Fibrational Theory of Behaviors and Observations

Bisimulation, Logic, and Games from Modalities

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About me

- Yuichi Komorida
- PhD student at Sokendai (AY 2018-2022)
- Formerly in KU as an undergraduate (AY 2013-2017)
- Planning the thesis ← today's topic!

1. About me
2. Technical context

   Coalgebra / Fibrational coinduction / Codensity lifting
3. Main results

   Codensity games / Expressivity of modal logic / Fiberedness of codensity lifting
4. Future research directions

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Context of the theory

- Theory of (observable) behaviors: coalgebra
- Fibrational theory of (observable) behaviors: fibrational coinduction
- Fibrational theory of behaviors and observations: **this thesis**

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Coalgebra [Rutten, 2000] etc.

$\mathbb{C}$: category $B \colon \mathbb{C} \to \mathbb{C}$

A B-coalgebra is a pair

$$(X \in \mathbb{C}, x \colon X \to BX)$$

We'll mainly consider $\mathbb{C} = \mathbf{Set}$.

- $\mathscr{P}$-coalgebras = Kripke frames
- $\mathscr{D} \times 2^{\mathsf{AP}}$-coalgebras = Markov chains
- LTS, (non-deterministic/deterministic/weighted) automata, and many others

probabilistic

weighted

Coalgebra morphism

- An arrow $f\colon X \to Y$ such that the diagram on the right commute
- It "preserves all observable behaviors"
- How to extract information in a way that it is preserved by any morphism? $\to$ fibrational coinduction

$$
\begin{tikzcd}
X \arrow[r, "f"'] \arrow[d, "x"'] & Y \arrow[d, "y"'] \\
BX \arrow[r, "Bf"'] & BY
\end{tikzcd}
$$

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Fibration

- Fibration: functor $p\colon \mathbb{E} \to \mathbb{C}$ satisfying

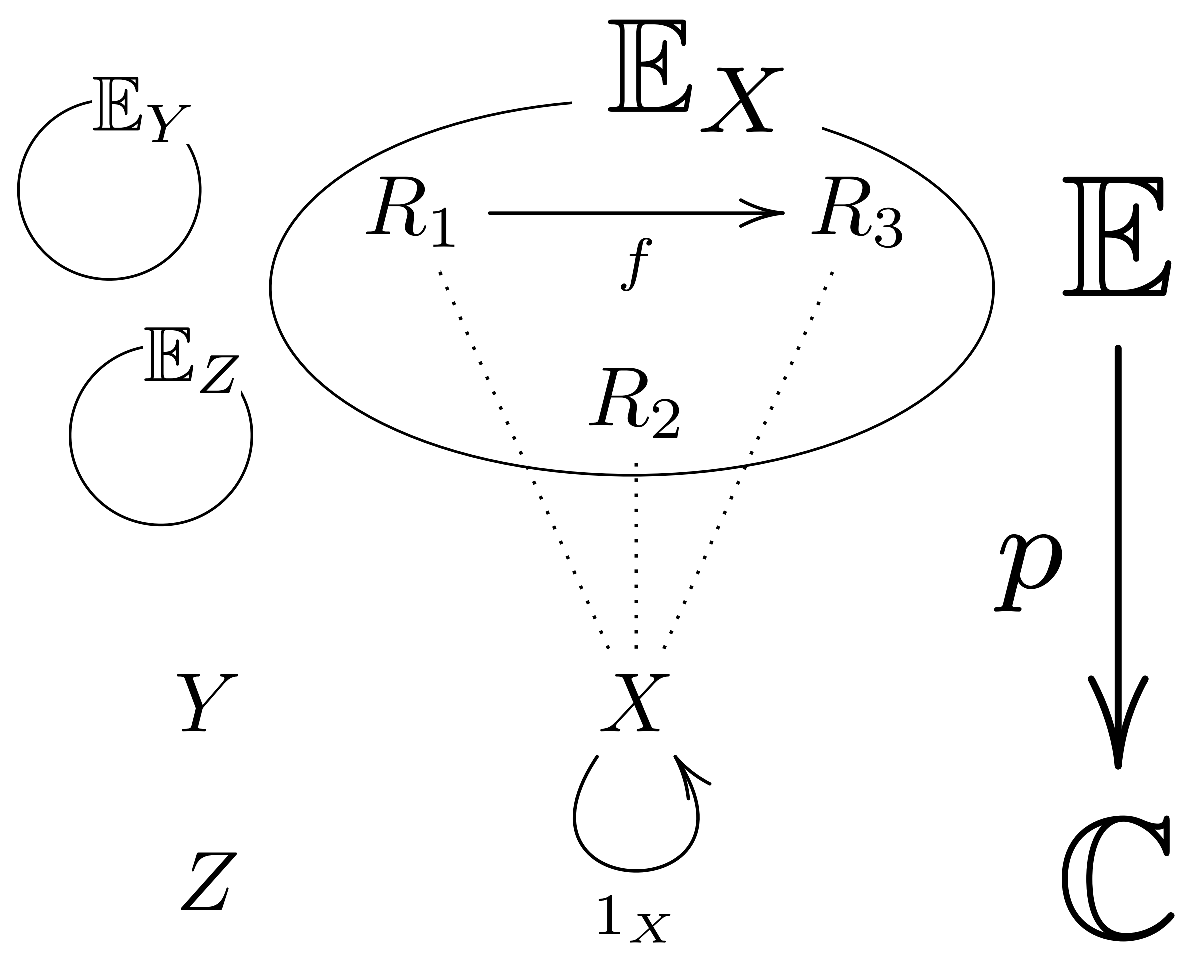
  cartesian lifting property.

