Injective Objects and Fibered Codensity Liftings

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Background

- Functor lifting along a fibration is used e.g. for bisimilarity and its generalizations
- Codensity lifting [Katsumata & Sato CALCO15] [Sprunger+ CMCS18] is a general method to obtain a functor lifting

Contribution

- When codensity lifting yields a fibered functor?
- We obtained the first general sufficient condition for that.
- We defined c-injective object to formulate it.

Background

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Coalgebra

$\mathbb{C}$: category $\quad F: \mathbb{C} \to \mathbb{C}$

An F-coalgebra is a pair

$$(X \in \mathbb{C},\ t: X \to FX)$$

How states behave

We'll mainly consider $\mathbb{C} = \mathbf{Set}$.

- $\mathcal{P}$-coalgebras = Kripke frames
- $\mathcal{D}$-coalgebras = Markov chains
- LTS, (non-deterministic/deterministic/weighted) automata, and many others

Bisimilarity

- Which states behave "the same"?
- For t: $X \rightarrow FX$, if $x{\sim}y$ holds, then "$t(x){\sim}t(y)$" should also hold
    - The greatest relation $\sim$ on X among such is the <u>bisimilarity relation</u>
- We need a map $2^{X\times X}\rightarrow 2^{FX\times FX}$
    - $\Rightarrow$ Functor lifting gives one!

Bisimulation metric

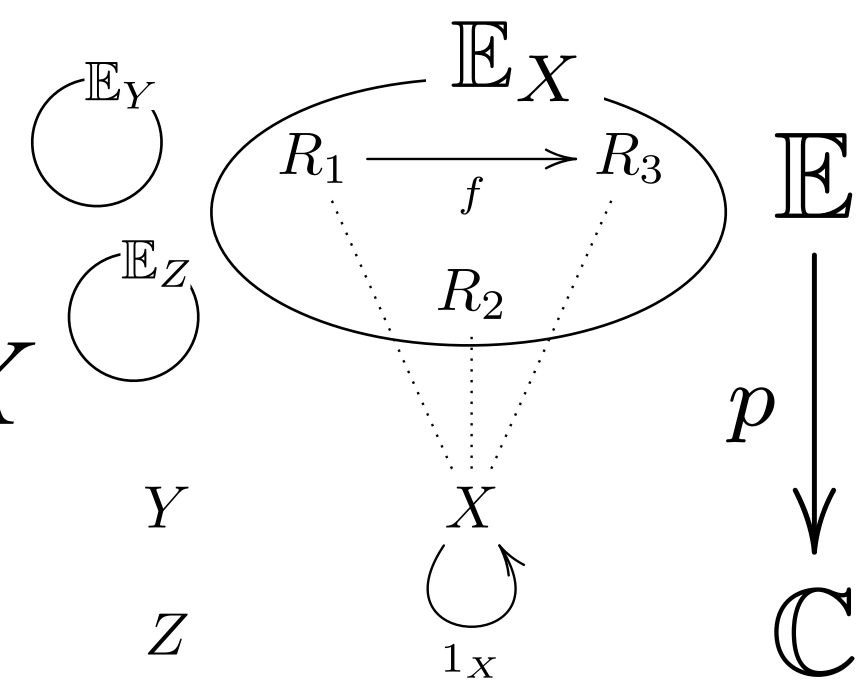
[Desharnais+,TCS318(3),2004]

- Which states ”behave alike”?
- Pseudometric on X
- We need a map
  - turns a pseudometric on X
  - into a pseudometric on FX
  - $\Rightarrow$ Functor lifting gives one!

Fibrations

I'll explain later

- Fibration: functor p: $\mathbb{E} \to \mathbb{C}$
  satisfying cartesian lifting property.
- $R \in \mathbb{E}$ is above $X \in \mathbb{C} \Leftrightarrow pR=X$
- Fiber $\mathbb{E}_X$ over $X \in \mathbb{C}$
  - object: $R \in \mathbb{E}$ above X
  - arrow: f in $\mathbb{E}$ s.t. $pf=1_X$



Cartesian lifting property

$$\begin{tikzcd} \mathbb{E} \arrow[d, "p"'] & & R \arrow[d, dotted, no head] \\ \mathbb{C} & X \arrow[r, "f"'] & Y \end{tikzcd}$$

$$f^* : \mathbb{E}_Y \to \mathbb{E}_X$$

Fibrations: example 1

- Category $\mathbf{ERel}$
  - object: set with binary rel.
  - arrow: relation-preserving map
- Forgetful func. U: $\mathbf{ERel} \to \mathbf{Set}$ is a fibration.
- Fiber $\mathbf{ERel}_X = 2^{X \times X}$

