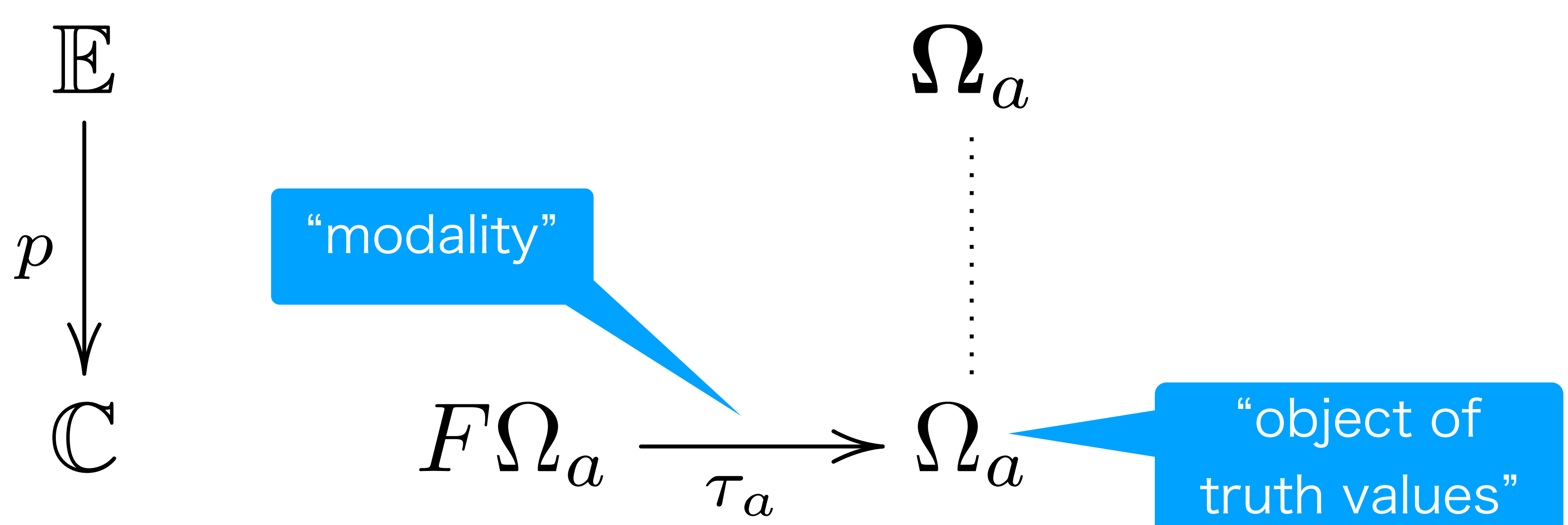
Non-example: $\mathbf{Set}^\to \to \mathbf{Set}$

Codensity Lifting: parameter

- A $\mathbf{CLat}_{\sqcap}$-fibration $p\colon \mathbb{E} \to \mathbb{C}$
- $F\colon \mathbb{C} \to \mathbb{C}$
- A family of pairs (called lifting parameter)

$$\left(\mathbf{\Omega}_a, \tau_a\colon F\mathbf{\Omega}_a \to \mathbf{\Omega}_a\right)_{a \in \mathbb{A}}$$

as in the diagram below:



Codensity Lifting

<u>Theorem</u> [Sprunger+ CMCS18]

The following defines

a lifting $F^{\mathbf{\Omega},\tau}$ of F:

$$F^{\mathbf{\Omega},\tau} R = \prod_{a \in \mathbb{A}, f} \left(\tau_a \circ F(pf)\right)^* \mathbf{\Omega}_a$$

where f ranges over $\mathbb{E}(R, \mathbf{\Omega}_a)$.

use "observation" a, f
and gather information

Codensity lifting: example

Use $\mathbf{ERel} \to \mathbf{Set}$, $\mathcal{D}$: $\mathbf{Set} \to \mathbf{Set}$,

$(\mathrm{Eq}_2, \mathrm{thr}_r : \mathcal{D}2 \to 2)_{r \in [0,1]}$

where Eq_2 is the equality relation

and $\mathrm{thr}_r(d) = \top \iff d(\{\top\}) \geq r.$

Threshold modality

$$(d, d') \in \mathcal{D}^{\mathrm{Eq}_2, \mathrm{thr}_r} X$$

$$\Leftrightarrow \forall r \in [0,1], \forall f : (X, R) \to (2, \mathrm{Eq}_2) \text{ rel.-pres.},$$

$$\mathrm{thr}_r((\mathcal{D}f)(d)) = \mathrm{thr}_r((\mathcal{D}f)(d'))$$

$$\Leftrightarrow \forall Y \subseteq X,\ R\text{-closed}, d(Y) = d(Y').$$

Predicate transformer in the prev. example appears!