- $R \in \mathbb{E}$ is above $X \in \mathbb{C} \iff pR = X$

- Fiber $\mathbb{E}_X$ over $X \in \mathbb{C}$

  object: $R \in \mathbb{E}$ above X

  arrow: f in $\mathbb{E}$ s.t. $pf = 1_X$



Cartesian lifting property

$$
\begin{tikzcd}
\mathbb{E} \arrow[d, "p"'] & & & R \arrow[d, dotted, no head] \\
\mathbb{C} & & X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

Cartesian lifting property

$$\begin{tikzcd} \mathbb{E} \arrow[d, "p"'] \\ \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} {\color{red}f^*R} \arrow[d, dotted, no head] & R \arrow[l, mapsto] \arrow[d, dotted, no head] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$${\color{red}f^* : \mathbb{E}_Y \to \mathbb{E}_X}$$

Cartesian lifting property

$$
\begin{tikzcd}
\mathbb{E} \arrow[d, "p"'] \\
\mathbb{C}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
{\color{red} f^* R} \arrow[d, dotted, no head] & R \arrow[l, mapsto] \arrow[d, dotted, no head] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

$${\color{red} f^* : \mathbb{E}_Y \to \mathbb{E}_X}$$

“Observe unknown X using known R through f”

Fibrations: example 1

- Category $\mathbf{EqRel}$
  - object: set with equivalence rel.
  - arrow: relation-preserving map
- Forgetful func. $U\colon \mathbf{EqRel} \to \mathbf{Set}$ is a fibration.
- Fiber $\mathbf{EqRel}_X$ is the set of all eq.rel.

Fibrations: example 1

$$
\begin{tikzcd}
\mathbf{EqRel} \arrow[d, "U"'] & f^*R \subseteq X \times X \arrow[d, dotted, no head] & R \subseteq Y \times Y \arrow[l, mapsto] \arrow[d, dotted, no head] \\
\mathbf{Set} & X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

$$f^*R = \{(x, x') \in X \times X \mid (f(x), f(x')) \in Y \times Y\}$$

Fibrations: example 2

- Category $\mathbf{PMet}_1$
  - object: set with [0,1]-valued pseudometric
  - arrow: non-expansive map
- Forgetful func. U: $\mathbf{PMet}_1 \to \mathbf{Set}$ is a fibration.
- Fiber $(\mathbf{PMet}_1)_X$ is the set of all [0,1]-valued pseudometrics on X

Fibrations: example 2

$$
\begin{tikzcd}
\mathbf{PMet}_1 \arrow[d, "U"'] \\
\mathbf{Set}
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*d\colon X \times X \to [0,1] \arrow[d, dotted, no head] & d\colon Y \times Y \to [0,1] \arrow[l, mapsto] \arrow[d, dotted, no head] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

$$f^*d(x, x') = d(f(x), f(x'))$$

$\mathbf{CLat}_\sqcap$-fibration

$\cdots$ is a fibration where

- each fiber $\mathbb{E}_X$ is a complete lattice
- each reindexing $f^*: \mathbb{E}_Y \to \mathbb{E}_X$ preserves meets

Examples: $\mathbf{EqRel} \to \mathbf{Set}$, $\mathbf{PMet}_1 \to \mathbf{Set}$,

$\mathbf{Pred} \to \mathbf{Set}$, $\mathbf{Meas} \to \mathbf{Set}$, $\mathbf{Top} \to \mathbf{Set}$, $\cdots$

A $\mathbf{CLat}_\sqcap$-fibration specifies a <u>form of information</u>

Remark on the “order”