Fibrations: example 1

$$\begin{tikzcd} \mathbf{ERel} \arrow[d, "U"'] & f^*R \subseteq X \times X \arrow[d, dotted, no head] & R \subseteq Y \times Y \arrow[l, mapsto] \arrow[d, dotted, no head] \\ \mathbf{Set} & X \arrow[r, "f"'] & Y \end{tikzcd}$$

$$f^*R = \{(x, x') \in X \times X \mid (f(x), f(x')) \in Y \times Y\}$$

Fibrations: example 2

- Category $\mathbf{PMet}_1$
  - object: set with [0,1]-valued pseudometric
  - arrow: non-expansive map
- Forgetful func. U: $\mathbf{PMet}_1 \to \mathbf{Set}$ is a fibration.
- Fiber $(\mathbf{PMet}_1)_X$ is the set of all [0,1]-valued pseudometrics on X

Fibrations: example 2

$$\begin{tikzcd}
\mathbf{PMet}_1 \arrow[d, "U"'] & f^*d\colon X \times X \to [0,1] \arrow[d, dotted, no head] & d\colon Y \times Y \to [0,1] \arrow[l, mapsto] \arrow[d, dotted, no head] \\
\mathbf{Set} & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$f^*d(x, x') = d(f(x), f(x'))$$

$\mathbf{CLat}_\sqcap$-fibration

$\cdots$ is a fibration where

- each fiber $\mathbb{E}_X$ is a complete lattice
- each pullback functor $f^*\colon \mathbb{E}_Y \to \mathbb{E}_X$ preserves meets

Examples: $\mathbf{ERel} \to \mathbf{Set}$, $\mathbf{PMet}_1 \to \mathbf{Set}$, $\mathbf{PreOrd} \to \mathbf{Set}$, $\mathbf{Top} \to \mathbf{Set}$, $\cdots$

Remark on the "order"

- Order in $\mathbb{E}_X$: $E \sqsubseteq E'$ means $E \to E'$
- In $\mathbf{ERel}$, $(X, R) \sqsubseteq (X, R')$ means $R \subseteq R'$
- In $\mathbf{PMet}_1$, it is "reversed"
  - $(X, d) \sqsubseteq (X, e)$ means, for each x_1, x_2, $d(x_1, x_2) \geq e(x_1, x_2)$
  - Meet $\sqcap$ means $\sup$ of the values

Functor lifting

$\dot{F}$ is called a lifting of F along p if $\cdots$

$$
\begin{tikzcd}
\mathbb{E} \arrow[r, "\dot{F}"'] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{E} \arrow[d, "p"'] \\
\mathbb{C} \arrow[r, "F"'] & \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, mapsto] \arrow[d, dash, dotted] & \dot{F}R \arrow[d, dash, dotted] \\
X \arrow[r, mapsto] & FX
\end{tikzcd}
$$

Fibrational coinduction

[Hermida & Jacobs 1998]

$p\colon \mathbb{E} \to \mathbb{C}$: fibration, $F\colon \mathbb{C} \to \mathbb{C}$,

$\dot{F}\colon \mathbb{E} \to \mathbb{E}$: lifting of F along p,

$t\colon X \to FX$ F-coalgebra

$$\begin{tikzcd} \mathbb{E} \arrow[d, "p"'] & \mathbb{E}_X \arrow[r, bend right, "\dot{F}"'] & \mathbb{E}_{FX} \arrow[l, bend right, "t^*"'] \\ \mathbb{C} & X \arrow[r, "t"'] & FX \end{tikzcd}$$

$$\nu(t^* \circ \dot{F})$$

greatest fixed point

Background

- Functor lifting along a fibration is used e.g. for bisimilarity and its generalizations
- **Codensity lifting** [Katsumata & Sato CALCO15] [Sprunger+ CMCS18] is a general method to obtain a functor lifting

Codensity lifting

Generalization:

Kantorovich distance

↓

Kantorovich lifting [Baldan et al. FSTTCS14]

↓

Codensity lifting [Katsumata & Sato CALCO15]

[Sprunger+ CMCS18]

Kantorovich distance

- $\mathscr{D} : \mathbf{Set} \to \mathbf{Set}$: the discrete prob. dist. functor
- $(X, d) \in \mathbf{PMet}_1$, $p, q \in \mathscr{D}X$

$$d^K(p, q) = \sup_f \left| \sum_x f(x)p(x) - \sum_x f(x)q(x) \right|$$

- f ranges over nonexpansive maps $(X, d) \to ([0,1], d_{\mathbb{R}})$

Kantorovich distance

- $e \colon \mathscr{D}[0,1] \to [0,1]$: **expected value function**
- $(X, d) \in \mathbf{PMet}_1$, $p, q \in \mathscr{D}X$

$$d^K(p,q) = \sup_f d_{\mathbb{R}} \left(e((\mathscr{D}f)(p)), e((\mathscr{D}f)(q)) \right)$$