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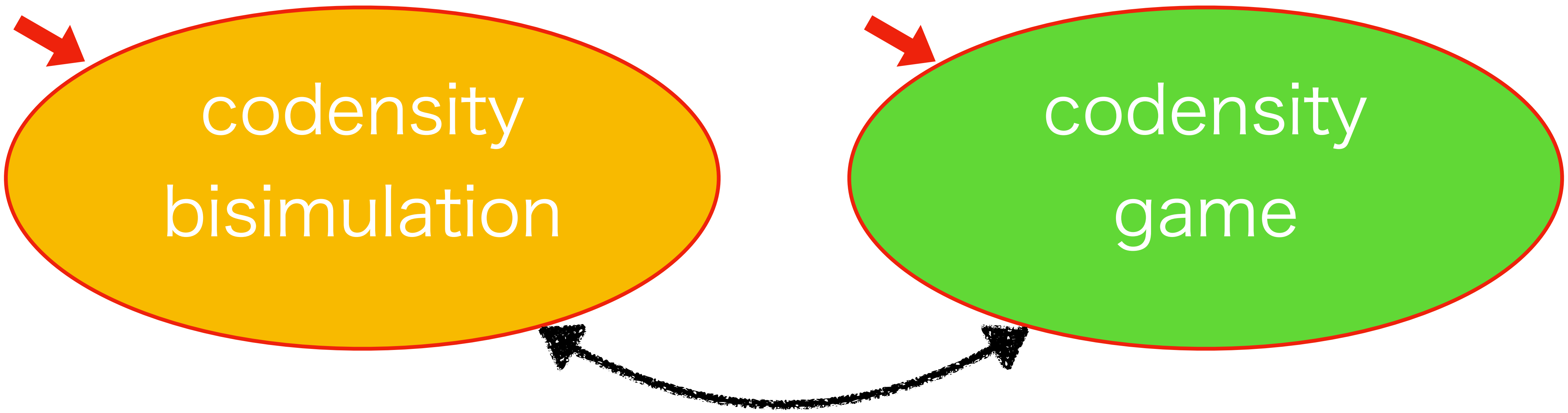

➡ **general "GFP=game" theorem**

- We use
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Codensity bisimilarity

Given $\mathbf{CLat}_\sqcap$-fibration $p\colon \mathbb{E} \to \mathbb{C}$,

$$F\colon \mathbb{C} \to \mathbb{C}, \quad (\mathbf{\Omega}_a, \tau_a : F\Omega_a \to \Omega_a)_{a \in \mathbb{A}},$$

F-coalgebra $t\colon X \to FX$,

we have a lifting $F^{\mathbf{\Omega},\tau}\colon \mathbb{E} \to \mathbb{E}$,

codensity lifting

and a predicate transformer

$$t^* \circ F^{\mathbf{\Omega},\tau} : \mathbb{E}_X \to \mathbb{E}_X .$$

fibrational coinduction

Codensity bisimilarity is $\nu(t^* \circ F^{\mathbf{\Omega},\tau}) \in \mathbb{E}_X .$

Codensity game

😇 makes some conjecture on the codensity bisimilarity

- From $R \in \mathbb{E}_X$

😈 challenges by an “observation”

😈 chooses $a \in \mathbb{A}$ and $f: X \to \Omega_a$ s.t.

$$R \not\sqsubseteq (\tau_a \circ Ff \circ t)^* \mathbf{\Omega}_a .$$

- From $(a \in \mathbb{A}, f: X \to \Omega_a)$

😇 chooses $R' \in \mathbb{E}_X$ s.t. $R' \not\sqsubseteq f^* \mathbf{\Omega}_a$.

😇 shows the “observation” is not appropriate, by another conjecture

Main theorem:
general “GFP=game” theorem

<u>Theorem</u>

$R \sqsubseteq \nu(t^* \circ F^{\Omega,\tau})$

GFP definition

if and only if

😇 has a winning strategy

starting from $R \in \mathbb{E}_X$.

game characterization

Remark

- To recover the game for bisimilarity relation on Markov chains (in the previous slides), we have to use a trick, “trimming.”
- We derived a new simple game for bisimulation metric for $\mathcal{D}$-coalgebras.