- Order in $\mathbb{E}_X$: $E \sqsubseteq E'$ means $E \to E'$
- In $\mathbf{EqRel}$, $(X, R) \sqsubseteq (X, R')$ means $R \subseteq R'$
- In $\mathbf{PMet}_1$, it is “reversed”
  - $(X, d) \sqsubseteq (X, e)$ means, for each x_1, x_2, $d(x_1, x_2) \geq e(x_1, x_2)$
  - Meet $\sqcap$ means $\sup$ of the values

Functor lifting

Def. $\overline{B}$ is called a lifting of B along p if $\cdots$

$$
\begin{tikzcd}
\mathbb{E} \arrow[r, "\overline{B}"'] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{E} \arrow[d, "p"'] \\
\mathbb{C} \arrow[r, "B"'] & \mathbb{C}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
R \arrow[r, mapsto] \arrow[d, dotted, no head] & \overline{B}R \arrow[d, dotted, no head] \\
X \arrow[r, mapsto] & BX
\end{tikzcd}
$$

$\overline{B}$ specifies how to extract $\mathbb{E}$-information from B-behaviors.

$\overline{B}$-coinductive predicate

[Hermida & Jacobs 1998]

$p\colon \mathbb{E} \to \mathbb{C}$: $\mathbf{CLat}_{\sqcap}$-fibration, $B\colon \mathbb{C} \to \mathbb{C}$,

$\overline{B}\colon \mathbb{E} \to \mathbb{E}$: lifting of B along p,

$x\colon X \to BX$ B-coalgebra

$$\nu(x^* \circ \overline{B})$$

greatest fixed point

$$\begin{tikzcd} \mathbb{E} \arrow[d, "p"'] & \mathbb{E}_X \arrow[r, bend right, "\overline{B}"'] & \mathbb{E}_{BX} \arrow[l, bend right, "x^*"'] \\ \mathbb{C} & X \arrow[r, "x"'] & BX \end{tikzcd}$$

Coalgebra morphisms preserve $\nu(x^* \circ \bar{B})$

$x\colon X \to BX$, $y\colon Y \to BY$, a coalgebra morphism $f\colon X \to Y$

- In the fiber $\mathbb{E}_X$, $\nu(x^* \circ \bar{B}) \sqsubseteq f^*\nu(y^* \circ \bar{B})$ holds.
- If $\bar{B}$ is fibered (explained later), $\nu(x^* \circ \bar{B}) = f^*\nu(y^* \circ \bar{B})$ holds.
- Examples
  - In $\mathbf{EqRel} \to \mathbf{Set}$, f preserves (resp. reflects) the equivalence.
  - In $\mathbf{PMet}_1 \to \mathbf{Set}$, f is non-expansive (resp. isometry).

- Coinductive predicate $\nu(x^* \circ \bar{B})$ depends only on "observable behaviors"
  - How to give an appropriate lifting $\bar{B}$?
  - What is "observation" here?
- $\rightarrow$ Use **codensity lifting**

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Codensity lifting

Generalization:

Kantorovich distance

↓

Kantorovich lifting [Baldan et al. FSTTCS14]

↓

Codensity lifting [Katsumata & Sato CALCO15]

[Sprunger+ CMCS18]

Kantorovich distance

- $\mathscr{D} : \mathbf{Set} \to \mathbf{Set}$: the discrete prob. dist. functor
- $(X, d) \in \mathbf{PMet}_1$, $p, q \in \mathscr{D}X$

$$d^K(p, q) = \sup_f \left| \sum_x f(x)p(x) - \sum_x f(x)q(x) \right|$$

- f ranges over nonexpansive maps $(X, d) \to ([0,1], d_{\mathbb{R}})$

  (intuitively it is an "observation")

Kantorovich distance

- $e : \mathscr{D}[0,1] \to [0,1]$: **expected value function**
- $(X, d) \in \mathbf{PMet}_1,\ p, q \in \mathscr{D}X$

$$d^K(p,q) = \sup_f d_{\mathbb{R}} \left(e((\mathscr{D}f)(p)), e((\mathscr{D}f)(q)) \right)$$

- f ranges over nonexpansive maps $(X, d) \to ([0,1], d_{\mathbb{R}})$
  (intuitively it is an “observation”)

Kantorovich lifting [Baldan et al. FSTTCS14]

- $B\colon \mathbf{Set} \to \mathbf{Set}$
- Use $\tau\colon B[0,1] \to [0,1]$
- $(X, d) \in \mathbf{PMet}_1$, $p, q \in FX$

$$d^{\uparrow B}(p,q) = \sup_f d_{\mathbb{R}} \left(\tau((Bf)(p)), \tau((Bf)(q))\right)$$