- f ranges over nonexpansive maps $(X, d) \to ([0,1], d_{\mathbb{R}})$

Kantorovich lifting

[Baldan et al. FSTTCS14]

- $F\colon \mathbf{Set} \to \mathbf{Set}$
- Use $\tau\colon F[0,1] \to [0,1]$
- $(X, d) \in \mathbf{PMet}_1$, $p, q \in FX$

$$d^{\uparrow F}(p, q) = \sup_f d_{\mathbb{R}} \left(\tau((Ff)(p)), \tau((Ff)(q))\right)$$

- f ranges over nonexpansive maps $(X, d) \to ([0,1], d_{\mathbb{R}})$

Kantorovich lifting

[Baldan et al. FSTTCS14]

- $F\colon \mathbf{Set} \to \mathbf{Set}$
- Use $\tau\colon F[0,1] \to [0,1]$
- $(X, d) \in \mathbf{PMet}_1$

$$d^{\uparrow F} = \bigsqcap_{f} (\tau \circ Ff)^* d_{\mathbb{R}}$$

- f ranges over nonexpansive maps $(X, d) \to ([0,1], d_{\mathbb{R}})$

Codensity lifting

[Katsumata & Sato CALCO15] [Sprunger+ CMCS18]

- $F\colon \mathbb{C} \to \mathbb{C}$, $p\colon \mathbb{E} \to \mathbb{C}$ ($\mathbf{CLat}_\sqcap$-fibration)
- Use $\tau\colon F\Omega \to \Omega$, $\mathbf{\Omega} \in \mathbb{E}_\Omega$
- $X \in \mathbb{C}, E \in \mathbb{E}_X$

$$F^{\mathbf{\Omega},\tau} E = \prod_f (\tau \circ Fpf)^* \mathbf{\Omega}$$

- f ranges over arrows $E \to \mathbf{\Omega}$ in $\mathbb{E}$
- A functor $F^{\mathbf{\Omega},\tau}\colon \mathbb{E} \to \mathbb{E}$ is defined

Contribution

- When codensity lifting yields a **fibered** functor?
- We obtained the first general sufficient condition for that.
- We defined c-injective object to formulate it.

Fiberedness

- Fibered lifting: functor lifting that interact well with pullbacks
- $F\colon \mathbb{C} \to \mathbb{C}$, $p\colon \mathbb{E} \to \mathbb{C}$ ($\mathbf{CLat}_{\sqcap}$-fibration)
- Lifting $\dot{F}\colon \mathbb{E} \to \mathbb{E}$ is fibered if
  - for any $f\colon X \to Y$ in $\mathbb{C}$ and $E \in \mathbb{E}_Y$,
  - $\dot{F}(f^*E) = (Ff)^*\dot{F}E$ holds.

Application of fiberedness

- Thm. If $\dot{F}$ is fibered, the coinductive predicate is stable under coalgebra morphisms:

For $t\colon X \to FX$, $u\colon Y \to FY$, and $f\colon X \to Y$,

if $u \circ f = Ff \circ t$ holds, then

$\nu(t^* \circ \dot{F}) = f^*\nu(u^* \circ \dot{F})$ holds.

Contribution

- When codensity lifting yields a fibered functor?
- We obtained the first general sufficient condition for that.
- We defined c-injective object to formulate it.

- Kantorovich lifting is always fibered [Baldan et al. FSTTCS14]
  - In that case fiberedness ⇔ preservation of isometries
- Codensity lifting …… ???

Property of [0,1]

[Baldan et al. FSTTCS14]

- f: $X \to Y$ and $(Y, d) \in \mathbf{PMet}_1$
- For any g, there exists h:

$$
\begin{tikzcd}
(X, f^*d) \arrow[dd, "f"'] \arrow[dr, "g"] & \\
\arrow[r, phantom, "\circlearrowleft"] & ([0, 1], d_{\mathbb{R}}) \\
(Y, d) \arrow[ur, dotted, "\exists h"'] &
\end{tikzcd}
$$

C-injective object

- $p\colon \mathbb{E} \to \mathbb{C}$ ($\mathbf{CLat}_{\sqcap}$-fibration)
- Def. $\mathbf{\Omega} \in \mathbb{E}_{\Omega}$ is c-injective if, for any $f\colon X \to Y$, $E \in \mathbb{E}_Y$, and $g\colon f^*E \to \mathbf{\Omega}$, the following h exists:

$$
\begin{tikzcd}
f^*E \arrow[d, "f"'] \arrow[rr, "g"] & \arrow[d, phantom, "\circlearrowright"] & \mathbf{\Omega} \\
E \arrow[urr, dotted, "\exists h"'] & &
\end{tikzcd}
$$

Main theorem

- Thm. If $\mathbf{\Omega} \in \mathbb{E}$ is c-injective, then the codensity lifting $F^{\mathbf{\Omega},\tau} \colon \mathbb{E} \to \mathbb{E}$ is a fibered lifting of F.