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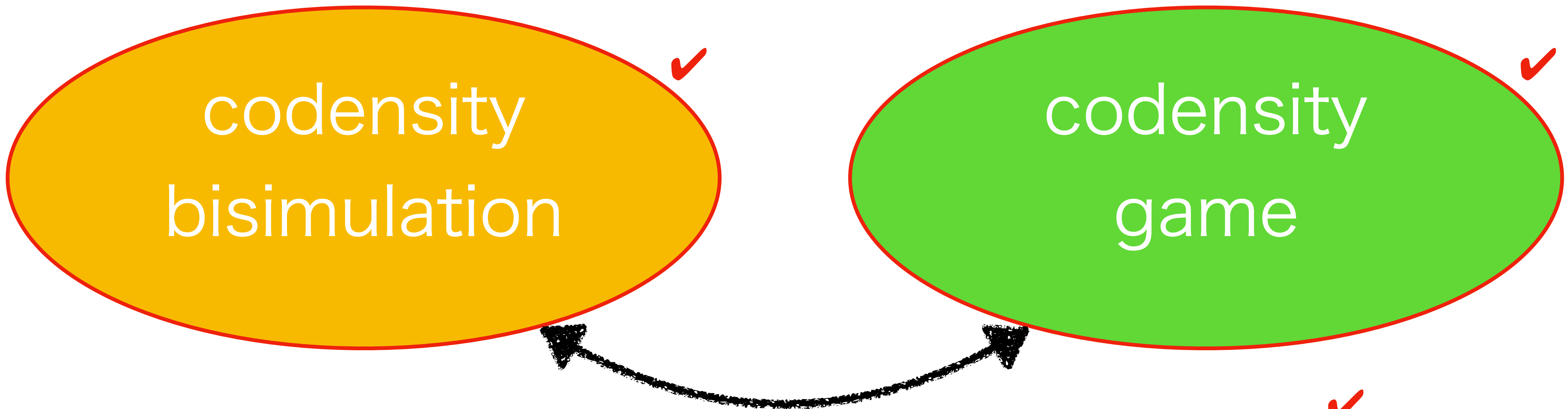

general "GFP=game" theorem ✔

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Future work

- Relation to modal logic
- Seek new useful bisimilarity-like notions
- Relation to another game for continuous lattices [Baldan+ POPL19]

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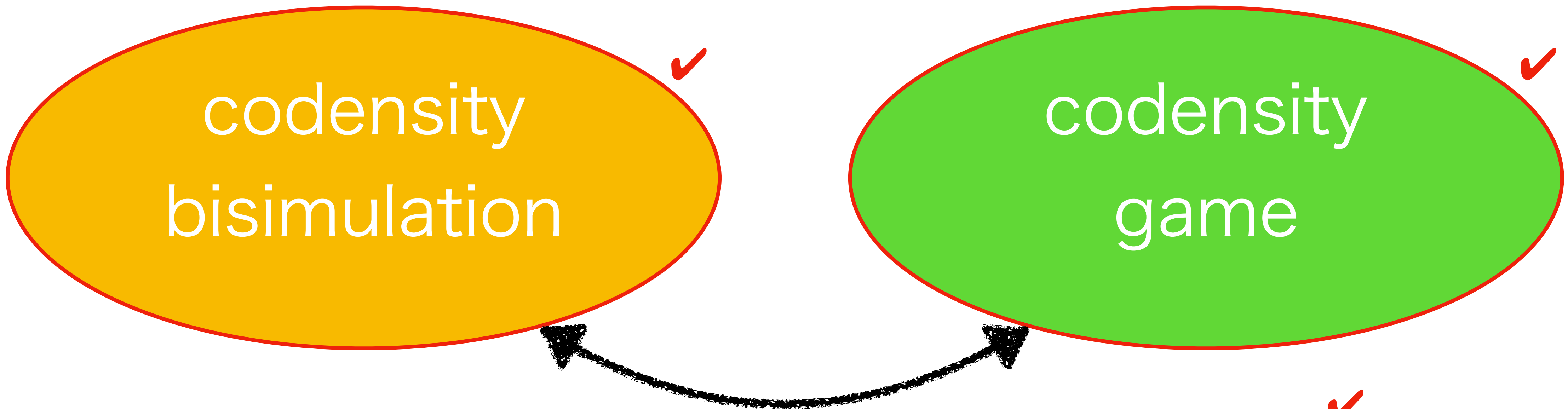

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Thank you!