- f ranges over nonexpansive maps $(X, d) \to ([0,1], d_{\mathbb{R}})$

  (intuitively it is an “observation”)

Kantorovich lifting [Baldan et al. FSTTCS14]

- $B \colon \mathbf{Set} \to \mathbf{Set}$
- Use $\tau \colon B[0,1] \to [0,1]$
- $(X, d) \in \mathbf{PMet}_1$

$$d^{\uparrow B} = \bigsqcap_{f} (\tau \circ Bf)^* d_{\mathbb{R}}$$

- f ranges over nonexpansive maps $(X, d) \to ([0,1], d_{\mathbb{R}})$

  (intuitively it is an "observation")

Codensity lifting

[Katsumata & Sato CALCO15] [Sprunger+ CMCS18]

- $B\colon \mathbb{C} \to \mathbb{C}$, $p\colon \mathbb{E} \to \mathbb{C}$ ($\mathbf{CLat}_\sqcap$-fibration)
- Use $\tau\colon B\Omega \to \Omega, \mathbf{\Omega} \in \mathbb{E}_\Omega$
- $X \in \mathbb{C}, E \in \mathbb{E}_X$

$$B^{\mathbf{\Omega},\tau} E = \bigsqcap_{f} (\tau \circ B(pf))^* \, \mathbf{\Omega}$$

- f ranges over arrows $E \to \mathbf{\Omega}$ in $\mathbb{E}$ (intuitively it is an "observation")
- A functor $B^{\mathbf{\Omega},\tau}\colon \mathbb{E} \to \mathbb{E}$ is defined

Codensity lifting (multiple parameters)

[Katsumata & Sato CALCO15] [Sprunger+ CMCS18]

- $B\colon \mathbb{C} \to \mathbb{C}$, $p\colon \mathbb{E} \to \mathbb{C}$ ($\mathbf{CLat}_{\sqcap}$-fibration)
- Use $(\tau_\lambda\colon B\Omega_\lambda \to \Omega_\lambda, \mathbf{\Omega}_\lambda \in \mathbb{E}_{\Omega_\lambda})_{\lambda\in\Lambda}$
- $X \in \mathbb{C}, E \in \mathbb{E}_X$

$$B^{\mathbf{\Omega},\tau}E = \bigsqcap_{\lambda,f} (\tau_\lambda \circ B(pf))^* \mathbf{\Omega}_\lambda$$

- f ranges over arrows $E \to \mathbf{\Omega}_\lambda$ in $\mathbb{E}$ (intuitively it is an “observation”)
- A functor $B^{\mathbf{\Omega},\tau}\colon \mathbb{E} \to \mathbb{E}$ is defined

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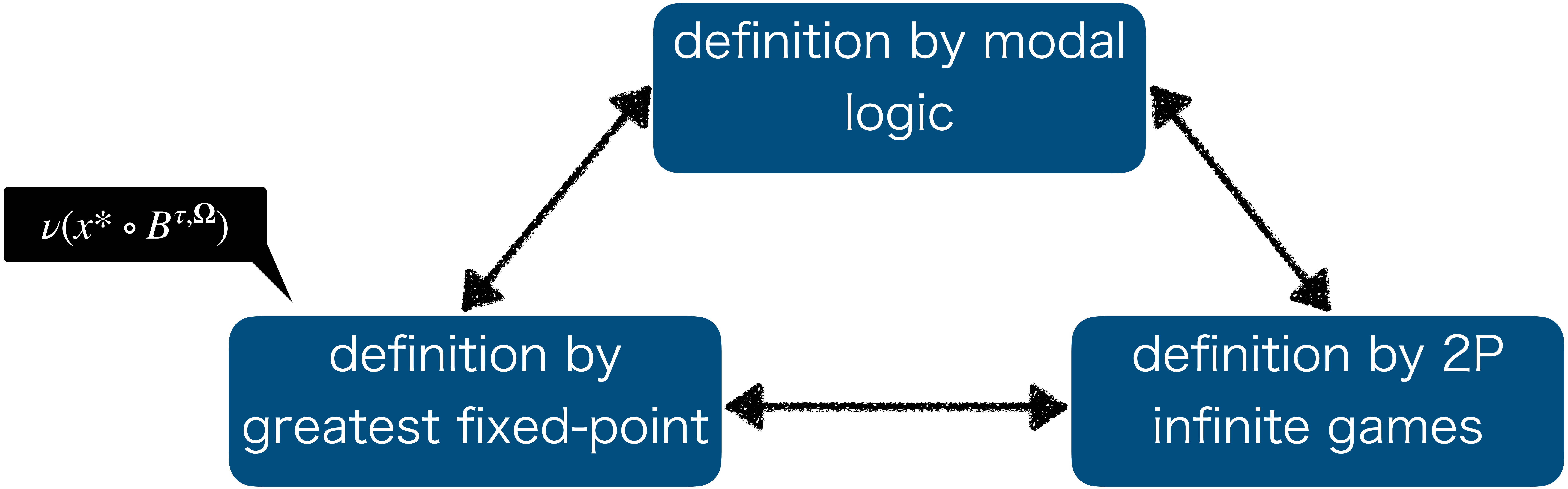
Codensity bisimilarity