Examples of fibered codensity liftings

TABLE VI
CODENSITY LIFTING OF FUNCTORS

	fibration $p : \mathbb{E} \to \mathbb{C}$	functor $F\colon \mathbb{C} \to \mathbb{C}$	obs. dom. $\mathbf{\Omega}$	modality τ	lifting $F^{\mathbf{\Omega},\tau}$ of F
1	$\mathbf{Pre} \to \mathbf{Set}$	powerset $\mathcal{P}$	$(2, \leq)$	$\diamond\colon \mathcal{P}2 \to 2$	lower preorder [14]
2	$\mathbf{Pre} \to \mathbf{Set}$	powerset $\mathcal{P}$	$(2, \geq)$	$\diamond\colon \mathcal{P}2 \to 2$	upper preorder [14]
3	$\mathbf{ERel} \to \mathbf{Set}$	powerset $\mathcal{P}$	$(2, \mathrm{Eq}_2)$	$\diamond\colon \mathcal{P}2 \to 2$	(for bisimulation, see Ex. III.3 & VII.2)
4	$\mathbf{EqRel} \to \mathbf{Set}$	powerset $\mathcal{P}$	$(2, \mathrm{Eq}_2)$	$\diamond\colon \mathcal{P}2 \to 2$	(for bisimulation, see Ex. III.2 & VII.2)
5	$\mathbf{PMet}_1 \to \mathbf{Set}$	subdistrib. $\mathcal{D}_{\leq 1}$	$([0,1], d_{[0,1]})$	$e\colon \mathcal{D}_{\leq 1}[0,1] \to [0,1]$	Kantorovich metric
6	$\mathbf{PMet}_1 \to \mathbf{Set}$	powerset $\mathcal{P}$	$([0,1], d_{[0,1]})$	$\inf\colon \mathcal{P}[0,1] \to [0,1]$	Hausdorff pseudometric (cf. Appendix C)
7	$U^*(\mathbf{PMet}_1) \to \mathbf{Meas}$	sub-Giry $\mathcal{G}_{\leq 1}$	$([0,1], d_{[0,1]})$	$e\colon \mathcal{G}_{\leq 1}[0,1] \to [0,1]$	Kantorovich metric
8†	$\mathbf{Pre} \to \mathbf{Set}$	powerset $\mathcal{P}$	$(2, \leq), (2, \geq)$	$\diamond\colon \mathcal{P}2 \to 2$	convex preorder [14]
9†	$\mathbf{EqRel} \to \mathbf{Set}$	subdistrib. $\mathcal{D}_{\leq 1}$	$(2, \mathrm{Eq}_2)$	$(\tau_r\colon \mathcal{D}_{\leq 1}2 \to 2)_{r\in[0,1]}$	(for prob. bisim., see §VIII-G)
10†	$\mathbf{Top} \to \mathbf{Set}$	$2 \times (\_)^{\Sigma}$	Sierpinski space	(see Ex. VI.5)	(for bisim. topology, see Ex. VI.5)

(Taken from [K. et al. LICS19])

Examples of fibered codensity liftings

fibration	Ω	c-injective?	examples
Pre→**Set**	(2,≦)	Yes	upper, lower, convex preorders
ERel→**Set**	(2,=)	No	(for bisimilarity)
EqRel→**Set**	(2,=)	Yes	(for bisimilarity)
$\mathbf{PMet}_1$→**Set**	$([0,1],d_R)$	Yes	Hausdorff and Kantorovich distances
$U^*(\mathbf{PMet}_1)$→**Meas**	$([0,1],d_R)$	No	Kantorovich distance
Top→**Set**	Sierpinski space	Yes	(for bisimulation topology)

Future work

- Application to modal logic (ongoing with C.Kupke and J.Rot)
  - In particular to the fibrational framework [Kupke & Rot CSL20 to appear]
- Study of c-injective objects in a unified way?
  - In $\mathbf{Top} \to \mathbf{Set}$, they are continuous lattices [Scott 1972]
  - In $\mathbf{PreOrd} \to \mathbf{Set}$, they are complete lattices [Banashewski & Bruns 1967]
  - In $\mathbf{PMet}_1 \to \mathbf{Set}$, they are called bounded hyperconvex spaces
  - S.Fujii recently identified in some other cases [Fujii arXiv 2019]