- Combine codensity lifting and fibrational coinduction
- For $x\colon X \to BX$, its <u>codensity bisimilarity</u> is $\nu(x^* \circ B^{\tau,\mathbf{\Omega}})$
- Examples: bisimilarity, simulation preorder, behavioral distance, safety predicate, $\cdots$

Contents of the thesis

- Explicit formulation of “observation” enables two other forms of definition of codensity bisimilarity:



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   Presented in LICS2019

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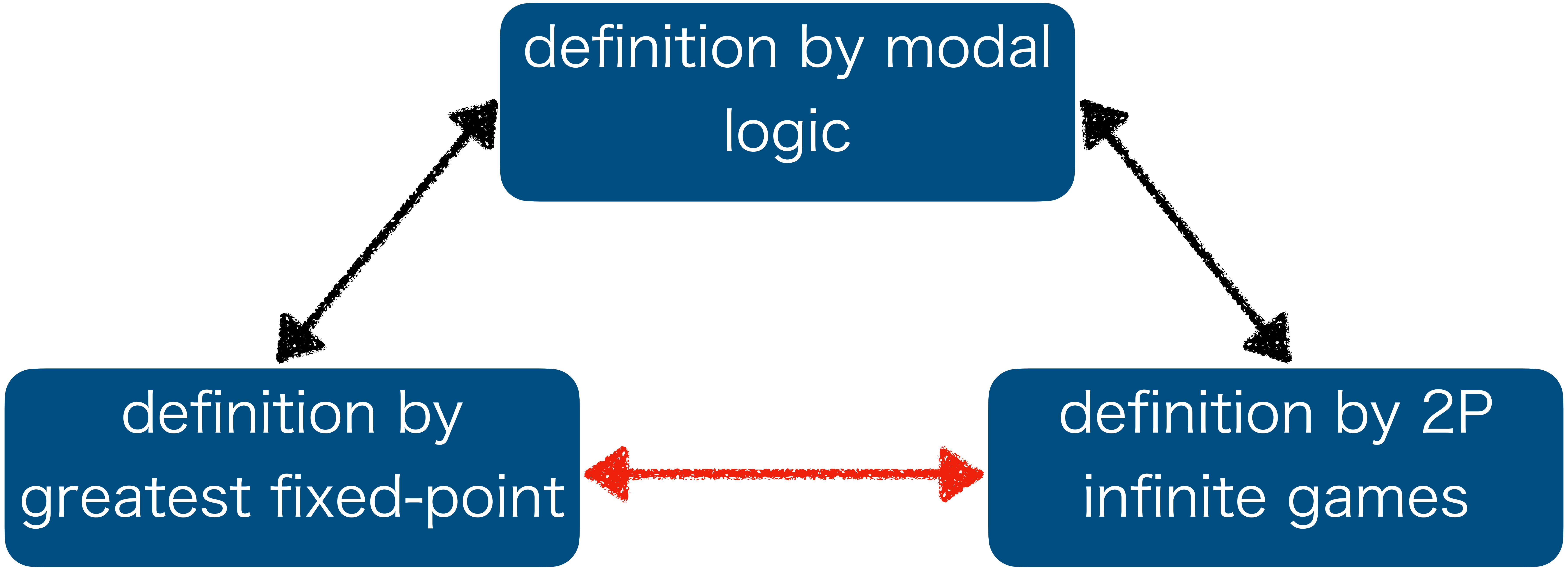
definition by modal logic
definition by greatest fixed-point
definition by 2P infinite games

Codensity game

- (possibly) infinite game by 2 players:
  - 😇, who tries to show “they are bisimilar” (in the case of bisimilarity)
  - 👿, who tries to disprove it
- 😇 wins if the game lasts indefinitely

Codensity game

😇 makes some conjecture on the codensity bisimilarity

- From $R \in \mathbb{E}_X$

👿 challenges by an “observation”

👿 chooses $\lambda \in \Lambda$ and $f\colon X \to \mathbf{\Omega}_\lambda$ s.t.

$$R \not\sqsubseteq x^*(\tau_\lambda \circ Bf)^*\mathbf{\Omega}_\lambda.$$

- From $(\lambda \in \Lambda, f\colon X \to \mathbf{\Omega}_\lambda)$

😇 chooses $R' \in \mathbb{E}_X$ s.t. $R' \not\sqsubseteq f^*\mathbf{\Omega}_\lambda$.

😇 shows the “observation” is not appropriate, by another conjecture

General "GFP=game" theorem

<u>Theorem</u>

$R \sqsubseteq \nu(x^* \circ B^{\Omega,\tau})$

if and only if

😇 has a winning strategy

starting from $R \in \mathbb{E}_X$.

GFP definition

game characterization

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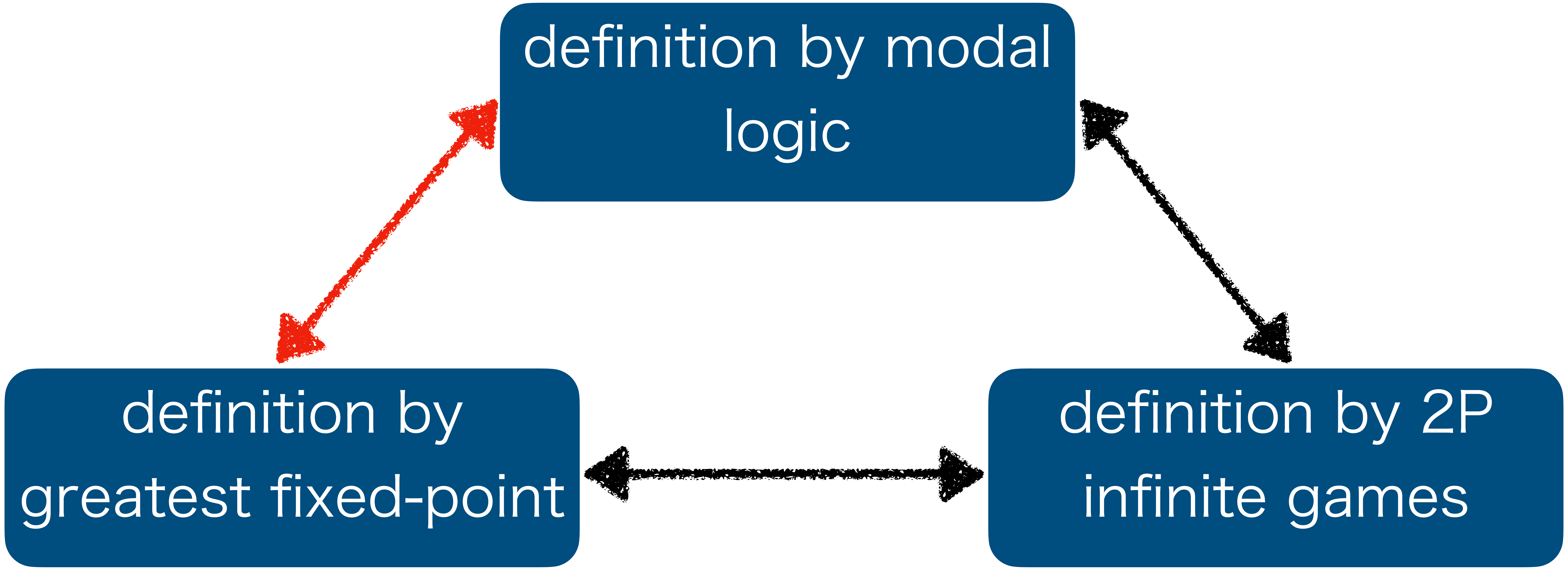
definition by modal logic
definition by greatest fixed-point
definition by 2P infinite games

Coalgebraic modal logic

- Formulas: $\varphi ::= \sigma(\varphi_1, \ldots, \varphi_n) \mid \heartsuit_\lambda \varphi'$
- Interpretation of a formula φ w.r.t. $x\colon X \to BX$ is $[\![\varphi]\!]_x\colon X \to \Omega$ in $\mathbb{C}$
- σ is interpreted by $f_\sigma\colon \Omega^n \to \Omega$ in $\mathbb{C}$
- A modality $\heartsuit_\lambda$ is interpreted

by B-algebra $\tau_\lambda\colon B\Omega \to \Omega$ ……

$$
\begin{tikzcd}
X \arrow[r, "{[\![\heartsuit_\lambda \varphi]\!]}"] \arrow[d, "x"'] & \Omega \\
BX \arrow[r, "{B[\![\varphi]\!]}"'] & B\Omega \arrow[u, "\tau_\lambda"']
\end{tikzcd}
$$

Recall logical distance···

$$
\begin{tikzcd}
(a,b) \mapsto |[\![\varphi]\!]_x(a) - [\![\varphi]\!]_x(b)| \arrow[d, dotted, no head] & (r,s) \mapsto |r - s| \arrow[d, dotted, no head] \\
X \arrow[r, "{[\![\varphi]\!]_x}"'] & {[0,1]}
\end{tikzcd}
$$

$$(a,b) \mapsto \sup_{\varphi} |[\![\varphi]\!](a) - [\![\varphi]\!](b)|$$

Fibrational logical equivalence

$$
\begin{tikzcd}
\mathbb{E} \arrow[d, "p"'] & & [\![\varphi]\!]_x^*\mathbf{\Omega} \arrow[d, dash, dotted] & & \mathbf{\Omega} \arrow[d, dash, dotted] \\
\mathbb{C} & & X \arrow[rr, "{[\![\varphi]\!]_x}"'] & & \Omega
\end{tikzcd}
$$

$$\mathsf{LE}(x) := \sqcap_\varphi \, [\![\varphi]\!]_x^*\mathbf{\Omega}$$

Adequacy & expressivity

Let $x\colon X \to BX$ be any B-coalgebra.

$\mathrm{Bisim}(x) := \nu(x^* \circ B^{\Omega,\tau})$ $\sqsubseteq ?$ $\sqsupseteq ?$ $\mathrm{LE}(x) = \sqcap_\varphi [\![\varphi]\!]_x^* \Omega$

The modal logic is $\cdots$

- adequate if $\mathrm{Bisim}(x) \sqsubseteq \mathrm{LE}(x)$ holds.
- expressive if $\mathrm{Bisim}(x) \sqsupseteq \mathrm{LE}(x)$ holds.

Two ways to observe the behavior

$\mathrm{Bisim}(x) = \nu(x^* \circ B^{\Omega,\tau})$

codensity lifting

$B^{\Omega,\tau} \colon \mathbb{E} \to \mathbb{E}$

Any observation

$f \colon X \to \Omega$

$\mathrm{LE}(x) = \sqcap_{\varphi} [\![\varphi]\!]_x^* \Omega$

Logical observation

$[\![\varphi]\!]_x \colon X \to \Omega$

modality $\tau_\lambda \colon B\Omega \to \Omega$

General adequacy

- Prop. If $\overline{B} = \overline{B}^{\mathbf{\Omega},\tau}$ holds, then the logic is adequate, that is, $\mathsf{Bisim}(x) \sqsubseteq \mathsf{LE}(x)$ holds for any $x \colon X \to BX$.

- (In the paper, the condition $\overline{B} = \overline{B}^{\mathbf{\Omega},\tau}$ is built in the definition itself; see Def.III.1 and Prop.III.13)

Two ways to observe the behavior

Bisim$(x) = \nu(x^* \circ \overline{B})$

codensity lifting

$B^{\Omega,\tau} \colon \mathbb{E} \to \mathbb{E}$

Any observation

$f \colon X \to \Omega$

LE$(x) = \sqcap_{\varphi} [\![\varphi]\!]_x^* \Omega$

Logical observation

$[\![\varphi]\!]_x \colon X \to \Omega$

modality $\tau_\lambda \colon B\Omega \to \Omega$

Approximating family of observations

- Def. $S \subseteq \mathbb{C}(X, \Omega)$ is an approximating family of observations if, for each $\lambda \in \Lambda$ and $h\colon \left(\sqcap_{k \in S} k^*\mathbf{\Omega} \right) \to \mathbf{\Omega}$, the following holds:
$$\sqcap_{l \in S, \mu \in \Lambda} (\tau_\mu \circ Bl)^*\mathbf{\Omega} \sqsubseteq (\tau_\lambda \circ B(ph))^*\mathbf{\Omega}\,.$$

- In other words, if each “legitimate observation” h gives no additional information, then S is an approximating family.

Main theorems

Let $x\colon X \to BX$ be a coalgebra. Assume $\overline{B} = \overline{B}^{\mathbf{\Omega},\tau}$.

- Thm. If the set $\{[\![\varphi]\!] \mid \varphi$ is a formula$\}$ is an approximating family, then the logic is expressive. (Knaster—Tarski form, Thm. IV.5)
- Thm. If, for each n, the set $\{[\![\varphi]\!] \mid \varphi$ is a formula of modal depth $\leq n\}$ is an approximating family, then the logic is expressive. (Kleene form, Thm.IV.7)

("Expressive" means $\mathsf{Bisim}(x) \sqsupseteq \mathsf{LE}(x)$, as mentioned)

General adequacy

- Prop. If $\overline{B} = \overline{B}^{\mathbf{\Omega},\tau}$ holds, then the logic is adequate, that is, $\mathsf{Bisim}(x) \sqsubseteq \mathsf{LE}(x)$ holds for any $x\colon X \to BX$.
- (In the paper, the condition $\overline{B} = \overline{B}^{\mathbf{\Omega},\tau}$ is built in the definition itself; see Def.III.1 and Prop.III.13)

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Presented in CMCS2020

Fiberedness

- Fibered lifting: functor lifting that interact well with reindexing
- $B\colon \mathbb{C} \to \mathbb{C}$, $p\colon \mathbb{E} \to \mathbb{C}$ ($\mathbf{CLat}_{\sqcap}$-fibration)
- Lifting $\bar{B}\colon \mathbb{E} \to \mathbb{E}$ is <u>fibered</u> if
  - for any $f\colon X \to Y$ in $\mathbb{C}$ and $E \in \mathbb{E}_Y$,
  - $\bar{B}(f^*E) = (Bf)^*\bar{B}E$ holds.

- Kantorovich lifting is always fibered [Baldan et al. FSTTCS14]
    - In that case fiberedness ⇔ preservation of isometries
- Codensity lifting …… ???

Property of [0,1]

[Baldan et al. FSTTCS14]

- $f\colon X \to Y$ and $(Y, d) \in \mathbf{PMet}_1$
- For any g, there exists h:

$$
\begin{tikzcd}
(X, f^* d) \arrow[dd, "f"'] \arrow[drr, "g"] & & \\
& \circlearrowright & ([0,1], d_{\mathbb{R}}) \\
(Y, d) \arrow[urr, dotted, "\exists h"'] & &
\end{tikzcd}
$$

C-injective object

- $p\colon \mathbb{E} \to \mathbb{C}$ ($\mathbf{CLat}_{\sqcap}$-fibration)
- Def. $\mathbf{\Omega} \in \mathbb{E}_{\Omega}$ is c-injective if, for any $f\colon X \to Y$, $E \in \mathbb{E}_Y$, and $g\colon f^*E \to \mathbf{\Omega}$, the following h exists:

$$\begin{tikzcd}
f^*E \arrow[dd, "f"'] \arrow[rrd, "g"] & & \\
 & \circlearrowright & \mathbf{\Omega} \\
E \arrow[rru, dotted, "\exists h"'] & &
\end{tikzcd}$$

Fiberedness theorem

- Thm. If $\mathbf{\Omega} \in \mathbb{E}$ is c-injective, then the codensity lifting $B^{\mathbf{\Omega},\tau}\colon \mathbb{E} \to \mathbb{E}$ is a fibered lifting of B.
- Cor. In this case, any coalgebra morphism $f\colon X \to Y$ from $x\colon X \to BX$ to $y\colon Y \to BY$ "reflects" the coinductive predicate: $\nu(x^* \circ B^{\mathbf{\Omega},\tau}) = f^*\nu(y^* \circ B^{\mathbf{\Omega},\tau})$ holds.

Examples of fibered codensity liftings

fibration	Ω	c-injective?	examples
Pre→**Set**	(2,≦)	Yes	upper, lower, convex preorders
ERel→**Set**	(2,=)	No	(for bisimilarity)
EqRel→**Set**	(2,=)	Yes	(for bisimilarity)
$\mathbf{PMet}_1$→**Set**	$([0,1],d_R)$	Yes	Hausdorff and Kantorovich distances
U*($\mathbf{PMet}_1$)→**Meas**	$([0,1],d_R)$	No	Kantorovich distance
Top→**Set**	Sierpinski space	Yes	(for bisimulation topology)

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Relations between codensity liftings

- How to relate different codensity bisimilarities? Any general theory?
  - Ex. Behavioral metric = 0 $\iff$ bisimilar
- A few preliminary results are in the (just accepted) journal version of [LICS2019]
- A satisfying form of result is: codensity lifting is a functor from some category to the functor category.

Elaborating examples in new fibrations

- "Behavioral topology" in $\mathbf{Top} \to \mathbf{Set}$: computability result using domain theory?
- "Behavioral uniformity" in $\mathbf{Unif} \to \mathbf{Set}$:
    - Stable under a "small" change of system?
    - "Asymptotic" behaviors can be read off?

Algorithm from fibrational game?

- Solve codensity games to calculate codensity bisimilarities?
  - The arena is infinite in many cases, so not trivial
- Reduction to some finite part? When can we do it?
- A new scheme to obtain a game from Wasserstein lifting?